ON THE NON-VANISHING OF GENERALIZED KATO CLASSES FOR ELLIPTIC CURVES OF RANK 2

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ABSTRACT. We prove the non-vanishing of the generalized Kato classes of Darmon–Rotger in cases where they lie in the pro-p Selmer group of elliptic curves $E/\mathbf{Q}$ of rank 2. In particular, we prove the first cases of a conjecture of Darmon–Rotger in this rank 2 setting.

CONTENTS

1. INTRODUCTION

1.1. **Motivating question.** Let E be a modular elliptic curve over $\mathbf{Q}$ of conductor N, and let $L(E,s)$ be the associated Hasse–Weil L-series. After the groundbreaking work of Gross–Zagier [GZ86] and Kolyvagin [Kol88], we know that

$$\operatorname{ord}_{s=1}L(E,s)\in\{0,1\}\quad\Longrightarrow\quad\operatorname{ord}_{s=1}L(E,s)=\operatorname{rank}_{\mathbf{Z}}E(\mathbf{Q}),\tag{1.1}$$

as predicted by the Birch–Swinnerton-Dyer conjecture. The proof of (1.1) relies on choosing a suitable auxiliary imaginary quadratic field K such that $\operatorname{ord}_{s=1}L(E/K,s)=1$; by the Gross–Zagier formula this ensures the non-torsionness of the associated Heegner point $y_K\in E(K)$, a point which descends to $E(\mathbf{Q})$ when $\operatorname{ord}_{s=1}L(E,s)=1$.

Given the desirability of extending these results to higher ranks, it is natural to ask:

Question 1.1. Suppose $\operatorname{ord}_{s=1}L(E,s)=2$, and choose an imaginary quadratic field K with

$$\operatorname{ord}_{s=1}L(E/K,s)=2.\tag{1.2}$$

Can one use K to produce canonical non-torsion points in $E(\mathbf{Q})$? Or at least, can one produce explicit nonzero classes in the pro-p Selmer group $\operatorname{Sel}(\mathbf{Q},T_pE)$ for suitable primes p?

Here $\operatorname{Sel}(\mathbf{Q},T_pE)$ denotes the inverse limit under the multiplication-by-p maps of the usual p^n-descent Selmer groups $\operatorname{Sel}_{p^n}(E/\mathbf{Q})\subset H^1(\mathbf{Q},E[p^n])$, thus sitting in the exact sequence

$$0\longrightarrow E(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Z}_p\longrightarrow\operatorname{Sel}(\mathbf{Q},T_pE)\longrightarrow T_p\text{Ш}(E/\mathbf{Q})\longrightarrow 0,$$

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where $T_p Ш(E/\mathbf{Q})$ should be trivial, since $Ш(E/\mathbf{Q})$ is expected to be finite. In this paper, for suitable primes p, we provide an affirmative answer to the second part of Question 1.1, with condition (1.2) replaced by its algebraic counterpart:

$$\mathrm{rank}_{\mathbf{Z}} E(K) = \mathrm{rank}_{\mathbf{Z}} E(\mathbf{Q}) = 2 \quad \text{and} \quad \# Ш(E/\mathbf{Q})[p^\infty] < \infty. \tag{1.3}$$

This is achieved by establishing certain cases of a recent conjecture by Darmon–Rotger which we now recall.

1.2. **A conjecture of Darmon–Rotger.** Following their groundbreaking work [DR17a] on the equivariant Birch and Swinnerton-Dyer conjecture, Darmon–Rotger [DR16] formulated a non-vanishing criterion for the *generalized Kato classes* introduced in [DR17a]. In this paper, we consider a special case in which their conjecture predicts that those classes have a bearing on the arithmetic of elliptic curves over $\mathbf{Q}$ of rank 2.

Let $E/\mathbf{Q}$ be an elliptic curve of conductor N, and let K be an imaginary quadratic field of discriminant prime to N. Fix a prime $p \nmid 2N$ of good ordinary reduction for E, and assume that $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K. Let $\chi : G_K \to \mathbf{C}^\times$ be a ring class character of conductor prime to Np with $\chi(\overline{\mathfrak{p}}) \neq \pm 1$, and set $\alpha := \chi(\overline{\mathfrak{p}})$, $\beta := \chi(\mathfrak{p})$.

Let f be the weight 2 newform associated with E by modularity [Wil95, TW95, BCDT01], so that $L(E,s) = L(f,s)$, and let g and h be the weight 1 theta series of χ and χ^{-1}, respectively. As explained in [DR16] (in which g and h can be more general weight 1 eigenforms), attached to the triple (f,g,h) and the prime p one has four *generalized Kato classes*

$$\kappa(f, g_\alpha, h_{\alpha^{-1}}),\ \kappa(f, g_\alpha, h_{\beta^{-1}}),\ \kappa(f, g_\beta, h_{\alpha^{-1}}),\ \kappa(f, g_\beta, h_{\beta^{-1}})\ \in \mathrm{H}^1(\mathbf{Q}, V_{fgh}), \tag{1.4}$$

where $V_{fgh} \simeq V_p E \otimes V_g \otimes V_h$ is the tensor product of the p-adic representations associated to f, g, and h. The class $\kappa(f, g_\alpha, h_{\alpha^{-1}})$ arises as the p-adic limit

$$\kappa(f, g_\alpha, h_{\alpha^{-1}}) = \lim_{\ell \to 1} \kappa(f, \boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$$

as $(\boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ runs over the classical weight $\ell \geqslant 2$ specializations of Hida families $(\boldsymbol{g}, \boldsymbol{h})$ passing through the p-stabilizations $(g_\alpha, h_{\alpha^{-1}})$ in weight 1, where

$$g_\alpha := g(q) - \beta g(q^p), \quad h_{\alpha^{-1}} := h(q) - \beta^{-1} h(q^p),$$

and $\kappa(f, \boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ is obtained from the p-adic étale Abel–Jacobi image of certain higher-dimensional Gross–Kudla–Schoen diagonal cycles [GK92, GS95] on triple products of modular curves.

One of the main results of [DR17a] is an "explicit reciprocity law" whereby the image of the class $\kappa(f, g_\alpha, h_{\alpha^{-1}})$ under the restriction map $\mathrm{loc}_p : \mathrm{H}^1(\mathbf{Q}, V_{fgh}) \to \mathrm{H}^1(\mathbf{Q}_p, V_{fgh})$ satisfies

$$\exp^*(\mathrm{loc}_p(\kappa(f, g_\alpha, h_{\alpha^{-1}}))) = 0 \iff L(1, f \otimes g \otimes h) = 0,$$

under the dual exponential map $\exp^*$ of Bloch–Kato [BK90]. As a result, the class $\kappa(f, g_\alpha, h_{\alpha^{-1}})$ (and similarly the other three classes (1.4)) lies in the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V_{fgh}) \subset \mathrm{H}^1(\mathbf{Q}, V_{fgh})$ if and only if the triple product L-series $L(s, f \otimes g \otimes h)$ vanishes at $s = 1$. One of the main conjectures of [DR16] went further to propose a criterion for when the classes (1.4) should generate a *non-trivial* subspace of $\mathrm{Sel}(\mathbf{Q}, V_{fgh})$; this should be the case precisely when $L(s, f \otimes g \otimes h)$ vanishes to order exactly 2 at $s = 1$.

In our setting, in light of the decomposition

$$V_{fgh} \simeq \left(V_p E \otimes \mathrm{Ind}_K^{\mathbf{Q}} \mathbb{1}\right) \oplus \left(V_p E \otimes \mathrm{Ind}_K^{\mathbf{Q}} \chi^2\right) \tag{1.5}$$

and the corresponding factorization

$$L(s, f \otimes g \otimes h) = L(E, s) \cdot L(E^K, s) \cdot L(E/K, \chi^2, s), \tag{1.6}$$

where $E^K/\mathbf{Q}$ is the twist of E by the quadratic character attached to K, the cases of the main conjectures of [DR16] relevant to us (and to the motivating question from the previous section) may be stated as follows (*cf.* [DR16, Conj. 3.2, §4.5.3]). Let

$$\kappa_{\alpha,\alpha^{-1}},\ \kappa_{\alpha,\beta^{-1}},\ \kappa_{\beta,\alpha^{-1}},\ \kappa_{\beta,\beta^{-1}} \in \mathrm{H}^1(\mathbf{Q}, V_pE) \tag{1.7}$$

be the image of the classes in (1.4) under the projection $\mathrm{H}^1(\mathbf{Q}, V_{fgh}) \to \mathrm{H}^1(\mathbf{Q}, V_pE)$ deduced from (1.5).

Conjecture 1.2 (Darmon–Rotger)**.** *Assume that $L(E^K, 1)$ and $L(E/K, \chi^2, 1)$ are both nonzero. Then the generalized Kato classes* (1.7) *generate a non-trivial subspace of* $\mathrm{Sel}(\mathbf{Q}, V_pE)$ *if and only if the following equivalent conditions hold:*

(a) $\mathrm{ord}_{s=1}L(E, s) = 2$*;*
(b) $\mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q}) = 2$ *and* $Ш(E/\mathbf{Q})[p^\infty]$ *is finite.*

Remark 1.3.

(i) Of course, the equivalence (a) $\Longleftrightarrow$ (b) is predicted by the Birch and Swinnerton-Dyer conjecture.
(ii) Conjecture 1.2 is a special case of [DR16, Conj. 3.2], and at the time of its formulation it was not known (even experimentally) in any single example (see [*loc.cit.*, §4.5.3]). As an application of the main result of this paper (Theorem 1 below), numerical examples supporting this conjecture will be presented in §6.

1.3. **Main result.** We keep the setting introduced in §1.2, and let $\bar{\rho}_{E,p} : G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathbf{F}_p}(E[p])$ be the mod p Galois representation associated to E. Write

$$N = N^+N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in K. Our main result towards Conjecture 1.2 is the following.

Theorem 1. *Assume that $L(E^K, 1)$ and $L(E/K, \chi^2, 1)$ are both nonzero, and that*

- $\bar{\rho}_{E,p}$ *is irreducible,*
- N^- *is square-free,*
- $\bar{\rho}_{E,p}$ *is ramified at every prime $q \mid N^-$.*

If $\mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q}) = 2$ *and* $Ш(E/\mathbf{Q})[p^\infty]$ *is finite, then the generalized Kato classes* (1.7) *generate a non-trivial subspace of* $\mathrm{Sel}(\mathbf{Q}, V_pE)$*. To be more precise, $\kappa_{\alpha,\alpha^{-1}}$ and $\kappa_{\beta,\beta^{-1}}$ are both nonzero, while $\kappa_{\alpha,\beta^{-1}} = \kappa_{\beta,\alpha^{-1}} = 0$.*

Remark 1.4. The hypotheses in Theorem 1 imply that E has root number $+1$ (by e.g. [Nek01]), that $L(E, 1) = 0$ (by e.g. [Kat04]), and that N^- is the product of an *odd* number of primes. Thus the elliptic curves $E/\mathbf{Q}$ in Theorem 1 satisfy $\mathrm{ord}_{s=1}L(E, s) \geqslant 2$. On the other hand, if the root number of E is $+1$ and $\bar{\rho}_{E,p}$ is irreducible and ramified at some prime q, by [BFH90] and [Vat03] there exist infinitely many imaginary quadratic fields K and ring class characters χ of prime-power conductor such that the following hold:

- q is inert in K,
- every prime factor of N/q splits in K,
- $L(E^K, 1) \neq 0$ and $L(E/K, \chi^2, 1) \neq 0$.

Therefore, by Theorem 1 the generalized Kato classes (1.7) of Darmon–Rotger provide an explicit construction of non-trivial Selmer classes for rank 2 elliptic curves having a flavor similar to the Heegner point construction for rank 1 elliptic curves. In §6, we will exhibit several numerical examples satisfying the hypotheses of Theorem 1, and this gives the first examples of the non-vanishing of generalized Kato classes for rank 2 elliptic curves.

Prior to this work, the only known construction of non-trivial classes in $\mathrm{Sel}(\mathbf{Q}, V_pE)$ for elliptic curves $E/\mathbf{Q}$ with $\mathrm{ord}_{s=1}L(E, s) \geqslant 2$ were due to Skinner–Urban (see [SU06, Urb13]). It would be very interesting to compare the two constructions.

Remark 1.5. Letting $\log_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to \mathbf{Q}_p$ denote the composition of the restriction map $\mathrm{loc}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p) \otimes_{\mathbf{Z}} \mathbf{Q}_p$ with the formal group logarithm $E(\mathbf{Q}_p) \otimes_{\mathbf{Z}} \mathbf{Q}_p \to \mathbf{Q}_p$, the proof of Theorem 1 yields the following expressions for the generalized Kato classes mod $\mathbf{Q}_p^\times$:

$$\kappa_{\alpha,\alpha^{-1}} = \log_p(Q) \cdot P - \log_p(P) \cdot Q, \quad \kappa_{\alpha,\beta^{-1}} = 0,$$
$$\kappa_{\beta,\alpha^{-1}} = 0, \quad \kappa_{\beta,\beta^{-1}} = \log_p(Q) \cdot P - \log_p(P) \cdot Q,$$

where (P, Q) is any $\mathbf{Q}$-basis of $E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$. In other words, under the hypotheses of Theorem 1 the classes (1.7) generate the p-adic line

$$\ker(\log_p) \subset \mathrm{Sel}(\mathbf{Q}, V_pE)$$

inside the 2-dimensional $\mathrm{Sel}(\mathbf{Q}, V_pE)$. This confirms a prediction made by Darmon–Rotger (see [DR16, §4.5.3]). Moreover, one can show that the classes $\kappa_{\alpha,\alpha^{-1}}$ and $\kappa_{\beta,\beta^{-1}}$ mod $\overline{\mathbf{Q}}^\times$ are independent of the auxiliary choice of ring class character χ (see Remark 5.1).

1.4. **Outline of the proof.** The proof of Theorem 1 is based on the study of a certain p-adic L-function attached to E/K that we know recall, and its relation with the generalized Kato classes that we establish in this paper. We outline the proof of the non-vanishing of

$$\kappa_{E,K} := \kappa_{\alpha,\alpha^{-1}} \in \mathrm{H}^1(\mathbf{Q}, V_pE);$$

changing the roles of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ yields the analogous result for $\kappa_{\beta,\beta^{-1}}$.

- *Step 1: Euler system construction of Bertolini–Darmon theta elements.*

Let Γ_∞ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K. By [BD96, CH18], building on various generalizations of Gross' explicit form of Waldspurger's special value formula [Wal85, Gro87], one can construct a p-adic L-function $\Theta_{f/K} \in \mathbf{Z}_p[\![\Gamma_\infty]\!]$ interpolating "square-roots" of the central critical values $L(E/K, \phi, 1)$, as ϕ runs over finite order character of Γ_∞. Starting with [BD96], the element $\Theta_{f/K}$ has been widely studied (see [Vat03, BD05, PW11] for a sample), but one didn't know its place in Perrin-Riou's vision [PR00] (or its natural extension [LZ14] to $\mathbf{Z}_p$-extensions other than the cyclotomic), whereby p-adic L-functions ought to arise as the image of p-adic families of special cohomology classes under generalized Coleman power series maps.

Letting $\kappa(f, \boldsymbol{gh})$ be the 1-dimensional p-adic family of cohomology classes $\kappa(f, \boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ used in the construction of $\kappa(f, g_\alpha, h_{\alpha^{-1}})$, in Section 4 we prove that

$$\mathrm{Col}^\eta(\mathrm{loc}_p(\kappa(f, \boldsymbol{gh})) = \Theta_{f/K} \cdot (\text{nonzero constant}), \tag{1.8}$$

where Col^η is a generalized Coleman power series map defined in terms of an "anticyclotomic" variant of Perrin-Riou's big exponential map [PR94]. The proof of (1.8) relies on work towards the "explicit reciprocity law" for diagonal cycles by Darmon–Rotger [DR17a], and reduces to it when combined with a factorization of the triple product p-adic L-function [Hsi17].

- *Step 2: Howard's p-adic height formula formula for derived heights.*

In light of the construction of $\kappa_{E,K}$ and the interpolation properties satisfied by Col^η and $\Theta_{f/K}$, the equality (1.8) specialized at the trivial character shows that

$$L(E, 1) = 0 \quad \Longrightarrow \quad \kappa_{E,K} \in \mathrm{Sel}(\mathbf{Q}, V_pE).$$

In other words, viewing it as an equality in the power series ring $\mathbf{Z}_p[\![T]\!]$, the above implication is what can be drawn from (1.8) evaluated at $T = 0$. To further deduce the non-vanishing of $\kappa_{E,K}$ we are led to look at *the leading term* of (1.8) at $T = 0$. To that end, let

$$\mathrm{Sel}(K, V_pE) = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset \cdots \supset S^{(\infty)} \tag{1.9}$$

be the filtration defined by Bertolini–Darmon [BD95] and Howard [How04], and let

$$h^{(r)} \colon S^{(r)} \times S^{(r)} \longrightarrow \mathbf{Q}_p$$

be the r-th *derived* anticyclotomic p-adic height pairing of [BD95, How04]. From the known properties of $h^{(r)}$, one can easily see that if

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = \mathrm{Sel}(K, V_pE) \quad \text{and} \quad \dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2,$$

as we have under our hypotheses, then (1.9) reduces to

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = S^{(1)} = S^{(2)} \cdots = S^{(r)} \quad \text{and} \quad S^{(r+1)} = S^{(r+2)} = \cdots = S^{(\infty)} = \{0\}$$

for some even integer $r \geqslant 2$. On the other hand, setting

$$\rho := \mathrm{ord}_{T=0}\Theta_{f/K}(T),$$

one can deduce that $r \geqslant \rho$ from the work of Skinner–Urban [SU14]; in particular, $\mathrm{Sel}(\mathbf{Q}, V_pE) = S^{(\rho)}$. Finally, combining Howard's generalization [How04] of Rubin's formula with Kobayashi's extension [Kob18] of Perrin-Riou's big exponential map for Lubin–Tate $\mathbf{Z}_p$-extensions, we shall prove that

$$h^{(\rho)}(\kappa_{E,K}, x) = \left(\frac{d}{dT}\right)^{\rho} \Theta_{f/K}(T)|_{T=0} \cdot \log_p(x) \cdot (\text{nonzero constant in } \overline{\mathbf{Q}}),$$

for all $x \in S^{(\rho)} = E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}_p$. Since our assumptions immediately imply that the map $\log_p$ is nonzero, the non-vanishing of $\kappa_{E,K}$ follows.

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2. Derived p-adic height pairings

2.1. **Notation and definitions.** In this section, we review the definition of the derived p-adic height pairings in [How04]. Let p be a rational prime and let K be a number field. Let Σ be a finite set of places of K containing all archimedean places and all places above p. Let K_Σ be the maximal algebraic extension of K unramified outside Σ and let $G_{K,\Sigma} = \mathrm{Gal}(K_\Sigma/K)$. Let K_∞/K be a $\mathbf{Z}_p$-extension in K_Σ. We assume that all primes above p are totally ramified in K_∞. Let $K_n \subset K_\infty$ be the subfield with $[K_n\colon K] = p^n$. Let $\Gamma_n = \mathrm{Gal}(K_n/K)$ and $\Gamma_\infty = \mathrm{Gal}(K_\infty/K)$. Let $\Lambda = \mathbf{Z}_p[\![\Gamma_\infty]\!]$ and let $\kappa_\Lambda\colon G_{K,\Sigma} \to \mathrm{Gal}(K_\infty/K) \to \Lambda^\times$ be the tautological character $\kappa_\Lambda(\sigma) = \sigma|_{K_\infty}$. Let $\iota\colon \Gamma_\infty \to \Gamma_\infty$ be the involution $\gamma \mapsto \gamma^{-1}$. For any Λ-module M and each integer k, let $M\{k\}$ be the $G_{K,\Sigma}$-module M on which $G_{K,\Sigma}$ acts via κ_Λ^k.

Let $\mathcal{O}$ be a local ring finitely generated over $\mathbf{Z}_p$ with maximal ideal $\mathfrak{m}$. Put $\Lambda_\mathcal{O} = \Lambda \otimes_{\mathbf{Z}_p} \mathcal{O}$. Denote by $\mathbf{Mod}_\mathcal{O}$ the category of $\mathcal{O}[G_{K,\Sigma}]$-modules finite free over $\mathcal{O}$. For T an object of $\mathbf{Mod}_\mathcal{O}$ we let $T_\Lambda = T \otimes_\mathcal{O} \Lambda_\mathcal{O}\{-1\}$ be the $G_{K,\Sigma}$-module $T \otimes_\mathcal{O} \Lambda_\mathcal{O}$ twisted by κ_Λ^{-1}. Let $\mathcal{K}$ be the localization of $\Lambda_\mathcal{O}$ at the prime $\mathfrak{m}\Lambda_\mathcal{O}$, and set $P := \mathcal{K}/\Lambda_\mathcal{O}$. Likewise we define $T_\mathcal{K} = T \otimes_\mathcal{O} \mathcal{K}\{-1\}$ and $T_P = T \otimes_\mathcal{O} P\{-1\}$. We shall denote the limits

$$\mathrm{H}^1(K_\infty, T) := \varinjlim_n \mathrm{H}^1(K_n, T), \qquad \widehat{\mathrm{H}}^1(K_\infty, T) := \varprojlim_n \mathrm{H}^1(K_n, T)$$

with respect to the restriction and corestriction maps, respectively. Let

$$\mathrm{pr}_{K_n}\colon \widehat{\mathrm{H}}^1(K_\infty, T) \to \mathrm{H}^1(K_n, T)$$

be the canonical projection map. Throughout, by [How04, Lem. 1.4] and Shapiro's lemma, we shall make use of the identification

$$\mathrm{H}^1(K, T_\Lambda) = \widehat{\mathrm{H}}^1(K_\infty, T).$$

Let $T^* = \mathrm{Hom}(T, \mathcal{O}(1))$ and denote by $e\colon T \times T^* \to \mathcal{O}(1)$ the canonical $G_{K,\Sigma}$-equivariant perfect paring. For any place v of K and finite extension L/K_v, let $\mathrm{inv}_L : \mathrm{H}^1(L, \mathcal{O}(1)) \simeq \mathcal{O}$ be the invariant map and let $\langle\, ,\,\rangle_L\colon \mathrm{H}^1(L,T)\times\mathrm{H}^1(L,T^*)\to\mathcal{O}$ be the perfect pairing $\langle z,w\rangle_L := \mathrm{inv}_L(z\cup w)$. Let

$$e_\Lambda\colon T_\Lambda\times T_\Lambda^*\to\Lambda_{\mathcal{O}}(1)$$

be the unique perfect $G_{K,\Sigma}$-equivariant pairing characterized by

$$e_\Lambda(t\otimes\lambda_1, s\otimes\lambda_2)=\lambda_1\lambda_2^\iota e_\Lambda(t,s)\quad\text{for all }\lambda_1,\lambda_2\in\Lambda_{\mathcal{O}}.$$

For every place v of K, define the bilinear pairing

$$\langle\, ,\,\rangle_{K_\infty,v}\colon \mathrm{H}^1(K_v,T_\Lambda)\times\mathrm{H}^1(K_v,T_\Lambda^*)\longrightarrow\mathrm{H}^2(K_v,\Lambda_{\mathcal{O}}(1))\simeq\Lambda_{\mathcal{O}},\quad \langle\mathbf{z},\mathbf{w}\rangle_{K_\infty,v}=\mathrm{inv}_v(e_\Lambda(\mathbf{z}\cup\mathbf{w})).$$

Fix a topological generator $\boldsymbol{\gamma}$ of Γ_∞ and set

$$g_n := \boldsymbol{\gamma}^{p^n}-1\in\Lambda.$$

Then $\Lambda_{\mathcal{O}}/(g_n)=\mathcal{O}[\Gamma_n]$. By definition, if $\mathbf{z}=(\mathbf{z}_n)\in\mathrm{H}^1(K_v,T_\Lambda)=\varprojlim_n\mathrm{H}^1(K_{n,v},T)$ and $\mathbf{w}=(\mathbf{w}_n)\in\mathrm{H}^1(K_v,T_\Lambda^*)=\varprojlim_n\mathrm{H}^1(K_{n,v},T^*)$, then we have

$$\langle\mathbf{z},\mathbf{w}\rangle_{K_\infty,v}\pmod{g_n}=\sum_{\tau\in\Gamma_n}\langle\mathbf{z}_n^{\tau^{-1}},\mathbf{w}_n\rangle_{K_{n,v}}\tau. \tag{2.1}$$

Let $\mathcal{F}=\left\{\mathrm{H}^1_{\mathcal{F}}(K_v,T_{\mathcal{K}})\right\}_{v\in\Sigma}$ be a Selmer structure on $T_{\mathcal{K}}$, namely a choice of $\mathcal{K}$-submodule $\mathrm{H}^1_{\mathcal{F}}(K_v,T_{\mathcal{K}})\subset\mathrm{H}^1(K_v,T_{\mathcal{K}})$ for every $v\in\Sigma$, and let $\mathrm{H}^1_{\mathcal{F}}(K_v,T_P)$ be the image of the natural map $\mathrm{H}^1_{\mathcal{F}}(K_v,T_{\mathcal{K}})\to\mathrm{H}^1(K_v,T_P)$ induced by the quotient $\mathcal{K}\to P$. Define the *Selmer module* $\mathrm{H}^1_{\mathcal{F}}(K,T_P)$ to be the kernel of the map

$$\mathrm{H}^1(G_{K,\Sigma},T_P)\to\prod_{v\in\Sigma}\mathrm{H}^1(K_v,T_P)/H^1_{\mathcal{F}}(K_v,T_P).$$

2.2. In this subsection, we suppose that $\mathfrak{m}^m=0$ for some positive integer m, namely that $\mathcal{O}$ is Artinian. By [How04, Lem. 1.2], we then have

$$K=\bigcup_{n=0}^{\infty}\Lambda_{\mathcal{O}}\frac{1}{g_n}.$$

Moreover, by [How04, Lem. 1.5] and Shapiro's lemma, there is a natural isomorphism

$$\eta_{\boldsymbol{\gamma}}\colon\mathrm{H}^1(K,T_P)=\varinjlim_n\mathrm{H}^1(K,T_\Lambda\otimes\Lambda_{\mathcal{O}}g_n^{-1}/\Lambda_{\mathcal{O}})\simeq\varinjlim_n\mathrm{H}^1(K_n,T_\Lambda/g_nT_\Lambda)=\mathrm{H}^1(K_\infty,T). \tag{2.2}$$

By definition, for $\mathbf{z}=\{\mathbf{z}_n\}\in\widehat{\mathrm{H}}^1(K_\infty,T)$ we have

$$\eta_{\boldsymbol{\gamma}}(\frac{\mathbf{z}}{\boldsymbol{\gamma}-1})=\mathrm{pr}_K(\mathbf{z})\in\mathrm{H}^1(K,T). \tag{2.3}$$

For each n, let $\mathrm{H}^1_{\mathcal{F}}(K_n,T)$ be the Selmer module consisting of classes $s\in\mathrm{H}^1(K_n,T)$ such that the image of s in $\mathrm{H}^1(K_\infty,T)$ belongs to $\eta_{\boldsymbol{\gamma}}(\mathrm{H}^1_{\mathcal{F}}(K,T_P))$. Thus

$$\mathrm{H}^1_{\mathcal{F}}(K_\infty,T)=\varinjlim_n\mathrm{H}^1_{\mathcal{F}}(K_n,T)=\eta_{\boldsymbol{\gamma}}(\mathrm{H}^1_{\mathcal{F}}(K,T_P)).$$

Let J be the augmentation ideal of $\Lambda_{\mathcal{O}}$, the principal ideal of $\Lambda_{\mathcal{O}}$ generated by $\boldsymbol{\gamma}-1$. For each positive integer r, put

$$Y_T^{(r)} := \mathrm{H}^1_{\mathcal{F}}(K_\infty,T)[J]\cap J^{r-1}\mathrm{H}^1_{\mathcal{F}}(K_\infty,T). \tag{2.4}$$

This defines a decreasing filtration $Y_T^{(1)}\supset Y_T^{(2)}\supset Y_T^{(2)}\cdots$.

Let $\mathcal{F}^\perp=\left\{\mathrm{H}^1_{\mathcal{F}^\perp}(K_v,T_K^*)\right\}_{v\in\Sigma}$ be the Selmer structure on T_K^* such that $\mathrm{H}^1_{\mathcal{F}^\perp}(K_v,T_K^*)$ and $\mathrm{H}^1_{\mathcal{F}}(K_v,T_K)$ are exact orthogonal complements under local Tate duality at every place $v\in\Sigma$. Let

$$[-,-]_{\mathrm{CT}}\colon\mathrm{H}^1_{\mathcal{F}}(K,T_P)\times\mathrm{H}^1_{\mathcal{F}^\perp}(K,T_P^*)\longrightarrow P$$

be the $\Lambda_{\mathcal{O}}$-adic Cassels–Tate pairing of [How04, Thm. 1.8]. The *r-th derived height pairing*

$$h^{(r)}_{\mathcal{O}}(-,-)\colon Y^{(r)}_T \times Y^{(r)}_{T^*} \to J^r/J^{r+1}$$

in [How04, Def. 2.2] is defined by

$$\begin{aligned} h^{(r)}_{\mathcal{O}}(z,w) :=& (\boldsymbol{\gamma}-1)^2 \cdot [\eta^{-1}_{\boldsymbol{\gamma}}(z), \eta^{-1}_{\boldsymbol{\gamma}}(w)]_{\mathrm{CT}} \\ =& (\boldsymbol{\gamma}-1)^{r+1} \cdot [\eta^{-1}_{\boldsymbol{\gamma}}(u), \eta^{-1}_{\boldsymbol{\gamma}}(w)]_{\mathrm{CT}}, \end{aligned}$$

writing $z = (\boldsymbol{\gamma}-1)^{r-1}u$ with $u \in \mathrm{H}^1_{\mathcal{F}}(K, T_P)$. Note that $[\eta^{-1}_{\boldsymbol{\gamma}}(u), \eta^{-1}_{\boldsymbol{\gamma}}(w)]_{\mathrm{CT}} \in (\boldsymbol{\gamma}-1)^{-1}\Lambda/\Lambda$, so we find that $h^{(r)}_{\mathcal{O}}(z,w)$ belongs to J^r/J^{r+1}. The following is a restatement of [How04, Thm. 2.5], which can be viewed as a generalization of Rubin's formula [Rub94, Thm. 3.2(ii)] (c.f. [Nek06, Prop. 11.5.11]).

Proposition 2.1. *Let $z \in Y^{(r)}_T$ and $w \in Y^{(r)}_{T^*}$. Suppose that there exist $\mathbf{z} \in \mathrm{H}^1(K, T_\Lambda)$ and $\mathbf{w}_\Sigma = (\mathbf{w}_v) \in \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathcal{F}^\perp}(K_v, T^*_\Lambda)$ such that $\mathrm{pr}_K(\mathbf{z}) = z$ and $\mathrm{pr}_{K_v}(\mathbf{w}_v) = \mathrm{loc}_v(w)$. Then*

$$h^{(r)}_{\mathcal{O}}(z,w) = -\sum_{v\in\Sigma} \langle \mathbf{z}, \mathbf{w}_v\rangle_{K_{\infty,v}} \pmod{J^{r+1}}.$$

Proof. Let $y = \eta^{-1}_{\boldsymbol{\gamma}}(z) \in \mathrm{H}^1_{\mathcal{F}}(K, T_P)$ and $t = \eta^{-1}_{\boldsymbol{\gamma}}(w) \in \mathrm{H}^1_{\mathcal{F}^\perp}(K, T^*_P)$. Choose cochains $\widetilde{y} \in C^1(G_{K,\Sigma}, T_{\mathcal{K}})$ and $\widetilde{t} \in C^1(G_{K,\Sigma}, T^*_{\mathcal{K}})$ lifting s and t, respectively, and let $\epsilon_0 \in C^2(G_{K,\Sigma}, P(1))$ be such that $d\epsilon_0 = d\widetilde{y} \cup \widetilde{t}$. Choose $\widetilde{t}_\Sigma \in \bigoplus_{v\in\Sigma} Z^1(G_{K_v}, T^*_{\mathcal{K}})$ to be a lifting of $\mathrm{loc}_\Sigma(\widetilde{t}) \in \bigoplus_{v\in\Sigma} Z^1(K_v, T^*_P)$. According to the definition of the Cassels–Tate pairing [How04, (2), p. 1321], we find that

$$h^{(r)}_{\mathcal{O}}(z,w) = (\boldsymbol{\gamma}-1)^2 \cdot [y,t]_{\mathrm{CT}} = (\boldsymbol{\gamma}-1)^2 \cdot \mathrm{inv}_\Sigma(\mathrm{loc}_\Sigma(\widetilde{y}) \cup \widetilde{t}_\Sigma - \mathrm{loc}_\Sigma(\epsilon_0)). \tag{2.5}$$

Now we let $\widetilde{\mathbf{z}} \in Z^1(G_{K,\Sigma}, T_\Lambda)$ and $\widetilde{\mathbf{w}}_\Sigma \in \bigoplus_{v\in\Sigma} Z^1(G_{K_v}, T^*_\Lambda)$ be cocycles representing $\mathbf{z}$ and $\mathbf{w}_\Sigma$. Then $\widetilde{y} = \widetilde{\mathbf{z}}/(\boldsymbol{\gamma}-1)$ and $\widetilde{t}_\Sigma = \widetilde{\mathbf{w}}_\Sigma/(\boldsymbol{\gamma}-1)$ are liftings of z and t_Σ. Invoking the formula (2.5) with $\epsilon_0 = 0$ ($d\widetilde{\mathbf{z}} = 0$), we obtain

$$\begin{aligned} h^{(r)}_{\mathcal{O}}(z,w) &= (\boldsymbol{\gamma}-1)^2 \cdot \mathrm{inv}_\Sigma(e_\Lambda(\frac{\mathrm{loc}_\Sigma(\widetilde{\mathbf{z}})}{\boldsymbol{\gamma}-1} \cup \frac{\widetilde{\mathbf{w}}_\Sigma}{\boldsymbol{\gamma}-1})) \in J^r/J^{r+1} \\ &= -\,\mathrm{inv}_\Sigma(e_\Lambda(\mathrm{loc}_\Sigma(\mathbf{z}) \cup \mathbf{w}_\Sigma)) = -\sum_{v\in\Sigma} \langle \mathbf{z}, \mathbf{w}_v\rangle_{K_{\infty,v}} \pmod{J^{r+1}}. \end{aligned}$$

This completes the proof. □

2.3. **Derived p-adic heights for elliptic curves.** Let E be an elliptic curve over K with good ordinary reduction at every place above p. Let $T = T_pE$ be the p-adic Tate module of E, and take Σ to consist of the set of archimedean places, the p-adic places, and the places where E has bad reduction. For every positive integers k and r, we consider the module $Y^{(r)}_{T_k}$ defined in (2.4) with $T_k = E[p^k]$, $\mathcal{O} = \mathbf{Z}/p^k\mathbf{Z}$ and the Selmer structure $\mathcal{F}$ in [How04, Def. 3.2]. Since $T^*_k = T_k$ and $\mathcal{F}^\perp = \mathcal{F}$ under the Weil pairing, we have the derived height pairing $h_{\mathbf{Z}/p^k\mathbf{Z}}$ on $Y^{(r)}_{T_k} \times Y^{(r)}_{T_k}$ defined in §2.2. The constructions of $Y^{(r)}_{T_k}$ and $h_{\mathbf{Z}/p^k\mathbf{Z}}$ are clearly compatible with k under the quotient map $\mathbf{Z}/p^{k+1}\mathbf{Z} \to \mathbf{Z}/p^k\mathbf{Z}$. Taking the inverse limit over k, we let

$$Y^{(r)}_T := \varprojlim Y^{(r)}_{T_k}, \quad h^{(r)} := \varprojlim h^{(r)}_{\mathbf{Z}/p^k\mathbf{Z}}.$$

According to [How04, Lem. 4.1] there is canonical isomorphism

$$Y^{(1)}_T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = \mathrm{Sel}(K, V_pE),$$

and we let $S_p^{(r)}(E/K)$ be subspace of $\mathrm{Sel}(K, V_pE)$ spanned by the image of $Y_T^{(r)}$ under the above isomorphism. By definition $S_p^{(1)}(E/K) = \mathrm{Sel}(K, V_pE)$. We thus obtain

$$h^{(r)} : S_p^{(r)}(E/K) \times S_p^{(r)}(E/K) \to J^r/J^{r+1} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

where J is the augmentation ideal of $\Lambda = \mathbf{Z}_p[\![\Gamma_\infty]\!]$.

Corollary 2.2. *Let $z, w \in S_p^{(r)}(E/K)$. Suppose that there exist a global class $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, T)$ and local classes $\mathbf{w}_v \in \varprojlim_n \mathrm{H}^1_{\mathrm{fin}}(K_{n,v}, T)$ for every $v \mid p$ such that $\mathrm{pr}_K(\mathbf{z}) = z$ and $\mathrm{pr}_{K_v}(\mathbf{w}_v) = \mathrm{loc}_v(w)$. Then*

$$h^{(r)}(z, w) = -\sum_{v|p} \langle \mathrm{loc}_v(\mathbf{z}), \mathbf{w}_v \rangle_{K_{\infty,v}} \pmod{J^{r+1}}.$$

Proof. This follows from Proposition 2.1 and the fact that $\mathrm{H}^1(K_{n,v}, T) \otimes \mathbf{Q}_p = 0$ for every place $v \nmid p$. □

3. Perrin-Riou's theory for Lubin–Tate formal groups

3.1. The purpose of this section is to compute the derived p-adic heights via Perrin-Riou's big exponential maps. We begin with a review of the generalization of Perrin-Riou's theory to Lubin–Tate formal groups developed in [Kob18]. Throughout we fix a completed algebraic closure $\mathbf{C}_p$ of $\mathbf{Q}_p$. Let $\mathbf{Q}_p^{\mathrm{ur}} \subset \mathbf{C}_p$ be the maximal unramified extension of $\mathbf{Q}_p$ and let $\mathrm{Fr} \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ur}}/\mathbf{Q}_p)$ be the absolute Frobenius element. Let F be an unramified finite extension of $\mathbf{Q}_p$ and $\mathcal{O} = \mathcal{O}_F$. Put

$$R := \mathcal{O}[\![X]\!].$$

Let $\mathcal{F} = \mathrm{Spf}\, R$ be a relative Lubin–Tate formal group of height one defined over $\mathcal{O}$. For each $n \in \mathbf{Z}$, set $\mathcal{F}^{(n)} := \mathcal{F} \times_{\mathrm{Spec}\,\mathcal{O}, \mathrm{Fr}^{-n}} \mathrm{Spec}\,\mathcal{O}$. Let $\varphi_{\mathcal{F}} \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}^{(-1)})$ be the Frobenius morphism. Then $\varphi_{\mathcal{F}}$ induces a homomorphism $\varphi_{\mathcal{F}} \colon R \to R$ defined by

$$\varphi_{\mathcal{F}}(f) := f^{\mathrm{Fr}} \circ \varphi_{\mathcal{F}}.$$

Here f^{Fr} denotes the conjugate of f by Fr. Let $\psi_{\mathcal{F}}$ be the left inverse of $\varphi_{\mathcal{F}}$ satisfying $\psi_{\mathcal{F}}\varphi_{\mathcal{F}}(f) = f$ and

$$\varphi_{\mathcal{F}}\psi_{\mathcal{F}}(f) = p^{-1} \sum_{x \in \mathcal{F}[p]} f(X \oplus_{\mathcal{F}} x).$$

Let $F_\infty = \bigcup_{n=1}^\infty F(\mathcal{F}[p^n])$ be the Lubin–Tate $\mathbf{Z}_p^\times$-extension associated with the formal group $\mathcal{F}$. For $n \geq 0$, let F_n be the subfield of F_∞ with $\mathrm{Gal}(F_n/F) \simeq (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times$ and set $F_{-1} = F$. Let $G_\infty = \mathrm{Gal}(F_\infty/F)$. There is a unique decomposition $G_\infty = \Delta \times \Gamma_\infty^{\mathcal{F}}$, where $\Delta \simeq \mathrm{Gal}(F_0/F)$ is the torsion subgroup of G_∞ and $\Gamma_\infty^{\mathcal{F}} \simeq \mathbf{Z}_p$. Let $\varepsilon_{\mathcal{F}} \colon G_\infty \xrightarrow{\sim} \mathbf{Z}_p^\times$ be the Lubin–Tate character. For $a \in \mathbf{Z}_p^\times$, there exists a unique formal power series $[a](X) \in R$ such that

$$[a]^{\mathrm{Fr}} \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}} \circ [a] \quad \text{and} \quad [a](X) \equiv aX \pmod{X^2}.$$

Then G_∞ acts on R by $\sigma \cdot f(X) := f([\varepsilon_{\mathcal{F}}(\sigma)](X))$. This makes R an $\mathcal{O}[\![G_\infty]\!]$-module.

Lemma 3.1. *$R^{\psi_{\mathcal{F}}=0}$ is free of rank one over $\mathcal{O}[\![G_\infty]\!]$.*

Proof. This is a standard fact. See [Kob18, Prop. 5.4]. □

Let $L \subset \mathbf{C}_p$ be a finite extension over $\mathbf{Q}_p$. Let V be a finite-dimensional L-vector space on which $G_{\mathbf{Q}_p}$ acts as a continuous L-linear crystalline p-adic Galois representation. Let $\mathbf{D}(V) = \mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)$ be the filtered φ-module associated with V over $\mathbf{Q}_p$ and set

$$\mathscr{D}_\infty(V) := \mathbf{D}(V) \otimes_{\mathbf{Z}_p} R^{\psi_{\mathcal{F}}=0} \simeq \mathbf{D}(V) \otimes_{\mathbf{Z}_p} \mathcal{O}[\![G_\infty]\!].$$

Let $d : R \to \Omega_R$ be the standard derivation. Fixing an invariant differential form $\omega_{\mathcal{F}} \in \Omega_R$, we denote by $\log_{\mathcal{F}} \in R\widehat{\otimes}\mathbf{Q}_p$ the logarithm associated with $\omega_{\mathcal{F}}$ satisfying $\log_{\mathcal{F}}(0) = 0$ and

$d\log_{\mathcal{F}} = \omega_{\mathcal{F}}$, and define $\partial\colon R \to R$ by $df = \partial f \cdot \omega_{\mathcal{F}}$. Let $\epsilon = (\epsilon_n) \in T_p\mathcal{F} = \varprojlim_n \mathcal{F}^{(n+1)}[p^{n+1}]$ be a basis of the Tate module of $\mathcal{F}$, where the inverse limit is taken with respect to the maps $\varphi^{\mathrm{Fr}^{-(n+1)}}\colon \mathcal{F}^{(n+1)}[p^{n+1}] \to \mathcal{F}^{(n)}[p^n]$. Following [Kob18, p. 42], to ϵ and $\omega_{\mathcal{F}}$ we associate the p-adic period $t_\epsilon \in B^+_{\mathrm{cris}}$ for $\mathcal{F}$ as follows. For each n, there exists an unique isomorphism $\varphi_n^\flat : \mathcal{F}^{(n)} \to \mathcal{F}$ such that $\varphi^{\mathrm{Fr}^{-1}} \circ \cdots \circ \varphi^{\mathrm{Fr}^{-(n-1)}} \circ \varphi^{\mathrm{Fr}^{-n}} = [p^n] \circ \varphi_n^\flat$. Put $w_n := \varphi_n^\flat(\epsilon_{n-1}) \in \mathcal{F}[p^n]$. We have $[p](w_n) = w_{n-1}$ by definition. Let $A_{\mathrm{inf}} = A_{\mathrm{inf}}(\mathcal{O}_{\mathbf{C}_p}/\mathcal{O}_F)$ and $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbf{C}_p}$ be the p-adic period ring and the canonical map defined in [Fon94, §1.2.2]. Then it is not difficult to show that there exists a unique sequence $(\widetilde{w}_n)$ in $\mathcal{F}(A_{\mathrm{inf}})$ such that $[p](\widetilde{w}_n) = \widetilde{w}_{n-1}$ and $\theta(\widetilde{w}_n) = w_n$. We set $t_\epsilon := \log_{\mathcal{F}}(\widetilde{w}_0) \in B^+_{\mathrm{cris}}$. This p-adic period t_ϵ satisfies

$$\mathbf{D}_{\mathrm{cris},F}(\varepsilon_{\mathcal{F}}) = F t_\epsilon^{-1}; \quad \varphi t_\epsilon = \varpi t_\epsilon, \tag{3.1}$$

where ϖ is the uniformizer in F such that $\varphi^*_{\mathcal{F}}(\omega_{\mathcal{F}}^{\mathrm{Fr}}) = \varpi \cdot \omega_{\mathcal{F}}$.

We shall fix an extension $\breve{\varepsilon}_{\mathcal{F}}\colon \mathrm{Gal}(F_\infty/\mathbf{Q}_p) \to L^\times$ of the Lubin-Tate character $\varepsilon_{\mathcal{F}}$. For each $j \in \mathbf{Z}$, let $V\langle j\rangle := V \otimes_L \breve{\varepsilon}^j_{\mathcal{F}}$ denote the Lubin–Tate twist of V. By definition, $\mathbf{D}_{\mathrm{cris},F}(V\langle j\rangle) = \mathbf{D}(V) \otimes_{\mathbf{Q}_p} F t_\epsilon^{-j}$. Define the derivation $\mathrm{d}_\epsilon : \mathscr{D}(V\langle j\rangle) \to \mathscr{D}(V\langle j-1\rangle)$ by

$$\mathrm{d}_\epsilon f := \eta t_\epsilon \otimes \partial g, \quad f = \eta \otimes g \in \mathbf{D}_{\mathrm{cris},F}(V\langle j\rangle) \otimes_{\mathcal{O}} R^{\psi_{\mathcal{F}}=0}$$

and the map

$$\widetilde{\Delta}\colon \mathscr{D}_\infty(V) \longrightarrow \bigoplus_{j\in\mathbf{Z}} \frac{\mathbf{D}_{\mathrm{cris},F}(V\langle -j\rangle)}{1-\varphi}, \quad f \mapsto (\partial^j f(0) t_\epsilon^j \pmod{(1-\varphi)}).$$

Remark 3.2. When $\mathcal{F} = \widehat{\mathbf{G}}_m$, we have $F_\infty = F(\zeta_{p^\infty})$, the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ is the corresponding Lubin-Tate character, $\varphi_{\widehat{\mathbf{G}}_m}(f) = f^{\mathrm{Fr}}((1+X)^p - 1)$, and $\psi_{\widehat{\mathbf{G}}_m}(f)$ is the unique power series such that

$$\varphi_{\widehat{\mathbf{G}}_m}\psi_{\widehat{\mathbf{G}}_m}(f) = p^{-1} \sum_{\zeta^p=1} f(\zeta(1+X)-1).$$

If we take $\omega_{\widehat{\mathbf{G}}_m}$ to be $(1+X)^{-1}dX$, then $\partial = (1+X)\frac{d}{dX}$ and $\log_{\widehat{\mathbf{G}}_m}$ is the usual logarithm $\log(1+X)$. We shall fix a sequence of p-power roots of unity $\{\zeta_{p^n}\}_{n=1,2,3,\ldots}$ with $\zeta_p^p = 1$ and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ and let $t \in \mathbf{D}(V)$ be the p-adic period corresponding to $(\zeta_{p^{n+1}} - 1) \in T_p\widehat{\mathbf{G}}_m$.

3.2. **Perrin-Riou's big exponential map and the Coleman map.** For a finite extension K over $\mathbf{Q}_p$, let

$$\exp_{K,V}\colon \mathbf{D}(V) \otimes_{\mathbf{Q}_p} K \to \mathrm{H}^1(K,V)$$

be Bloch-Kato's exponential map defined in [BK90, §3]. In this subsection, we recall the main properties of Perrin-Riou's big exponential map $\Omega_{V,h}$ which interpolates $\exp_{F,V\langle j\rangle}$ as j runs over non-negative integers j. We begin with some notation. Let $V^* := \mathrm{Hom}_L(V,L)(1)$ be the Kummer dual of V and denote by

$$[-,-]_V : \mathbf{D}(V) \otimes K \times \mathbf{D}(V) \otimes K \to K \otimes_{\mathbf{Q}_p} L$$

be the K-linear paring induced by the canonical pairing $\langle\,,\,\rangle_{\mathrm{dR}}\colon \mathbf{D}(V^*) \times \mathbf{D}(V) \to L$. Let $\exp^*_{K,V}\colon \mathrm{H}^1(K,V) \to \mathbf{D}(V) \otimes K$ be the dual exponential map characterized uniquely by

$$\mathrm{Tr}_{K/\mathbf{Q}_p}([x, \exp^*_{K,V}(y)]_V) = \langle \exp_{K,V^*}(x), y\rangle_V, \quad x \in \mathrm{H}^1(K,V^*),\, y \in \mathbf{D}(V^*) \otimes K.$$

Choosing a $G_{\mathbf{Q}_p}$-stable $\mathcal{O}_L$-lattice T inside V, we let

$$\widehat{\mathrm{H}}^1(F_\infty,T) = \varprojlim_n \mathrm{H}^1(F_n,T); \quad \widehat{\mathrm{H}}^1(F_\infty,V) = \widehat{\mathrm{H}}^1(F_\infty,T) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the Iwasawa cohomology $\mathbf{Z}_p[\![G_\infty]\!]$-modules associated with V. We denote by

$$\mathrm{Tw}_j : \widehat{\mathrm{H}}^1(F_\infty,V) \simeq \widehat{\mathrm{H}}^1(F_\infty,V\langle j\rangle)$$

the twisting map by $\breve{\varepsilon}_{\mathcal{F}}^{j}$. For a non-negative real number r and any subfield K in $\mathbf{C}_p$, we put

$$\mathscr{H}_{r,K}(X) = \left\{ \sum_{n\geqslant 0,\tau\in\Delta} c_{n,\tau}\cdot\tau\cdot X^n \in K[\Delta][\![X]\!] \mid \sup_n |c_{n,\tau}|_p\, n^{-r} < \infty \text{ for all } \tau\in\Delta \right\},$$

where $|\cdot|_p$ is the normalized absolute value of K with $|p|_p = p^{-1}$. Let $\boldsymbol{\gamma}$ be a topological generator of $\Gamma_\infty^{\mathcal{F}}$. Define $\mathscr{H}_{r,K}(G_\infty)$ to be the set of elements $f(\boldsymbol{\gamma}-1)$ for $f\in\mathscr{H}_{r,K}(X)$. By definition, $\mathscr{H}_{0,K}(G_\infty) = \mathcal{O}_K[\![G_\infty]\!]\otimes\mathbf{Q}_p$. Put

$$\mathscr{H}_{\infty,K}(G_\infty) = \bigcup_{r\geq 0}\mathscr{H}_{r,K}(G_\infty).$$

For $n\geq -1$, we define a map

$$\Xi_{n,V}\colon \mathbf{D}(V)\otimes_{\mathbf{Q}_p}\mathscr{H}_{\infty,F}(X)\to\mathbf{D}(V)\otimes_{\mathbf{Q}_p}F_n$$

by

$$\Xi_{n,V}(G) := \begin{cases} p^{-(n+1)}\varphi^{-(n+1)}(G^{\mathrm{Fr}^{-(n+1)}}(\epsilon_n)) & \text{if } n\geq 0,\\ (1-p^{-1}\varphi^{-1})(G(0)) & \text{if } n=-1.\end{cases}$$

In what follows, we let h be a positive integer such that $\mathbf{D}(V) = \mathrm{Fil}^{-h}\,\mathbf{D}(V)$ and assume that $\mathrm{H}^0(F_\infty, V) = 0$. The following results on the construction of the big exponential maps and the explicit interpolation ormulae are due to Perrin-Riou and Colmez in the case $\mathcal{F} = \widehat{\mathbf{G}}_m$ and extended by Kobayashi and Shaowei Zhang for relative Lubin-Tate formal groups of height one.

Theorem 3.3 (Perrin-Riou, Colmez, Kobayashi, Shaowei Zhang)**.** *Let $\widetilde{\Lambda} := \mathbf{Z}_p[\![G_\infty]\!]$. There exists the big exponential map*

$$\Omega_{V,h}^{\epsilon} : \mathscr{D}_\infty(V)^{\widetilde{\Delta}=0} \longrightarrow \widehat{\mathrm{H}}^1(F_\infty, T)\otimes_{\widetilde{\Lambda}}\mathscr{H}_{\infty,\mathbf{Q}_p}(G_\infty)$$

which is characterized by the following interpolation property: $\Omega_{V,h}$ is a map of $\widetilde{\Lambda}$-modules and for every $g\in\mathscr{D}_\infty(V)^{\widetilde{\Delta}=0}$ with $j\geq 1-h$, we have

$$\mathrm{pr}_{F_n}(\mathrm{Tw}_j\circ\Omega_{V,h}^{\epsilon}(g)) = (-1)^{h+j-1}(h+j-1)!\cdot\exp_{F_n,V\langle j\rangle}(\Xi_{n,V\langle j\rangle}(\mathrm{d}_\epsilon^{-j}G))\in\mathrm{H}^1(F_n, V\langle j\rangle),$$

while if $j\leq -h$, then

$$\exp^*_{F_n,V\langle j\rangle}(\mathrm{pr}_{F_n}(\mathrm{Tw}_j\circ\Omega_{V,h}^{\epsilon}(g))) = \frac{1}{(-h-j)!}\cdot\Xi_{n,V\langle j\rangle}(\mathrm{d}_\epsilon^{-j}G))\in\mathbf{D}(V\langle j\rangle)\otimes_{\mathbf{Q}_p}F_n,$$

where $G\in\mathbf{D}(V)\otimes_{\mathbf{Q}_p}\mathscr{H}_{h,F}(X)$ is a solution of the equation

$$(1-\varphi\otimes\varphi_{\mathcal{F}})G = g.$$

Moreover, if $D_{[s]}$ is a φ-invariant $\mathbf{Q}_p$-subspace of $\mathbf{D}(V)$ such that all eigenvalues of φ on $D_{[s]}$ have the p-adic valuation s, then $\Omega_{V,h}^{\epsilon}$ maps $(D_{[s]}\otimes R^{\psi=0})^{\widetilde{\Delta}=0}$ into $\widehat{\mathrm{H}}^1(F_\infty,T)\otimes_{\widetilde{\Lambda}}\mathscr{H}_{s+h,\mathbf{Q}_p}(G_\infty)$.

Proof. In the case $\mathcal{F} = \widehat{\mathbf{G}}_m$, the construction of the big exponential map $\Omega_{V,h}$ and the interpolation at $j\geq 1-h$ are due to Perrin-Riou [PR94, §3.2.3 Théorème, §3.2.4 (i)] and interpolation formula at $j\leq -h$ is a consequence of the *explicit reciprocity formula* proved by Colmez [Col98, THÉORÈME IX.4.5] (c.f. [Ber03, Theorem II.10]). The ideas of Perrin-Riou and Colmez can applied to general relative Lubin-Tate formal groups of height one with some necessary modifications. More details can be found in [Kob18, Appendix] for the construction of $\Omega_{V,h}^{\epsilon}$ and the interpolation at $j\geq 1-h$ and [Zha04, Theorem 6.2] for the explicit reciprocity formula. □

To introduce the Coleman map, we further assume the following hypothesis:

$$\mathscr{D}_\infty(V)^{\widetilde{\Delta}=0} = \mathscr{D}_\infty(V). \tag{3.2}$$

For simplicity, we shall write $\mathscr{H}_K$ for $\mathscr{H}_{\infty,K}(G_\infty)$ in the sequel. We let

$$[-,-]_V : \mathbf{D}(V^*) \otimes_{\mathbf{Q}_p} \mathscr{H}_F \times \mathbf{D}(V) \otimes_{\mathbf{Q}_p} \mathscr{H}_F \to L \otimes_{\mathbf{Q}_p} \mathscr{H}_F$$

be the pairing defined by $[\eta_1 \otimes \lambda_1, \eta_2 \otimes \lambda_2]_V = \langle \eta_1, \eta_2 \rangle_{\mathrm{dR}} \otimes \lambda_1 \lambda_2^\iota$ for all $\lambda_1, \lambda_2 \in \mathscr{H}_F$. To each $e \in R^{\psi_{\mathcal{F}}=0}$ and a generator ϵ of the Tate module of $\mathcal{F}$, we associate the unique $\mathcal{O}_L[\![G_\infty]\!]$-linear Coleman map $\mathrm{Col}_e^\epsilon \colon \widehat{\mathrm{H}}^1(F_\infty, V^*) \to \mathbf{D}(V^*) \otimes_{\mathbf{Q}_p} \mathscr{H}_F$ which is characterized by

$$\mathrm{Tr}_{F/\mathbf{Q}_p}([\mathrm{Col}_e^\epsilon(\mathbf{z}), \eta]_V) = \langle \mathbf{z}, \Omega_{V,h}^\epsilon(\eta \otimes e) \rangle_{F_\infty} \in L \otimes_{\mathbf{Q}_p} \mathscr{H}_{\mathbf{Q}_p} \tag{3.3}$$

for all $\eta \in \mathbf{D}(V)$. Let $\mathcal{Q}$ be the completion of $\mathbf{Q}_p^{\mathrm{ur}}$ in $\mathbf{C}_p$ and let $\mathcal{W}$ be the ring of integers of $\mathcal{Q}$. Let $F_n^{\mathrm{ur}} := F_n \mathbf{Q}_p^{\mathrm{ur}}$. Let $\sigma_0 \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be an element such that $\sigma_0|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$ is the absolute Frobenius element. Fix an isomorphism $\rho : \widehat{\mathbf{G}}_m \simeq \mathcal{F}$ defined over $\mathcal{W}$ and let $\rho : \mathcal{W}[\![T]\!] \simeq R \otimes_{\mathcal{O}} \mathcal{W}$ be the map defined by $\rho(f) = f \circ \rho^{-1}$. We have

$$\varphi_{\mathcal{F}} \circ \rho = \rho^{\mathrm{Fr}} \circ \varphi_{\widehat{\mathbf{G}}_m}.$$

Let $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$ be a generator over $\mathcal{O}[\![G_\infty]\!]$ and write $\rho(1+X) = h_{\boldsymbol{e}} \cdot \boldsymbol{e}$ for some $h_{\boldsymbol{e}} \in \mathcal{W}[\![G_\infty]\!]$. This implies that $\boldsymbol{e}(0) \in \mathcal{O}^\times$. Now we fix $\epsilon := (\epsilon_n)_{n=0,1,\ldots}$ to be the generator of the Tate module of $\mathcal{F}$ given by

$$\epsilon_n = \rho^{\mathrm{Fr}^{-(n+1)}}(\zeta_{p^{n+1}} - 1) \in \mathcal{F}^{(n+1)}[p^{n+1}].$$

Put $\widetilde{G}_\infty := \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$. Let $\eta \in \mathbf{D}(V)$ such that $\varphi\eta = \alpha\eta$ of slope s (i.e. $|\alpha|_p = p^{-s}$). For every $\mathbf{z} \in \widehat{\mathrm{H}}^1(F_\infty, V^*)$, we define

$$\mathrm{Col}^\eta(\mathbf{z}) := \sum_{j=1}^{[F:\mathbf{Q}_p]} \left[\mathrm{Col}_{\boldsymbol{e}}^\epsilon(\mathbf{z}^{\sigma_0^{-j}}), \eta\right] \cdot h_{\boldsymbol{e}} \cdot \sigma_0^j \in \mathscr{H}_{s+h,L\mathcal{Q}}(\widetilde{G}_\infty).$$

Here $[-,-] : \mathbf{D}(V^*) \otimes \mathscr{H}_{\mathcal{Q}} \times \mathbf{D}(V) \to \mathscr{H}_{L\mathcal{Q}}$ is the image of $[-,-]_V$ under the natural map $L \otimes_{\mathbf{Q}_p} \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{L\mathcal{Q}}$. For any integer j, put

$$\mathbf{z}_{-j,n} := \mathrm{pr}_{F_n}(\mathrm{Tw}_{-j}(\mathbf{z})) \in \mathrm{H}^1(F_n, V^*\langle -j \rangle).$$

For a finite order character χ of $\widetilde{G}_\infty$ and $n \geq -1$, we say χ has conductor p^{n+1} if n is the smallest integer such that χ factors through $\mathrm{Gal}(F_n/\mathbf{Q}_p)$.

Theorem 3.4. *Suppose that* $\mathrm{Fil}^{-1}\mathbf{D}(V) = \mathbf{D}(V)$ *and* $h = 1$. *Let* ψ *be a* p*-adic character of* $\widetilde{G}_\infty$ *such that* $\psi = \chi \breve{\varepsilon}_{\mathcal{F}}^j$ *with* χ *a finite order character of conductor* p^{n+1}. *If* $j < 0$, *then*

$$\mathrm{Col}^\eta(\mathbf{z})(\psi) = \frac{(-1)}{(-j-1)!} \times \begin{cases} \left[\log_{F,V^*\langle -j \rangle} \mathbf{z}_{-j,n} \otimes t^{-j}, (1-p^{j-1}\varphi^{-1})(1-p^{-j}\varphi)^{-1}\eta\right] & \text{if } n=-1, \\ p^{(n+1)(j-1)}\boldsymbol{\tau}(\psi) \sum_{\tau \in \mathrm{Gal}(F_n/\mathbf{Q}_p)} \chi^{-1}(\tau) \left[\log_{F_n,V^*\langle -j \rangle} \mathbf{z}_{-j,n}^\tau \otimes t^{-j}, \varphi^{-(n+1)}\eta\right] & \text{if } n \geqslant 0. \end{cases}$$

If $j \geqslant 0$, *then*

$$\mathrm{Col}^\eta(\mathbf{z})(\psi) = j!(-1)^j \times \begin{cases} \left[\exp^*_{F,V^*\langle -j \rangle} \mathbf{z}_{-j,n} \otimes t^{-j}, (1-p^{j-1}\varphi^{-1})(1-p^{-j}\varphi)^{-1}\eta\right] & \text{if } n=-1, \\ p^{(n+1)(j-1)}\boldsymbol{\tau}(\psi) \sum_{\tau \in \mathrm{Gal}(F_n/\mathbf{Q}_p)} \chi^{-1}(\tau) \left[\exp^*_{F_n,V^*\langle -j \rangle} \mathbf{z}_{-j,n}^\tau \otimes t^{-j}, \varphi^{-(n+1)}\eta\right] & \text{if } n \geqslant 0. \end{cases}$$

Here $\boldsymbol{\tau}(\psi)$ is the Gauss sum defined by

$$\boldsymbol{\tau}(\psi) := \sum_{\tau\in\mathrm{Gal}(F_n^{\mathrm{ur}}/F^{\mathrm{ur}})} \psi\varepsilon_{\mathrm{cyc}}^{-j}(\tau\sigma_0^{n+1})\zeta_{p^{n+1}}^{\tau\sigma_0^{n+1}}.$$

Proof. This follows from the explicit reciprocity formula in Theorem 3.3 and the computation and in [Kob18, Thm. 5.10] (*cf.* [LZ14, Theorem 4.15]). □

3.3. **The derived p-adic heights and the Coleman map.** Let E be an elliptic curve over $\mathbf{Q}$ with good ordinary reduction at p. Let $V = T_pE \otimes_{\mathbf{Z}_p} L$. We have $\mathrm{Fil}^{-1}\mathbf{D}(V) = \mathbf{D}(V)$ and $V^* = V$. Let ω_E be the Néron differential of E regarded as an element in $\mathbf{D}(\mathrm{H}^1_{\mathrm{et}}(E_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p))$ by the de Rham comparison. We fix an embedding $\iota_p\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ once an for all. For any number field $H \subset \overline{\mathbf{Q}}$, let $\hat{H}$ be the completion of $\iota_p(H)$ in $\mathbf{C}_p$. Let K be an imaginary quadratic field. Let $\mathfrak{p}$ be the prime of K above p induced from ι_p. We suppose that $p = \mathfrak{p}\overline{\mathfrak{p}}$ is split in K throughout. Let K_∞/K be the anticyclotomic $\mathbf{Z}_p$-extension over K, $\Gamma_\infty := \mathrm{Gal}(K_\infty/K)$ and $\hat{\Gamma}_\infty = \mathrm{Gal}(\hat{K}_\infty/\mathbf{Q}_p)$. For any integer c, let H_c be the ring class field of K conductor c. Now we choose an integer $c > 1$ prime to c and let $F = \hat{H}_c$. Let $\xi \in K$ be a generator of $\mathfrak{p}^{[F:\mathbf{Q}_p]}$ and let F_∞ be the Lubin–Tate $\mathbf{Z}_p$-extension over F associated with $\xi/\overline{\xi}$. By [Kob18, Prop. 3.7], $F_\infty = \bigcup_{n=0}^\infty \hat{H}_{cp^n}$, and hence F_∞ is a finite extension of $\hat{K}_\infty$. Moreover, the hypothesis (3.2) holds since $\mathbf{D}(V)^{\varphi^{[F:\mathbf{Q}_p]}=(\xi/\overline{\xi})^j} = \{0\}$ for any integer j by the facts that the φ-eigenvalues of $\mathbf{D}(V)$ are p-Weil numbers while $\xi/\overline{\xi}$ is a 1-Weil number. By Lemma 3.1, $R^{\psi_{\mathcal{F}}=0}$ is a free $\mathcal{O}_F[\![G_\infty]\!]$-module of rank one. Let $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$ be a generator such that $\boldsymbol{e}(0) = 1$. Let $\alpha_p \in \mathbf{Z}_p^\times$ be the p-adic unit eigenvalue of the Frobenius φ on $\mathbf{D}(V)$. We let $\eta \in \mathbf{D}(V) = \mathbf{D}(\mathrm{H}^1_{\mathrm{et}}(E_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p)) \otimes \mathbf{D}(L(1))$ be a φ-eigenvector of slope -1 such that

$$\varphi\eta = p^{-1}\alpha_p \cdot \eta; \quad \langle \eta, \omega_E \otimes t^{-1}\rangle_{\mathrm{dR}} = 1.$$

Applying the big exponential map $\Omega^\epsilon_{V,1}$ in Theorem 3.3, we define

$$\mathbf{w}^\eta = \Omega^\epsilon_{V,1}(\eta\otimes\boldsymbol{e}) \in \widehat{\mathrm{H}}^1(F_\infty, V). \tag{3.4}$$

The following lemma is a standard fact.

Lemma 3.5. *We have*

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \exp_{F,V}\left(\frac{1-p^{-1}\varphi^{-1}}{1-\varphi}\eta\right) \in \mathrm{H}^1(F, V).$$

Proof. Let $g = \eta\otimes\boldsymbol{e}$ and let $G(X) \in \mathbf{D}(V)\otimes\mathscr{H}_{1,\mathcal{Q}}(X)$ such that $(1-\varphi\otimes\varphi_{\mathcal{F}})G = g$. Then we have

$$G(\epsilon_0) = \eta\otimes\boldsymbol{e}(\epsilon_0) - \eta + (1-\varphi)^{-1}\eta.$$

The equation $\psi_{\mathcal{F}}\boldsymbol{e}(X) = 0$ implies

$$\sum_{\zeta\in\mathcal{F}^{\mathrm{Fr}^{-1}}[p]} \boldsymbol{e}^{\mathrm{Fr}^{-1}}(X\oplus_{\mathcal{F}}\zeta) = 0.$$

It follows that

$$\mathrm{Tr}_{F_0/F}(G^{\mathrm{Fr}^{-1}}(\epsilon_0)) = \sum_{\tau\in\mathrm{Gal}(F_0/F)} \eta\otimes\boldsymbol{e}(\epsilon_0^\tau) - \eta + (1-\varphi)^{-1}\eta = \frac{p\varphi-1}{1-\varphi}\eta,$$

and hence

$$\begin{aligned}\mathrm{pr}_F(\mathbf{w}^\eta) =& \mathrm{cor}_{F_0/F}(\Xi_{0,V}(G)) = \exp_{F,V}\mathrm{Tr}_{F_0/F}\left(p^{-1}\varphi^{-1}(G^{\mathrm{Fr}^{-1}}(\epsilon_0))\right)\\ =& \exp_{F,V}\left((1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}\eta\right).\end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let $\mathbf{Q}_p^{\mathrm{cyc}}$ be the cyclotomic $\mathbf{Z}_p^\times$-extension of $\mathbf{Q}_p$. Let $\sigma_{\mathrm{cyc}} \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be the Frobenius such that $\sigma_{\mathrm{cyc}}|_{\mathbf{Q}_p^{\mathrm{cyc}}} = 1$ and $\sigma_{\mathrm{cyc}}|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$. For each $\mathbf{z} \in \widehat{\mathrm{H}}^1(\hat{K}_\infty, V)$, we have*

$$\langle \mathbf{z}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty} = \mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\mathbf{z})) \sum_{i=1}^{[F:\mathbf{Q}_p]} \frac{\sigma_{\mathrm{cyc}}^i|_{\hat{K}_\infty}}{[F_\infty : \hat{K}_\infty]\cdot h_{\boldsymbol{e}}^{\mathrm{Fr}^i}} \in \mathcal{W}[\![\hat{\Gamma}_\infty)]\!] \otimes \mathbf{Q}_p.$$

Proof. We first recall that for every $e \in (R \otimes_{\mathcal{O}} \mathcal{W})^{\psi_{\mathcal{F}}=0}$, the big exponential map $\Omega^\epsilon_{V,1}(\eta\otimes e)$ in Theorem 3.3 is given by

$$\Omega^\epsilon_{V,1}(\eta\otimes e) = (\exp_{F_n,V}(\Xi_{n,V}(G_e)))_{n=0,1,2,\dots},$$

where $G_e \in \mathbf{D}(V)\otimes \mathscr{H}_{1,\mathcal{Q}}(X)$ is a solution of $(1-\varphi\otimes\varphi_{\mathcal{F}})G_e = \eta\otimes e$. By the definition of G_e, we verify that

$$\begin{aligned}\Xi_{n,V}(G_e) &= p^{-(n+1)}(\varphi^{-(n+1)}\otimes 1)G_e^{\mathrm{Fr}^{-(n+1)}}(\epsilon_n)\\ &= \sum_{m=0}^{\infty}(p\varphi)^{-(n+1)}\varphi^m\eta\otimes e^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})\\ &= \sum_{m=0}^{n+1}(p\varphi)^{-(n+1)}\varphi^m\eta\otimes e^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m}) + p^{-(n+1)}(1-\varphi\otimes\mathrm{Fr})^{-1}(\eta\otimes e(0)).\end{aligned} \tag{3.5}$$

Put $z_n = \mathrm{pr}_{\hat{K}_n}(\mathbf{z})$ and $\hat{G}_n := \mathrm{Gal}(F_n/F)$. Following the computation in [Kob18, Thm. 5.10], we find that $\left[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^\epsilon_e(\mathbf{z})),\eta\right]$ is given by

(3.6)

$$\sum_{m=0}^{\infty}\left[\sum_{\gamma\in\hat{G}_n}\exp^*_{\hat{K}_n,V}(z_n^{\gamma^{-1}\sigma_0^{n+1-m}})\gamma, \sum_{\tau\in\hat{G}_n}(p\varphi)^{-(n+1)}\varphi^m\eta\otimes e^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})^{\tau\sigma_0^{n+1-m}}\tau|_{\hat{K}_n}\right].$$

On the other hand,

$$\mathrm{pr}_{\hat{K}_n}(\langle \mathbf{z}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty}) = \frac{1}{[F_\infty:\hat{K}_\infty]}\sum_{j=1}^{[F:\mathbf{Q}_p]}\mathrm{pr}_{\hat{K}_n}(\langle \mathbf{z}^{\sigma_0^{-j}}, \mathbf{w}^\eta\rangle_{F_\infty})\sigma_0^j|_{\hat{K}_\infty},$$

and $\mathrm{pr}_{\hat{K}_n}(\langle \mathbf{z}^{\sigma_0^{-j}}, \mathbf{w}^\eta\rangle_{F_\infty})$ equals

$$\begin{aligned}&\sum_{\gamma\in\hat{G}_n}\langle z_n^{\sigma_0^{-j}\gamma^{-1}}, \exp_{F_n,V}(\Xi_{n,V}(G_{\boldsymbol{e}})\rangle_{F_n}\gamma|_{\hat{K}_\infty} = \mathrm{Tr}_{F_n/\mathbf{Q}_p}\left(\left[\sum_{\gamma\in\hat{G}_n}\exp^*_{\hat{K}_n,V}(z_n^{\sigma_0^{-j}\gamma^{-1}})\gamma|_{\hat{K}_\infty}, \Xi_{n,V}(G_{\boldsymbol{e}})\right]_V\right)\\ &= \sum_{m=0}^{\infty}\sum_{i=1}^{[F:\mathbf{Q}_p]}\left[\sum_{\gamma\in\hat{G}_n}\exp^*_{\hat{K}_n,V}(z_n^{\gamma^{-1}\sigma_0^{i-j+n+1-m}})\gamma, \sum_{\tau\in\hat{G}_n}(p\varphi)^{-(n+1)}\varphi^m\eta\otimes\boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})^{\tau\sigma_0^{i+n+1-m}}\tau|_{\hat{K}_n}\right]\\ &= \sum_{i=1}^{[F:\mathbf{Q}_p]}\left[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\mathbf{z}^{\sigma_0^{-j}})^{\sigma_0^i}),\eta\right].\end{aligned}$$

It follows immediately that

$$\begin{aligned}\mathrm{pr}_{\hat{K}_n}(\langle \mathbf{z}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty}) &= \frac{1}{[F_\infty:\hat{K}_\infty]}\sum_{j=1}^{[F:\mathbf{Q}_p]}\sum_{i=1}^{[F:\mathbf{Q}_p]}\left[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\mathbf{z}^{\sigma_0^{-j}})^{\sigma_0^i}),\eta\right]\sigma_0^j\\ &= \frac{1}{[F_\infty:\hat{K}_\infty]}\sum_{i=1}^{[F:\mathbf{Q}_p]}(\mathrm{Col}^\eta(\mathbf{z}))^{\sigma_0^i}\cdot\frac{1}{h_{\boldsymbol{e}}^{\sigma_0^i}}.\end{aligned} \tag{3.7}$$

By definition,

$$\mathrm{Col}^\eta(\mathbf{z}) = \sum_{j=1}^{[F:\mathbf{Q}_p]} \left[\mathrm{Col}^\epsilon_{g_\rho}(\mathbf{z}^{\sigma_0^{-j}}), \eta\right] \sigma_0^j$$

with $g_\rho = \rho(1+X)$. From (3.6) with $e = g_\rho$ and the fact that $g_\rho^{\sigma_0^{m-n-1}}(\epsilon_{n-m}) = \zeta_{p^{n+1-m}} \in \mathbf{Q}_p^{\mathrm{cyc}}$, we deduce that

$$\left[\mathrm{Col}^\epsilon_{g_\rho}(\mathbf{z}^{\sigma_0^{-j}})^{\sigma_0^i}, \eta\right] = \left[\mathrm{Col}^\epsilon_{g_\rho}(\mathbf{z}^{\sigma_0^{-j}\sigma_{\mathrm{cyc}}^i}), \eta\right],$$

so $(\mathrm{Col}^\eta(\mathbf{z}))^{\sigma_0^i} = \mathrm{Col}^\eta(\mathbf{z}) \cdot \sigma_{\mathrm{cyc}}^i$. Now the lemma follows from (3.7). □

Now we give a formula of the derived p-adic heights over K_∞ in terms of the Coleman map over F_∞. For every prime v of K above p, let $\mathrm{H}^1_{\mathrm{fin}}(K_v, V)$ be the Bloch–Kato *finite* subspace of $\mathrm{H}^1(K_v, V)$ and set

$$\log_{\omega_E, v} = \langle \log_{K_v,V}(-), \omega_E \otimes t^{-1}\rangle_{\mathrm{dR}} : \mathrm{H}^1_{\mathrm{fin}}(K_v, V) \longrightarrow L. \tag{3.8}$$

Since p is of good reduction for E, we have $\mathrm{H}^1_{\mathrm{exp}}(K_v, V) = \mathrm{H}^1_{\mathrm{fin}}(K_v, V)$ by [BK90, Cor. 3,8,4], where $\mathrm{H}^1_{\mathrm{exp}}(K_v, V) \subset \mathrm{H}^1(K_v, V)$ is the image of $\exp_{K_v,V}$. To simplify the notation, we write $\mathrm{Col}^\eta(-)$ for $\mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(-))$ in what follows.

Proposition 3.7. *Let $z, x \in S_p^{(r)}(E/K) \otimes_{\mathbf{Q}_p} L$. Suppose that there exists $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$ such that $\mathrm{pr}_K(\mathbf{z}) = z$ and that $\mathrm{Col}^\eta(\mathrm{loc}_v(\mathbf{z})) \in J^r\mathcal{W}[\![\hat{\Gamma}_\infty]\!] \otimes \mathbf{Q}_p$ for some $v \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Then we have the p-adic height formula*

$$\begin{aligned} h^{(r)}(z, x) = &-\frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot \frac{1}{[F_\infty : \hat{K}_\infty]} \\ &\times \left(\log_{\omega_E,\mathfrak{p}}(x)\mathrm{Col}^\eta(\mathrm{loc}_\mathfrak{p}(\mathbf{z})) + \log_{\omega_E,\mathfrak{p}}(\overline{x})\mathrm{Col}^\eta(\mathrm{loc}_\mathfrak{p}(\overline{\mathbf{z}}))\right) \pmod{J^{r+1}\mathcal{W}[\![\hat{\Gamma}_\infty]\!] \otimes \mathbf{Q}_p}. \end{aligned}$$

Here $\overline{x}$ and $\overline{\mathbf{z}}$ are the complex conjugates of x and $\mathbf{z}$.

Proof. Let $\mathbf{w}_p := \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta) \in \widehat{\mathrm{H}}^1_{\mathrm{fin}}(\hat{K}_\infty, V)$. Since $\dim_{\mathbf{Q}_p} \mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) = 1$, we can write

$$\mathrm{loc}_\mathfrak{p}(x) = c \cdot \mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_p) = c \cdot \mathrm{cor}_{F/\mathbf{Q}_p}(\mathrm{pr}_F(\mathbf{w}^\eta))$$

for some $c \in \mathbf{Q}_p$. By Lemma 3.5,

$$\langle \log_{\mathbf{Q}_p,V}(\mathrm{loc}_\mathfrak{p}(x)), \omega_E \otimes t^{-1}\rangle_{\mathrm{dR}} = c[F : \mathbf{Q}_p] \cdot \langle \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi}\eta, \omega_E \otimes t^{-1}\rangle_{\mathrm{dR}}.$$

Since $\varphi\eta = p^{-1}\alpha_p\eta$, this shows that

$$c = \frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot [F : \mathbf{Q}_p]^{-1} \cdot \log_{\omega_E,\mathfrak{p}}(x).$$

Applying Corollary 2.2, we find that that

$$\begin{aligned} h^{(r)}(z, x) = &-(1 - p^{-1}\alpha_p)(1 - \alpha_p^{-1})^{-1}[F : \mathbf{Q}_p]^{-1} \\ &\times \left(\log_{\omega_E,\mathfrak{p}}(x) \cdot \langle \mathrm{loc}_\mathfrak{p}(\mathbf{z}), \mathbf{w}_p\rangle_{\hat{K}_\infty} + \log_{\omega_E,\mathfrak{p}}(\overline{x}) \cdot \langle \mathrm{loc}_\mathfrak{p}(\overline{\mathbf{z}}), \mathbf{w}_p\rangle_{\hat{K}_\infty}\right) \pmod{J^{r+1}}. \end{aligned}$$

Since $\rho(1 + X) = h_{\boldsymbol{e}} \cdot \boldsymbol{e}$ and $\boldsymbol{e}(0) = 1$, we find that $1 = \boldsymbol{e}(0) \cdot (h_{\boldsymbol{e}}|_{\gamma=1})$ and hence $h_{\boldsymbol{e}} \equiv 1 \pmod{J}$. The assertion follows from the above equation and Lemma 3.6. □

4. Euler system construction of theta elements

In this section we prove Theorem 4.7, recovering the square-root anticyclotomic p-adic L-functions of Bertolini–Darmon [BD96] (in the definite case) as the image of a p-adic family of the diagonal cycles of Darmon–Rotger [DR17a] via Perrin-Riou's exponential map for a certain Lubin–Tate $\mathbf{Z}_p$-extension.

4.1. **Ordinary Λ-adic forms.** Fix a prime $p>2$. Let $\mathbb{I}$ be a normal domain finite flat over $\Lambda := \mathcal{O}[\![1+p\mathbf{Z}_p]\!]$, where $\mathcal{O}$ is the ring of integers of a finite extension $L/\mathbf{Q}_p$. We say that a point $x \in \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ is *locally algebraic* if its restriction to $1+p\mathbf{Z}_p$ is given by $x(\gamma)=\gamma^{k_x}\epsilon_x(\gamma)$ for some integer k_x, called the *weight* of x, and some finite order character $\epsilon_x : 1+p\mathbf{Z}_p \to \mu_{p^\infty}$; we say that x is *arithmetic* if it has weight $k_x \geqslant 2$. Let $\mathfrak{X}_{\mathbb{I}}^+$ be the set of arithmetic points.

Fix a positive integer N prime to p, and let $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character modulo Np. Let $S^o(N,\chi,\mathbb{I})$ be the space of *ordinary* $\mathbb{I}$*-adic cusp forms* of tame level N and branch character χ, consisting of formal power series

$$\boldsymbol{f}(q)=\sum_{n=1}^{\infty} a_n(\boldsymbol{f})q^n \in \mathbb{I}[\![q]\!]$$

such that for every $x \in \mathfrak{X}_{\mathbb{I}}^+$ the specialization $\boldsymbol{f}_x(q)$ is the q-expansion of a p-ordinary cusp form $\boldsymbol{f}_x \in S_{k_x}(Np^{r_x+1}, \chi\omega^{2-k_x}\epsilon_x)$. Here $r_x \geqslant 0$ is such that $\epsilon_x(1+p)$ has exact order p^{r_x}, and $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mu_{p-1}$ is the Teichmüller character.

We say that $\boldsymbol{f} \in S^o(N,\chi,\mathbb{I})$ is a *primitive Hida family* if for every $x \in \mathfrak{X}_{\mathbb{I}}^+$ we have that $\boldsymbol{f}_x$ is an ordinary p-stabilized newform (in the sense of [Hsi17, Def. 2.4]) of tame level N. Given a primitive Hida family $\boldsymbol{f} \in S^o(N,\chi,\mathbb{I})$, and writing $\chi=\chi'\chi_p$ with χ' (resp. χ_p) a Dirichlet modulo N (resp. p), there is a primitive $\boldsymbol{f}^\iota \in S^o(N,\chi_p\overline{\chi}',\mathbb{I})$ with Fourier coefficients

$$a_\ell(\boldsymbol{f}^\iota)=\begin{cases} \overline{\chi}'(\ell)a_\ell(\boldsymbol{f}) & \text{if } \ell \nmid N, \\ a_\ell(\boldsymbol{f})^{-1}\chi_p\omega^2(\ell)\langle\ell\rangle_{\mathbb{I}}\ell^{-1} & \text{if } \ell \mid N, \end{cases}$$

having the property for every $x \in \mathfrak{X}_{\mathbb{I}}^+$ the specialization $\boldsymbol{f}_x^\iota$ is the p-stabilized newform attached to the character twist $\boldsymbol{f}_x \otimes \overline{\chi}'$.

By [Hid86] (*cf.* [Wil88, Thm. 2.2.1]), attached to every primitive Hida family $\boldsymbol{f} \in S^o(N,\chi,\mathbb{I})$ there is a continuous $\mathbb{I}$-adic representation $\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\operatorname{Frac} \mathbb{I})$ which is unramified outside Np, and such that for every prime $\ell \nmid Np$,

$$\operatorname{tr} \rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell)=a_\ell(\boldsymbol{f}), \quad \det \rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell)=\chi\omega^2(\ell)\langle\ell\rangle_{\mathbb{I}}\ell^{-1},$$

where $\langle\ell\rangle \in \mathbb{I}^\times$ is the image of $\ell\omega^{-1}(\ell)$ under the natural map $1+p\mathbf{Z}_p \to \mathcal{O}[\![1+p\mathbf{Z}_p]\!]^\times = \Lambda^\times \to \mathbb{I}^\times$. In particular, letting $\langle\varepsilon_{\mathrm{cyc}}\rangle_{\mathbb{I}} : G_{\mathbf{Q}} \to \mathbb{I}^\times$ be defined by $\langle\varepsilon_{\mathrm{cyc}}\rangle_{\mathbb{I}}(\sigma)=\langle\varepsilon_{\mathrm{cyc}}(\sigma)\rangle_{\mathbb{I}}$, it follows that $\rho_{\boldsymbol{f}}$ has determinant $\chi_{\mathbb{I}}^{-1}\varepsilon_{\mathrm{cyc}}^{-1}$, where $\chi_{\mathbb{I}} : G_{\mathbf{Q}} \to \mathbb{I}^\times$ is given by $\chi_{\mathbb{I}} := \sigma_\chi\langle\varepsilon_{\mathrm{cyc}}\rangle^{-2}\langle\varepsilon_{\mathrm{cyc}}\rangle_{\mathbb{I}}$ with σ_χ the Galois character sending $\mathrm{Frob}_\ell \mapsto \chi(\ell)^{-1}$. Moreover, by [Wil88, Thm. 2.2.2] the restriction of $\rho_{\boldsymbol{f}}$ to $G_{\mathbf{Q}_p}$ is given by

$$\rho_{\boldsymbol{f}}|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \psi_{\boldsymbol{f}} & * \\ 0 & \psi_{\boldsymbol{f}}^{-1}\chi_{\mathbb{I}}^{-1}\varepsilon_{\mathrm{cyc}}^{-1} \end{pmatrix} \tag{4.1}$$

where $\psi_{\boldsymbol{f}} : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ is the unramified character with $\psi_{\boldsymbol{f}}(\mathrm{Frob}_p)=a_p(\boldsymbol{f})$.

4.2. **Triple product p-adic L-function.** Let

$$(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in S^o(N_{\boldsymbol{f}},\chi_{\boldsymbol{f}},\mathbb{I}_{\boldsymbol{f}}) \times S^o(N_{\boldsymbol{g}},\chi_{\boldsymbol{g}},\mathbb{I}_{\boldsymbol{g}}) \times S^o(N_{\boldsymbol{h}},\chi_{\boldsymbol{h}},\mathbb{I}_{\boldsymbol{h}})$$

be a triple of primitive Hida families. Set

$$\mathcal{R} := \mathbb{I}_{\boldsymbol{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{I}_{\boldsymbol{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{I}_{\boldsymbol{h}},$$

which is a finite extension of the three-variable Iwasawa algebra $\mathcal{R}_0 := \Lambda\hat{\otimes}_{\mathcal{O}}\Lambda\hat{\otimes}_{\mathcal{O}}\Lambda$, and define the weight space $\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ for the triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the $\boldsymbol{f}$-dominated *unbalanced range* by

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} := \left\{(x,y,z) \in \mathfrak{X}_{\mathbb{I}_{\boldsymbol{f}}}^+ \times \mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\mathrm{cls}} \times \mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\mathrm{cls}} : \ k_x \geqslant k_y+k_z \text{ and } k_x \equiv k_y+k_z \pmod 2\right\}, \tag{4.2}$$

where $\mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\mathrm{cls}} \supset \mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^+$ (and similarly $\mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\mathrm{cls}}$) is the set of locally algebraic points in $\operatorname{Spec} \mathbb{I}_{\boldsymbol{g}}(\overline{\mathbf{Q}}_p)$ for which $\boldsymbol{g}_x(q)$ is the q-expansion of a classical modular form.

For $\boldsymbol{\phi} \in \{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}\}$ and a positive integer N prime to p and divisible by $N_{\boldsymbol{\phi}}$, define the space of Λ-*adic test vectors* $S^o(N, \chi_{\boldsymbol{\phi}}, \mathbb{I}_{\boldsymbol{\phi}})[\boldsymbol{\phi}]$ to be the $\mathbb{I}_{\boldsymbol{\phi}}$-submodule of $S^o(N, \chi_{\boldsymbol{\phi}}, \mathbb{I}_{\boldsymbol{\phi}})$ generated by $\{\boldsymbol{\phi}(q^d)\}$, as d ranges over the positive divisors of $N/N_{\boldsymbol{\phi}}$.

For the next result, set $N := \mathrm{lcm}(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$, and consider the following hypothesis:

(Σ^-) for some $(x, y, z) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, we have $\varepsilon_q(\boldsymbol{f}_x^\circ, \boldsymbol{g}_y^\circ, \boldsymbol{h}_z^\circ) = +1$ for all $q \mid N$.

Here $\varepsilon_q(\boldsymbol{f}_x^\circ, \boldsymbol{g}_y^\circ, \boldsymbol{h}_z^\circ)$ denotes the local root number of the Kummer self-dual twist of the Galois representations attached to the newforms $\boldsymbol{f}_x^\circ$, $\boldsymbol{g}_y^\circ$, and $\boldsymbol{h}_z^\circ$ corresponding to $\boldsymbol{f}_x$, $\boldsymbol{g}_y$, and $\boldsymbol{h}_z$, respectively.

Theorem 4.1. *Assume that the residual representation* $\bar{\rho}_{\boldsymbol{f}}$ *satisfies*

(CR) $\bar{\rho}_{\boldsymbol{f}}$ *is absolutely irreducible and* p*-distinguished,*

and that, in addition to (Σ^-)*, the triple* $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ *satisfies*

(ev) $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \omega^{2a}$ *for some* $a \in \mathbf{Z}$*,*

(sq) $\gcd(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$ *is square-free.*

Then there exist Λ*-adic test vectors* $(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star)$ *and an element*

$$\mathscr{L}_p^f(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star) \in \mathcal{R}$$

such that for all $(x, y, z) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ *of weight* (k, ℓ, m)*:*

$$\nu_{(x,y,z)}(\mathscr{L}_p^f(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star)^2) = \frac{\Gamma(k,\ell,m)}{2^{\alpha(k,\ell,m)}} \cdot \frac{\mathcal{E}(\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z)^2}{\mathcal{E}_0(\boldsymbol{f}_x)^2 \cdot \mathcal{E}_1(\boldsymbol{f}_x)^2} \cdot \prod_{q \mid N} c_q \cdot \frac{L(c, \boldsymbol{f}_x^\circ, \boldsymbol{g}_y^\circ, \boldsymbol{h}_z^\circ)}{\pi^{2(k-2)} \cdot \|\boldsymbol{f}_x^\circ\|^2},$$

where:

- $c = (k + \ell + m - 2)/2$,
- $\Gamma(k, \ell, m) = (c-1)! \cdot (c-m)! \cdot (c-\ell)! \cdot (c+1-\ell-m)!$,
- $\alpha(k, \ell, m) \in \mathcal{R}$ *is a linear form in the variables* k, ℓ, m,
- $\mathcal{E}(\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z) = (1 - \frac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^c})$,
- $\mathcal{E}_0(\boldsymbol{f}_x) = (1 - \frac{\beta_{\boldsymbol{f}_x}}{\alpha_{\boldsymbol{f}_x}})$, $\mathcal{E}_1(\boldsymbol{f}_x) = (1 - \frac{\beta_{\boldsymbol{f}_x}}{p\alpha_{\boldsymbol{f}_x}})$,

and $\|\boldsymbol{f}_x^\circ\|^2$ *is the Petersson norm of* $\boldsymbol{f}_x^\circ$ *on* $\Gamma_0(N_{\boldsymbol{f}})$*.*

Proof. See [Hsi17, Thm. A]. More specifically, the construction of $\mathscr{L}_p^f(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star)$ under hypotheses (CR), (ev), and (sq) is given in [Hsi17, §3.6] (where it is denoted $\mathscr{L}_{\boldsymbol{F}}^{\boldsymbol{f}}$), and the proof of its interpolation property assuming (Σ^-) is contained in [Hsi17, §7]. □

4.3. **Triple tensor product of big Galois representations.** Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families with $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \omega^{2a}$ for some $a \in \mathbf{Z}$. For $\boldsymbol{\phi} \in \{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}\}$, let $V_{\boldsymbol{\phi}}$ be the natural lattice in $(\mathrm{Frac}\,\mathbb{I}_{\boldsymbol{\phi}})^2$ realizing the Galois representation $\rho_{\boldsymbol{\phi}}$ in the étale cohomology of modular curves (see [Oht00]), and set

$$\mathbb{V}_{\boldsymbol{fgh}} := V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}.$$

This has rank 8 over $\mathcal{R}$, and by hypothesis its determinant can be written as $\det \mathbb{V}_{\boldsymbol{fgh}} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$ for a p-ramified Galois character $\mathcal{X}$ taking the value $(-1)^a$ on complex conjugation. Similarly as in [How07, Def. 2.1.3], we then define the *critical twist*

$$\mathbb{V}_{\boldsymbol{fgh}}^\dagger := \mathbb{V}_{\boldsymbol{fgh}} \otimes \mathcal{X}^{-1}.$$

More generally, for any multiple N of $N_{\boldsymbol{\phi}}$ one can define Galois modules $V_{\boldsymbol{\phi}}(N)$ by working in tame level N; these split non-canonically into a finite direct sum of the $\mathbb{I}_{\boldsymbol{\phi}}$-adic representations $V_{\boldsymbol{\phi}}$ (see [DR17a, §1.5.3]), and they define $\mathbb{V}_{\boldsymbol{fgh}}^\dagger(N)$ for any N divisible by $\mathrm{lcm}(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$.

If f is a classical specialization of $\boldsymbol{f}$ with associated p-adic Galois representation V_f, we let $\mathbb{V}_{f,\boldsymbol{gh}}$ be the quotient of $\mathbb{V}_{\boldsymbol{fgh}}$ given by

$$\mathbb{V}_{f,\boldsymbol{gh}} := V_f \otimes_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathbb{I}} V_{\boldsymbol{h}}.$$

Denote by $\mathbb{V}^{\dagger}_{f,\boldsymbol{gh}}$ the corresponding quotient of $\mathbb{V}^{\dagger}_{\boldsymbol{fgh}}$, and by $\mathbb{V}^{\dagger}_{f,\boldsymbol{gh}}(N)$ its level N counterparts.

4.4. **Theta elements and factorization.** We recall the factorization proven in [Hsi17, §8]. Let $f \in S_2(pN_f)$ be a p-stabilized newform of tame level N_f defined over $\mathcal{O}$, let $f^\circ \in S_2(N_f)$ be the associated newform, and let $\alpha_p = \alpha_p(f) \in \mathcal{O}^\times$ be the U_p-eigenvalue of f. Let K be an imaginary quadratic field of discriminant prime D_K. Write

$$N_f = N^+N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert or ramified) in K, and choose an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ = \mathbf{Z}/N^+\mathbf{Z}$.

Although Theorem 4.2 below is also available without this condition, we assume that $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C} \simeq \overline{\mathbf{Q}}_p$. Let $\Gamma_\infty = \mathrm{Gal}(K_\infty/K)$ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K, fix a topological generator $\boldsymbol{\gamma} \in \Gamma_\infty$, and identity $\mathcal{O}[\![\Gamma_\infty]\!]$ with the one-variable power series ring $\mathcal{O}[\![T]\!]$ via $\boldsymbol{\gamma} \mapsto 1+T$. For any prime-to-$p$ ideal $\mathfrak{a}$ of K, let $\sigma_{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in the Galois group of the ray class field $K(p^\infty)/K$ of conductor p^∞ under the *geometrically* normalized reciprocity law map.

Theorem 4.2. *Let χ be a ring class character of K of conductor $c\mathcal{O}_K$ with values in $\mathcal{O}$, and assume that:*

(i) $(pN_f, cD_K) = 1$,
(ii) *N^- is the square-free product of an odd number of primes,*
(iii) *$\bar{\rho}_f$ is absolutely irreducible and p-distinguished,*
(iv) *if $q \mid N^-$ is a prime with $q \equiv 1 \pmod p$, then $\bar{\rho}_f$ is ramified at q.*

There exists a unique element $\Theta_{f/K,\chi}(T) \in \mathcal{O}[[T]]$ such that for every p-power root of unity $\zeta \in \overline{\mathbf{Q}}_p$:

$$\Theta_{f/K,\chi}(\zeta-1)^2 = \frac{p^n}{\alpha_p^{2n}} \cdot \mathcal{E}_p(f,\chi,\zeta)^2 \cdot \frac{L(f^\circ/K \otimes \chi\epsilon_\zeta, 1)}{(2\pi)^2 \cdot \Omega_{f^\circ,N^-}} \cdot u_K^2 \sqrt{D_K} \chi\epsilon_\zeta(\sigma_{\mathfrak{N}^+}) \cdot \varepsilon_p,$$

where:

- *$n \geqslant 0$ is such that ζ has exact order p^n,*
- *$\epsilon_\zeta : \Gamma_\infty \to \mu_{p^\infty}$ be the character defined by $\epsilon_\zeta(\boldsymbol{\gamma}) = \zeta$,*
- $\mathcal{E}_p(f,\chi,\zeta) = \begin{cases} (1-\alpha_p^{-1}\chi(\mathfrak{p}))(1-\alpha_p\chi(\overline{\mathfrak{p}})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$
- *$\Omega_{f^\circ,N^-} = 4 \cdot \|f^\circ\|^2_{\Gamma_0(N_{f^\circ})} \cdot \eta^{-1}_{f^\circ,N^-}$ is the Gross period of f°,*
- *$\sigma_{\mathfrak{N}^+} \in \Gamma_\infty$ is the image of $\mathfrak{N}^+$ under the geometrically normalized Artin's reciprocity map,*
- *$u_K = |\mathcal{O}_K^\times|/2$, and $\varepsilon_p \in \{\pm 1\}$ is the local root number of f° at p.*

Proof. See [BD96] for the first construction, and [CH18, Thm. A] for the stated interpolation property. ☐

When χ is the trivial character, we write $\Theta_{f/K,\chi}(T)$ simply as $\Theta_{f/K}(T)$. Suppose now that f is the specialization of a primitive Hida family $\boldsymbol{f} \in S^o(N_f, \mathbb{I})$ with branch character $\chi_{\boldsymbol{f}} = \mathbb{1}$ at an arithmetic point $x_1 \in \mathfrak{X}^+_{\mathbb{I}}$ of weight 2. Let $\ell \nmid pN_f$ be a prime split in K, let χ be a ring class character of K of conductor $\ell^m\mathcal{O}_K$ for some even $m > 0$. Set $C = D_K\ell^{2m}$ and let

$$\boldsymbol{g} = \boldsymbol{\theta}_\chi(S_2) \in S^o(C, \omega^{-1}\eta_{K/\mathbf{Q}}, \mathcal{O}[\![S_2]\!]), \quad \boldsymbol{h} = \boldsymbol{\theta}_{\chi^{-1}}(S_3) \in S^o(C, \omega^{-1}\eta_{K/\mathbf{Q}}, \mathcal{O}[\![S_3]\!])$$

be the primitive CM Hida families constructed in [Hsi17, §8.3], where $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to K. The p-adic triple product L-function of Theorem 4.1 for this triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is an element in $\mathcal{R}=\mathbb{I}[\![S_2,S_3]\!]$; in the following we let

$$\mathscr{L}_p^f(\breve{f}^\star,\breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star)\in\mathcal{O}[\![S]\!]$$

denote the restriction to the "line" $S=S_2=S_3$ of its image under the specialization map at x_1.

Let $\mathbb{K}_\infty$ be the $\mathbf{Z}_p^2$-extension of K, and let $K_{\mathfrak{p}^\infty}$ denote the $\mathfrak{p}$-ramified $\mathbf{Z}_p$-extension in $\mathbb{K}_\infty$, with Galois group $\Gamma_{\mathfrak{p}^\infty}=\mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)$. Let $\gamma_{\mathfrak{p}}\in\Gamma_{\mathfrak{p}^\infty}$ be a topological generator, and for each variable T let $\Psi_T:\mathrm{Gal}(\mathbb{K}_\infty/K)\to\mathcal{O}[\![T]\!]^\times$ be the universal character defined by

$$\Psi_T(\sigma)=(1+T)^{l(\sigma)},\quad\text{where }\sigma|_{K_{\mathfrak{p}^\infty}}=\gamma_{\mathfrak{p}}^{l(\sigma)}. \tag{4.3}$$

Denoting by c the action of the non-trivial automorphism of $K/\mathbf{Q}$, the character Ψ_T^{1-c} factors through Γ_∞ and yields an identification $\mathcal{O}[\![\Gamma_\infty]\!]\simeq\mathcal{O}[\![T]\!]$ corresponding to the topological generator $\gamma_{\mathfrak{p}}^{1-c}$. Let p^b be the order of the p-part of the class number of K. Hereafter, we shall fix $\mathbf{v}\in\overline{\mathbf{Z}}_p^\times$ such that $\mathbf{v}^{p^b}=\varepsilon_{\rm cyc}(\gamma_{\mathfrak{p}}^{p^b})\in 1+p\mathbf{Z}_p$. Let $K(\chi,\alpha_p)$ be the finite extension of K by adjoining the values of χ and α_p.

Proposition 4.3. *Set $T=\mathbf{v}^{-1}(1+S)-1$. Then*

$$\mathscr{L}_p^f(\breve{f}^\star,\breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star)=\pm\Psi_T^{c-1}(\sigma_{\mathfrak{N}^+})\cdot\Theta_{f/K}(T)\cdot C_{f,\chi}\cdot\sqrt{L^{\rm alg}(f/K\otimes\chi^2,1)},$$

where $C_{f,\chi}\in K(\chi,\alpha_p)^\times$ and

$$L^{\rm alg}(f/K\otimes\chi^2,1):=\frac{L(f/K\otimes\chi^2,1)}{\pi^2\Omega_{f^\circ,N^-}}\in K(\chi).$$

Proof. This is the factorization formula of [Hsi17, Prop. 8.1] specialized to $S=S_2=S_3$, using the interpolation property for $\Theta_{f/K,\chi^2}(T)$ at $\zeta=1$. □

Remark 4.4. The factorization of Proposition 4.3 reflects the decomposition of Galois representations

$$\mathbb{V}^\dagger_{f,\boldsymbol{gh}}=\big(V_f(1)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-c}\big)\oplus\big(V_f(1)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\chi^2\big). \tag{4.4}$$

4.5. **Euler system construction of theta elements.** For the rest of the paper, assume that f, $\boldsymbol{g}=\boldsymbol{\theta}_\chi(S)$, and $\boldsymbol{h}=\boldsymbol{\theta}_{\chi^{-1}}(S)$ are as in §4.4, viewing the latter two in $S^o(C,\omega^{-1}\eta_{K/\mathbf{Q}},\mathcal{O}[\![S]\!])$. Keeping the notations from §4.3, by [DR16, §1] there exists a class

$$\kappa(f,\boldsymbol{gh})\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f,\boldsymbol{gh}}(N)) \tag{4.5}$$

constructed from twisted diagonal cycles on the triple product of modular curves of tame level N (we shall briefly recall the construction of this class in Theorem 4.6 below), where we may take $N=\mathrm{lcm}(N_f,C)$.

Every triple of test vectors $\breve{\boldsymbol{F}}=(\breve{f},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$ defines a Galois-equivariant projection

$$\mathrm{pr}_{\breve{\boldsymbol{F}}}:\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f,\boldsymbol{gh}}(N))\longrightarrow\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f,\boldsymbol{gh}})$$

and we let

$$\kappa(\breve{f},\breve{\boldsymbol{g}}\breve{\boldsymbol{h}}):=\mathrm{pr}_{\breve{\boldsymbol{F}}}(\kappa(f,\boldsymbol{gh}))\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f,\boldsymbol{gh}}). \tag{4.6}$$

Since Ψ_T^{1-c} gives the universal character of $\Gamma_\infty=\mathrm{Gal}(K_\infty/K)$, by (4.4) and Shapiro's lemma we have the equalities

$$\begin{aligned}\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f,\boldsymbol{gh}})&=\mathrm{H}^1(\mathbf{Q},V_f(1)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-c})\oplus\mathrm{H}^1(\mathbf{Q},V_f(1)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\chi^2)\\&=\widehat{\mathrm{H}}^1(K_\infty,V_f(1))\oplus\mathrm{H}^1(K,V_f(1)\otimes\chi^2).\end{aligned} \tag{4.7}$$

Let g and h be the weight 1 eigenform θ_χ and $h = \theta_{\chi^{-1}}$, respectively, so that the specialization of $(\boldsymbol{g}, \boldsymbol{h})$ at $T = 0 (\iff S = \mathbf{v} - 1)$ is a p-stabilization of the pair (g, h).

Lemma 4.5. *Assume that* $L(1, f \otimes g \otimes h) = 0$ *and that* $L(f/K \otimes \chi^2, 1) \neq 0$. *Then for every choice of test vectors* $\breve{\boldsymbol{F}} = (\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ *we have:*

(1) $\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{h}}) \in \widehat{\mathrm{H}}^1(K_\infty, V_f(1))$.
(2) $\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{h}})) = 0 \in \widehat{\mathrm{H}}^1(K_{\infty,\overline{\mathfrak{p}}}, V_f(1))$.

Proof. Let $\boldsymbol{\kappa} = \kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{h}})$ and for $? \in \{f, \boldsymbol{g}, \boldsymbol{h}\}$, let $F^0V_?$ be the rank one subspace of $V_?$ fixed by the inertia group at p. By (4.7), in order to prove (1) it suffices to show that some specialization of κ has trivial image in $\mathrm{H}^1(K, V_f(1) \otimes \chi^2)$. Let

$$\kappa_{\breve{f},\breve{g}\breve{h}} := \boldsymbol{\kappa}|_{S=\mathbf{v}-1} \in \mathrm{H}^1(\mathbf{Q}, V_{fgh}) = \mathrm{H}^1(K, V_f(1)) \oplus \mathrm{H}^1(K, V_f(1) \otimes \chi^2),$$

where $V_{fgh} := V_f(1) \otimes V_g \otimes V_h$. As noted in [DR17a, p. 634], the Selmer group $\mathrm{Sel}(\mathbf{Q}, V_{fgh}) \subset \mathrm{H}^1(\mathbf{Q}, V_{fgh})$ is given by

$$\mathrm{Sel}(\mathbf{Q}, V_{fgh}) = \ker\big(\mathrm{H}^1(\mathbf{Q}, V_{fgh}) \xrightarrow{\partial_p \circ \mathrm{loc}_p} \mathrm{H}^1(\mathbf{Q}_p, V_f^-(1) \otimes V_g \otimes V_h)\big),$$

where ∂_p is the natural map induced by the projection $V_f \twoheadrightarrow V_f^- := V_f/F^0V_f$, and so

$$\mathrm{Sel}(\mathbf{Q}, V_{fgh}) = \mathrm{Sel}(K, V_f(1)) \oplus \mathrm{Sel}(K, V_f(1) \otimes \chi^2). \tag{4.8}$$

Thus the implications $L(1, f \otimes g \otimes h) = 0 \Longrightarrow \kappa_{\breve{f},\breve{g}\breve{h}} \in \mathrm{Sel}(\mathbf{Q}, V_{fgh})$ and $L(f/K \otimes \chi^2, 1) = 0 \Longrightarrow \mathrm{Sel}(K, V_f(1) \otimes \chi^2) = 0$, which follow from [DR17a, Thm. C] and [BD05, Cor. 4] (see also [CH15, Thm. 1]), respectively, imply the result.

We proceed to prove (2). We know that the local class $\mathrm{loc}_p(\boldsymbol{\kappa})$ belongs to $\mathrm{H}^1(\mathbf{Q}_p, F^+\mathbb{V}^\dagger_{f\boldsymbol{gh}})$, where

$$F^+\mathbb{V}^\dagger_{f\boldsymbol{gh}} := \big(F^0V_f(1) \otimes F^0V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}} + F^0V_f(1) \otimes V_{\boldsymbol{g}} \otimes F^0V_{\boldsymbol{h}} + V_f(1) \otimes F^0V_{\boldsymbol{g}} \otimes F^0V_{\boldsymbol{h}}\big) \otimes \mathcal{X}^{-1}$$

is a rank four subspace of $\mathbb{V}^\dagger_{f\boldsymbol{gh}}$ (see [DR17a, Cor. 2.3]). In our case where $(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\psi\chi}, \boldsymbol{\theta}_{\psi\chi^{-1}})$, we have

$$F^+\mathbb{V}^\dagger_{f\boldsymbol{gh}} = V_f(1) \otimes \Psi_T^{1-c} + F^0V_f(1) \otimes (\chi^2 \oplus \chi^{-2}),$$

where Ψ_T is viewed as a character of $G_{\mathbf{Q}_p}$ via the embedding $K \hookrightarrow \mathbf{Q}_p$ induced by $\mathfrak{p}$. From part (1) of the lemma, it follows that

$$\begin{aligned} \mathrm{loc}_p(\boldsymbol{\kappa}) &= (\mathrm{loc}_{\mathfrak{p}}(\boldsymbol{\kappa}), \mathrm{loc}_{\mathfrak{p}}(\overline{\boldsymbol{\kappa}})) \in \mathrm{H}^1(K_{\mathfrak{p}}, V_f(1) \otimes \Psi_T^{1-c}) \oplus \{0\} \\ &\subset \mathrm{H}^1(K_{\mathfrak{p}}, V_f(1) \otimes \Psi_T^{1-c}) \oplus \mathrm{H}^1(K_{\mathfrak{p}}, V_f(1) \otimes \Psi_T^{c-1}) = \mathrm{H}^1(\mathbf{Q}_p, V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \Psi_T^{1-c}). \end{aligned}$$

We thus conclude that $\mathrm{loc}_{\mathfrak{p}}(\overline{\boldsymbol{\kappa}}) = 0$, and hence $\mathrm{loc}_{\overline{\mathfrak{p}}}(\boldsymbol{\kappa}) = 0$. ∎

From now on, assume that $f^\circ \in S_2(N_f)$ is the newform corresponding to an elliptic curve $E/\mathbf{Q}$ with good ordinary reduction at p. In particular, $V_f(1) \simeq V_pE \otimes_{\mathbf{Q}_p} L$, and under the conditions in Lemma 4.5 we have a class $\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{h}}) \in \widehat{\mathrm{H}}^1(K_\infty, V_pE \otimes L)$.

The following key theorem is a variant of the "explicit reciprocity law" of [DR17a, Thm. 5.3] specialized to our setting.

Theorem 4.6 (Darmon–Rotger)**.** *Assume that* $L(1, f \otimes g \otimes h) = 0$ *and that* $L(f/K \otimes \chi^2, 1) \neq 0$. *Then* $\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa(\breve{f}^\star, \breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star)) = 0$ *and*

$$\mathscr{L}_p^f(\breve{f}^\star, \breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star) = \alpha_p/2 \cdot (1 - \alpha_p^{-1}a_p(\boldsymbol{g})a_p(\boldsymbol{h})^{-1}) \cdot \mathrm{Col}^\eta(\mathrm{loc}_{\mathfrak{p}}(\kappa(\breve{f}^\star, \breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star))), \tag{4.9}$$

where $\breve{\boldsymbol{F}}^\star = (\breve{f}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star)$ *is the triple of test vectors from Theorem 4.1.*

Proof. We begin by briefly recalling from [DR17a, §1] the construction of the class $\kappa(f,\boldsymbol{gh})$ in (4.5). In the following, all references are to [DR17a] unless otherwise stated.

Consider the triple product of modular curves over $\mathbf{Q}$:

$$W_{s,s} := X_0(Np) \times X_s \times X_s,$$

where $X_0(Np)$ and X_s are the classical modular curves attached to the congruence subgroups $\Gamma_0(Np)$ and $\Gamma_1(Np^s)$, respectively, and the module for the latter is the one such that the cusp $i\infty$ is defined over $\mathbf{Q}$. The group $G_s^{(N)} := (\mathbf{Z}/Np^s\mathbf{Z})^\times$ acts on X_s by the diamond operators $\langle a; b\rangle$ ($a \in (\mathbf{Z}/N\mathbf{Z})^\times$, $b \in (\mathbf{Z}/p^s\mathbf{Z})^\times$), and we let

$$W_s := W_{s,s}/D_s$$

be the quotient of $W_{s,s}$ by the action of the subgroup $D_s \subset G_s^{(N)} \times G_s^{(N)}$ consisting of elements of the form $(\langle a; b\rangle, \langle a; b^{-1}\rangle)$. Let ${}^\flat\Delta_{s,s} \in \mathrm{CH}^2(W_{s,s})(\mathbf{Q}(\zeta_s))$ be the class in the Chow group defined by the "twisted diagonal cycle" defined in (41), and let ${}^\flat\Delta_s \in \mathrm{CH}^2(W_s)(\mathbf{Q}(\zeta_s))$ denote its natural image under the projection $\mathrm{pr}_s : W_{s,s} \to W_s$. By Proposition 1.4, after applying the correspondence $\varepsilon_{s,s}$ in (47) the cycle $\Delta_{s,s}$ becomes null-homologous, and so

$$\Delta_s := \varepsilon_{s,s}({}^\flat\Delta_s) \in \mathrm{CH}^2(W_s)_0(\mathbf{Q}(\zeta_s)),$$

letting $\varepsilon_{s,s}$ still denote the linear endomorphism of $\mathrm{CH}^2(W_s)$ defined by the above correspondence. Let $\varepsilon_s : G_{\mathbf{Q}} \to (\mathbf{Z}/p^s\mathbf{Z})^\times$ be the mod p^s cyclotomic character, and let $X_s^\dagger$ be the twist of X_s by the cocycle $\sigma \in G_{\mathbf{Q}} \mapsto \langle 1; \epsilon_s(\sigma)\rangle$. By Proposition 1.6, we may alternatively view

$$\Delta_s \in \mathrm{CH}^2(W_s^\dagger)_0(\mathbf{Q}),$$

where $W_s^\dagger$ the quotient of $W_{s,s}^\dagger := X_0(Np) \times X_s \times X_s^\dagger$ be a diamond action defined as before.

Consider the étale Abel–Jacobi map

$$\mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^2(W_s^\dagger)_0(\mathbf{Q}) \longrightarrow \mathrm{H}^1(\mathbf{Q}, \mathrm{H}^3_{\mathrm{et}}(W^\dagger_{s\,/\overline{\mathbf{Q}}}, \mathbf{Z}_p)(2))$$

Let $e_{\mathrm{ord}} = \lim_n U_p^{n!}$ be Hida's ordinary projector. Set

$$V_{s,s}^{\mathrm{ord}} := \mathrm{H}^1_{\mathrm{et}}(X_0(Np)_{/\overline{\mathbf{Q}}}, \mathbf{Z}_p) \otimes e_{\mathrm{ord}}(\mathrm{H}^1(X_{s\,/\overline{\mathbf{Q}}}, \mathbf{Z}_p)(1)) \otimes e_{\mathrm{ord}}(\mathrm{H}^1(X^\dagger_{s\,/\overline{\mathbf{Q}}}, \mathbf{Z}_p)(1)), \tag{4.10}$$

and let $V_s^{\mathrm{ord}} := (V_{s,s}^{\mathrm{ord}})_{D_s}$ denote the D_s-coinvariants. Let $\varpi_2 : X_{s+1} \mapsto X_s$ be the degeneracy map given by $\tau \mapsto p\tau$ on the complex upper half plane, which naturally defines

$$(\varpi_{2,2})_* = (1, \varpi_2, \varpi_2)_* : V_{s+1,s+1}^{\mathrm{ord}} \longrightarrow V_{s,s}^{\mathrm{ord}}. \tag{4.11}$$

Let $\tilde{\kappa}_s \in \mathrm{H}^1(\mathbf{Q}, V_s^{\mathrm{ord}})$ be given by the image of $\mathrm{AJ}_{\mathrm{et}}(\Delta_s)$ under the composite map

$$\mathrm{H}^1(\mathbf{Q}, \mathrm{H}^3_{\mathrm{et}}(W^\dagger_{s\,/\overline{\mathbf{Q}}}, \mathbf{Z}_p)(2)) \xrightarrow{\varepsilon_{s,s}\mathrm{pr}_{s,*}^{-1}\varepsilon_{s,s}} \mathrm{H}^1(\mathbf{Q}, \mathrm{H}^3_{\mathrm{et}}(W^\dagger_{s\,/\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{D_s}(2))$$
$$\xrightarrow{(1,e_{\mathrm{ord}},e_{\mathrm{ord}})\mathrm{pr}_{1,1,1}} \mathrm{H}^1(\mathbf{Q}, (V_{s,s}^{\mathrm{ord}})_{D_s}) = \mathrm{H}^1(\mathbf{Q}, V_s^{\mathrm{ord}}),$$

where the first arrow is defined by Lemma 1.8, and $\mathrm{pr}_{1,1,1}$ is the projection onto the $(1,1,1)$-component in the Künneth decomposition for $H^3_{\mathrm{et}}(W^\dagger_{s\,/\overline{\mathbf{Q}}}, \mathbf{Z}_p)$. By Proposition 1.9, we have

$$(\varpi_{2,2})_*(\tilde{\kappa}_{s+1}) = (1, U_p, 1)(\tilde{\kappa}_s),$$

and hence we obtain the compatible family

$$\kappa_\infty := \varprojlim_s (1, U_p, 1)^{-s}(\tilde{\kappa}_s) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}_\infty^{\mathrm{ord}}), \quad \text{where } \mathbb{V}_\infty^{\mathrm{ord}} := \varprojlim_{\mathrm{s}} \mathrm{V}_{\mathrm{s}}^{\mathrm{ord}},$$

the limit being with respect to the maps induced by (4.11). The triple $(f, \boldsymbol{g}, \boldsymbol{h})$ defines a natural projection $\varpi_{f,\boldsymbol{g},\boldsymbol{h}} : \mathbb{V}_\infty^{\mathrm{ord}} \to \mathbb{V}^\dagger_{f,\boldsymbol{gh}}(N)$, and following Definition 1.15 one sets

$$\kappa(f, \boldsymbol{gh}) := \varpi_{f,\boldsymbol{gh}}(\kappa_\infty) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f,\boldsymbol{gh}}(N)).$$

This is the class in (4.5) above. Now, to prove the equality (4.9) in the theorem, it suffices to show that both sides agree at infinitely many points. Thus let $x \in \mathfrak{X}_{\mathbb{I}}^+$ have weight 2 with $\zeta := \epsilon_x(1+p) \in \mu_{p^\infty}$ a primitive p^s-th root of unity, and set

$$\kappa(f, \boldsymbol{g}_x \boldsymbol{h}_x) := \kappa(f, \boldsymbol{g}\boldsymbol{h})|_{T=\zeta\mathbf{v}-1}.$$

Directly from the definitions (*cf.* Proposition 2.5), we have

$$\kappa(f, \boldsymbol{g}_x \boldsymbol{h}_x) = a_p(\boldsymbol{g}_x)^{-s} \cdot \varpi_{f,\boldsymbol{g}_x,\boldsymbol{h}_x}(\mathrm{AJ}_{\mathrm{et}}(\Delta_s)) \in \mathrm{H}^1(\mathbf{Q}, V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N)), \tag{4.12}$$

where $V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N)$ is the $(f, \boldsymbol{g}_x, \boldsymbol{h}_x)$-isotypic component of (4.10), and $\varpi_{f,\boldsymbol{g}_x,\boldsymbol{h}_x}$ is the projection to that component. By Corollary 2.3 and (77), the image of $\kappa(f, \boldsymbol{g}_x\boldsymbol{h}_x)$ in the local cohomology group $\mathrm{H}^1(\mathbf{Q}_p(\zeta), V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N))$ lands in the Bloch–Kato finite subspace $\mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p(\zeta), V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N)) \subset \mathrm{H}^1(\mathbf{Q}, V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N))$, and so we may consider the image $\log_p(\kappa(f, \boldsymbol{g}_x\boldsymbol{h}_x))$ of this restriction under the Bloch–Kato logarithm map

$$\log_p : \mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p(\zeta), V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N)) \longrightarrow (\mathrm{Fil}^0 D_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N))^\vee,$$

where $D_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N) := (B_{\mathrm{cris}} \otimes V_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N))^{G_{\mathbf{Q}_p(\zeta)}}$ and the dual is with respect to the de Rham pairing $\langle\ ,\ \rangle_{\mathrm{dR}}$. By the de Rham comparison isomorphism, we have

$$D_{f\boldsymbol{g}_x\boldsymbol{h}_x}(N) \simeq \mathrm{H}^1_{\mathrm{dR}}(X_0(Np)_{/\mathbf{Q}_p(\zeta)})[f] \times \mathrm{H}^1_{\mathrm{dR}}(X_{s/\mathbf{Q}_p(\zeta)})(1)[\boldsymbol{g}_x] \times \mathrm{H}^1_{\mathrm{dR}}(X_{s/\mathbf{Q}_p(\zeta)})(1)[\boldsymbol{g}_x].$$

As in p. 639, attached to the test vectors $(\breve{f}, \breve{\boldsymbol{g}}_x, \breve{\boldsymbol{h}}_x)$ one has the de Rham classes $(\eta^\circ_{\breve{f}^*}, \omega^\circ_{\breve{\boldsymbol{g}}_x}, \omega^\circ_{\breve{\boldsymbol{h}}_x})$, and comparing Proposition 2.10 and Corollary 2.11 we deduce from (4.12) that

$$\begin{aligned} \langle \log_p(\kappa(f, \boldsymbol{g}_x\boldsymbol{h}_x)), \eta^\circ_{\breve{f}^*} \otimes \omega^\circ_{\breve{\boldsymbol{g}}_x}\omega^\circ_{\breve{\boldsymbol{h}}_x}\rangle_{\mathrm{dR}} &= a_p(\boldsymbol{g}_x)^{-s}\langle \mathrm{AJ}_p(\Delta_s), \eta^\circ_{\breve{f}^*} \otimes \omega^\circ_{\breve{\boldsymbol{g}}_x}\omega^\circ_{\breve{\boldsymbol{h}}_x}\rangle_{\mathrm{dR}} \\ &= \mathcal{E}(f, \boldsymbol{g}_x, \boldsymbol{h}_x) \cdot \mathfrak{g}(\epsilon_x) \cdot \alpha_p^{s-1} a_p(\boldsymbol{g}_x)^{-s} a_p(\boldsymbol{h}_x)^{-s} \cdot \breve{f}^*(\breve{\boldsymbol{g}}_x \breve{H}^\iota_x), \end{aligned}$$

where $\breve{H}^\iota_x = d^{-1}\breve{\boldsymbol{h}}^\iota_x$ is the primitive of $\breve{\boldsymbol{h}}^\iota_x$ given by part (3) of Corollary 4.5, and $\mathcal{E}(f, \boldsymbol{g}_x, \boldsymbol{h}_x) = -2(1 - \alpha_p^{-1} a_p(\boldsymbol{g}_x) a_p(\boldsymbol{h}_x)^{-1})^{-1}$. Consider the formal q-expansion

$$\breve{\boldsymbol{H}}^\iota(q) := \sum_{p \nmid n} \langle n^{-1} \rangle a_n(\breve{\boldsymbol{h}}) q^n.$$

Taking $(\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ to be the test vectors $\breve{\boldsymbol{F}}^\star$ from Theorem 4.1 above, the construction in [Hsi17, §3.6] yields $\mathscr{L}_p^f(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{h}}) = \breve{f}^*(\breve{\boldsymbol{g}}\breve{\boldsymbol{H}}^\iota)$. Since by construction $\breve{\boldsymbol{H}}^\iota$ specializes at x to $\breve{H}^\iota_x$, we thus see as in the proof of Theorem 4.16 that

$$\langle \log_p(\kappa(f, \boldsymbol{g}_x\boldsymbol{h}_x)), \eta^\circ_{\breve{f}^*} \otimes \omega^\circ_{\breve{\boldsymbol{g}}_x}\omega^\circ_{\breve{\boldsymbol{h}}_x}\rangle_{\mathrm{dR}} = \mathcal{E}(f, \boldsymbol{g}_x, \boldsymbol{h}_x) \cdot \mathfrak{g}(\epsilon_x) \cdot \alpha_p^{s-1} a_p(\boldsymbol{g}_x)^{-s} a_p(\boldsymbol{h}_x)^{-s} \cdot \mathscr{L}_p^f(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{h}})(x). \tag{4.13}$$

On the other hand, letting $\psi_x := \Psi_T|_{T=\zeta\mathbf{v}-1}$, we obtain that $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ are theta series attached to the characters $(\chi\psi_x^{-1}, \chi^{-1}\psi_x^{-1})$ of G_K with $a_p(\boldsymbol{g}_x) = \chi\psi_x^{-1}(\sigma_{\overline{\mathfrak{p}}})$ and $a_p(\boldsymbol{h}_x) = \chi^{-1}\psi_x^{-1}(\sigma_{\overline{\mathfrak{p}}})$. Moreover, we have

$$\epsilon_x|_{G_{\mathbf{Q}_p}} = \psi_x^{1+c}|_{G_{K_\mathfrak{p}}} \cdot \varepsilon_{\mathrm{cyc}}^{-1}; \quad \psi_x^{c-1} = \phi_x \breve{\varepsilon}_{\mathcal{F}}^{-1}$$

for some finite order character ϕ_x of $\mathrm{Gal}(F_\infty/\mathbf{Q}_p)$, viewing the character in the left-hand side of this equality as character on $\mathrm{Gal}(F_\infty/F)$ by composition with $\mathrm{Gal}(F_\infty/F) \subseteq \mathrm{Gal}(F_\infty/\mathbf{Q}_p) \twoheadrightarrow \mathrm{Gal}(K_{\infty,\mathfrak{p}}/K_\mathfrak{p}) \subset \Gamma_\infty$. Setting $\eta = \eta^\circ_{\breve{f}^*} \otimes t^{-1}$ and $\mathbf{z}_x = \mathrm{loc}_\mathfrak{p}(\kappa(\breve{f}^\star, \breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star))_x$, we thus see that

$$\begin{aligned} \langle \log_p(\kappa(f, \boldsymbol{g}_x\boldsymbol{h}_x)), \eta^\circ_{\breve{f}^*} \otimes \omega^\circ_{\breve{\boldsymbol{g}}_x}\omega^\circ_{\breve{\boldsymbol{h}}_x}\rangle_{\mathrm{dR}} &= \langle \log_p(\mathbf{z}_x) \otimes t, \eta\rangle_{\mathrm{dR}} \\ &= \mathfrak{g}(\epsilon_x) \cdot \alpha_p^s a_p(\boldsymbol{g}_x)^{-s} a_p(\boldsymbol{h}_x)^{-s} \cdot \mathrm{Col}^\eta(\mathbf{z}_x)(\psi_x^{c-1}), \end{aligned} \tag{4.14}$$

using Theorem 3.4 with $j = -1$ for the last equality. Comparing (4.13) with (4.14) and letting s vary, the result follows. □

The following theorem gives a Euler system construction of $\Theta_{f/K}(T)$ from a p-adic family of diagonal cycles.

Theorem 4.7. *With notations and assumptions as in Theorem 4.6, we have*

$$\mathrm{Col}^\eta(\mathrm{loc}_\mathfrak{p}(\kappa(\breve{f}^\star,\breve{\boldsymbol{g}}^\star\breve{\boldsymbol{h}}^\star))) = \pm\Psi_T^{c-1}(\sigma_{\mathfrak{N}^+})\cdot\Theta_{f/K}(T)\cdot\sqrt{L^{\mathrm{alg}}(E/K\otimes\chi^2,1)}\cdot\frac{2C_{f,\chi}}{\alpha_p(1-\alpha_p^{-1}\chi(\overline{\mathfrak{p}})^2)},$$

where $C_{f,\chi}\in K(\chi,\alpha_p)^\times$ is the non-zero algebraic number as in Proposition 4.3.

Proof. Note that $a_p(\boldsymbol{g})a_p(\boldsymbol{h})^{-1}=\chi(\overline{\mathfrak{p}})^2$. The theorem thus follows immediately from Proposition 4.3 and Theorem 4.6. □

4.6. **Generalized Kato classes.** Set $\alpha=\chi(\overline{\mathfrak{p}})$, and denote by $(g_\alpha,h_{\alpha^{-1}})$ the weight 1 forms obtained by specializing the Hida families $(\boldsymbol{g},\boldsymbol{h})$ at $S=\mathbf{v}-1$. Thus g_α (resp. $h_{\alpha^{-1}}$) is the p-stabilization of the theta series $g=\theta_\chi$ (resp. $h=\theta_{\chi^{-1}}$) having U_p-eigenvalue α (resp. α^{-1}). By specialization, the $\mathcal{O}[\![S]\!]$-adic class in (4.6) yields the class

$$\kappa(f,g_\alpha,h_{\alpha^{-1}}):=\kappa(\breve{f},\breve{\boldsymbol{g}}\breve{\boldsymbol{h}})|_{S=\mathbf{v}-1}\in\mathrm{H}^1(\mathbf{Q},V_{fgh}),$$

where $V_{fgh}:=V_f\otimes V_g\otimes V_h$. Setting $\beta=\chi(\mathfrak{p})$ and alternatively changing the roles of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ in the construction $\boldsymbol{g}$ and $\boldsymbol{h}$ we thus obtain the four *generalized Kato classes*

$$\kappa(f,g_\alpha,h_{\alpha^{-1}}),\ \kappa(f,g_\alpha,h_{\beta^{-1}}),\ \kappa(f,g_\beta,h_{\alpha^{-1}}),\ \kappa(f,g_\beta,h_{\beta^{-1}})\in\mathrm{H}^1(\mathbf{Q},V_{fgh}). \tag{4.15}$$

From now on, we assume that $\alpha\neq\pm1$, so that the four classes (4.15) are *a priori* distinct. Recall that f is the p-stabilization of the newform associated to an elliptic curve $E/\mathbf{Q}$, so that $V_f(1)=V_pE$, and let $\kappa_{\alpha,\alpha^{-1}},\kappa_{\alpha,\beta^{-1}},\kappa_{\beta,\alpha^{-1}},\kappa_{\beta,\beta^{-1}}\in\mathrm{H}^1(K,V_pE)$ be the image of the classes (4.15) under the map $\mathrm{H}^1(\mathbf{Q},V_{fgh})\to\mathrm{H}^1(K,V_pE)$ induced by (1.5).

Corollary 4.8. *Assume that $L(E/K,1)=0$ and that $L(f/K\otimes\chi^2,1)\neq0$. Then:*

(1) $\kappa_{\alpha,\alpha^{-1}},\kappa_{\beta,\beta^{-1}}\in\mathrm{Sel}(K,V_pE\otimes L)$,
(2) $\kappa_{\alpha,\beta^{-1}}=\kappa_{\beta,\alpha^{-1}}=0$.

Proof. By the factorization (1.6), the inclusions in part (1) follow from the proof of Lemma 4.5. To see part (2), we make use of the 3-variable generalized Kato class $\boldsymbol{\kappa}:=\boldsymbol{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(S_1,S_2,S_2)\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{\boldsymbol{fgh}})$ defined in [DR17b, §3.7 (119)] attached to the triple $\boldsymbol{f}=\boldsymbol{f}(S_1)$, $\boldsymbol{g}=\boldsymbol{\theta}_{\psi\chi}(S_2)$ and $\boldsymbol{h}=\boldsymbol{\theta}_{\psi\chi}(S_3)$. Thus $\kappa(f,g_\alpha,h_{\beta^{-1}})$ is the specialization $\boldsymbol{\kappa}((1+p)^2-1,\mathbf{v}-1,\mathbf{v}-1)$. Let

$$\boldsymbol{\kappa}':=\boldsymbol{\kappa}((1+p)^2-1,\mathbf{v}(1+T)-1,\mathbf{v}(1+T)^{-1}-1)\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{\boldsymbol{fgh}^\iota}),$$

where $\mathbb{V}^\dagger_{\boldsymbol{fgh}^\iota}=V_p(E)\otimes\left(\mathrm{Ind}_K^{\mathbf{Q}}\chi^2\oplus\mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-c}\right)$. Similarly as in the proof of Lemma 4.5, by [DR17b, Prop. 3.28] the class $\mathrm{loc}_p(\boldsymbol{\kappa}')$ belongs to $\mathrm{H}^1(\mathbf{Q}_p,F^+\mathbb{V}^\dagger_{\boldsymbol{fgh}^\iota})$, where

$$F^+\mathbb{V}^\dagger_{\boldsymbol{fgh}}=V_pE\otimes\chi^{-2}+F^0V_pE\otimes(\Psi_T^{1-c}\oplus\Psi_T^{1-c}).$$

It follows that the projection $\boldsymbol{\kappa}'_V$ of $\boldsymbol{\kappa}'$ into $\widehat{\mathrm{H}}^1(K_\infty,V_pE)$ is crystalline at p, and hence $\boldsymbol{\kappa}'_V$ is a Selmer class for V_pE over the anticyclotomic tower K_∞/K. Since the space of such universal norms is trivial by Cornut–Vatsal [CV05] (the sign of E/K is $+1$ in our case), this shows that $\boldsymbol{\kappa}'_V=0$ and therefore $\kappa(f,g_\alpha,h_{\beta^{-1}})=\kappa_{\alpha,\beta^{-1}}=0$. The vanishing of $\kappa_{\beta,\alpha^{-1}}$ is shown in the same manner. □

5. Proof of the main result

Proof of Theorem 1. We begin by noting that our hypotheses imply that for every place v of K above p the restriction map

$$\mathrm{loc}_v : \mathrm{Sel}(K, V_pE) \longrightarrow \mathrm{H}^1(K_v, V_pE)$$

is nonzero. Indeed, the finiteness of $Ш(E/K)[p^\infty]$ (which follows from our hypotheses that $Ш(E/\mathbf{Q})[p^\infty]$ is finite and that $L(E^K, 1) \neq 0$, so that $Ш(E^K/\mathbf{Q})[p^\infty]$ is also finite by [Kol88]) implies that $\mathrm{Sel}(K, V_pE) \simeq E(K) \otimes_{\mathbf{Z}} \mathbf{Q}_p$, and the claim follows from the inclusion $E(K) \subset E(K_v)$ and the injectivity of the local Kummer map $E(K_v) \otimes_{\mathbf{Z}} \mathbf{Q}_p \to \mathrm{H}^1(K_v, V_pE)$. In particular, setting $V = V_pE \otimes_{\mathbf{Q}_p} L$, the composite map

$$\log_{\omega_E,\mathfrak{p}} : \mathrm{Sel}(K, V) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \mathrm{H}^1_{\mathrm{fin}}(K_{\mathfrak{p}}, V) \to L$$

is nonzero, where the second arrow is given by (3.8).

Now let $S = \mathrm{Sel}(K, V_pE) \otimes_{\mathbf{Q}_p} L$. Let $S^{(r)} := S_p^{(r)}(E/K) \otimes_{\mathbf{Q}_p} L$ be the subspaces in §2.3 and $S^{(\infty)}$ be the subspace of anticyclotomic universal norms. By [How04, Thm. 4.2] we have the filtration

$$S = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset S^{(r+1)} \supset \cdots \supset S^{(\infty)}, \tag{5.1}$$

and $S^{(r+1)}$ is the null space of r-th derived height pairing $h^{(r)} \colon S^{(r)} \times S^{(r)} \to J^r/J^{r+1} \otimes L$. By Cornut–Vatsal [CV05], we have $S^{(\infty)} = \{0\}$. Since $L(E^K, 1) \neq 0$, by Kolyvagin's work [Kol88] (or alternatively, Kato's [Kat04]) we have $\mathrm{Sel}(\mathbf{Q}, V_pE^K) = \{0\}$, and so letting $S^+ \simeq \mathrm{Sel}(\mathbf{Q}, V_pE) \otimes_{\mathbf{Q}_p} L$ denote the subspace of S fixed under the action of complex conjugation, we have

$$S = S^+.$$

By the second part of [How04, Thm. 4.2], it follows that $h^{(1)}$ is identically zero on S, and so $S = S^{(2)}$, and more generally $S^{(r)} = S^{(r+1)}$ for every odd $r \geqslant 1$.

In particular, the above show that under our hypotheses we have

$$\dim_L S = \dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

Combined with parts (c) and (d) of [How04, Cor. 4.3], it follows that (5.1) reduces to

$$S = S^{(1)} = S^{(2)} = \cdots = S^{(r)} \quad \text{and} \quad S^{(r+1)} = \cdots = S^{(\infty)} = \{0\} \tag{5.2}$$

for some *even* $r \geqslant 2$ and the derived height $h^{(r)}$ is a non-degenerate pairing on $S^{(r)}$. Let X_∞ be the Pontrjagin dual of $\mathrm{Sel}_{p^\infty}(E/K_\infty)$, which is known to be Λ-torsion [BD05]. Let $J \subset \Lambda$ be the augmentation ideal, and fix a pseudo-isomorphism

$$X_\infty \sim M \oplus M', \quad \text{with} \quad M \simeq (\Lambda/J)^{e_1} \oplus (\Lambda/J^2)^{e_2} \oplus \cdots \tag{5.3}$$

and M' a torsion Λ-module with characteristic ideal prime to J. By [How04, Cor. 4.3(c)] we have $e_i = \dim_{\mathbf{Q}_p}(S^{(i)}/S^{(i+1)})$; letting $\mathcal{L}_p \in \Lambda$ be a generator of the principal ideal $\mathrm{char}_\Lambda(X_\infty)$, combining (5.2) and (5.3) this shows that

$$\mathrm{ord}_J \mathcal{L}_p = 2r.$$

On the other hand, by our hypotheses on $\bar{\rho}_{E,p}$ the divisibility in the Iwasawa main conjecture due to Skinner–Urban [SU14] (see [*loc.cit.*, §3.6.3]) implies that $(\Theta_{f/K}^2) \subset (\mathcal{L}_p)$, and so

$$r \geqslant \rho := \mathrm{ord}_J(\Theta_{f/K}). \tag{5.4}$$

Let $\bar{\theta}_{f/K}$ be the *leading coefficient* of $\Theta_{f/K}$ defined by

$$\bar{\theta}_{f/K} := \Theta_{f/K}(T) \pmod{J^{\rho+1}} \in J^\rho/J^{\rho+1}.$$

From (5.2) and (5.4) we see that $S = S^{(\rho)}$. Thus combining the derived p-adic height formula in Proposition 3.7, Theorem 4.7 and part (2) of Lemma 4.5, we deduce that for every $x \in S^{(\rho)} = E(\mathbf{Q}) \otimes_{\mathbf{Z}} L$ we have

$$h^{(\rho)}(\kappa_{\alpha,\alpha^{-1}}, x) = \frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot \bar{\theta}_{f/K} \cdot \log_{\omega_E,\mathfrak{p}}(x) \cdot C, \tag{5.5}$$

where α_p is the p-adic unit root of $X^2 - a_p(E)X + p = 0$, and C is a non-zero algebraic number with $C^2 \in K(\chi, \alpha_p)^\times$. Since as noted above our hypotheses imply that $\bar{\theta}_{f/K} \neq 0$ and the map $\log_{\omega_E,\mathfrak{p}}$ is nonzero, we see that $r = \rho$ and the non-vanishing of $\kappa_{\alpha,\alpha^{-1}}$ follows. □

Remark 5.1. By the height formula (5.5), we can deduce that the class $\kappa_{\alpha,\alpha^{-1}} \mod \overline{\mathbf{Q}}^\times$ actually only depends on K and is independent of the auxiliary choice of ring class character χ, and that as elements in $E(\mathbf{Q}) \otimes_{\mathbf{Z}} L$ we have

$$\kappa_{\alpha,\alpha^{-1}} = C \cdot \frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot \frac{\bar{\theta}_{f/K}}{h^{(\rho)}(P, Q)} \cdot \big(P \otimes \log_p Q - Q \otimes \log_p P\big)$$

with a basis (P, Q) of $E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$.

6. Numerical examples

In the following examples, we consider elliptic curves $E/\mathbf{Q}$ with $\mathrm{rank}_{\mathbf{Z}} E(\mathbf{Q}) = 2$ and conductor $N \in \{q, 2q\}$ where q is a prime, $\Delta > 0$ a square-free integer such that q is inert in $K = \mathbf{Q}(\sqrt{-\Delta})$ with class number one and the L-series of the quadratic twist E^K does not vanish at 1, and $p \geq 5$ is an odd good ordinary prime for E which is split in K. For every such triple $(E, p, -\Delta)$, letting $f \in S_2(\Gamma_0(N))$ be the newform associated with E, we give numerical examples where the associated theta element

$$\Theta_{E/K}(T) = \Theta_{f/K}(T) \in \mathbf{Z}_p[\![T]\!]$$

vanishes to order exactly 2 at $T = 0$. When that is the case, by the work of Bertolini–Darmon [BD95, BD05] on the anticyclotomic Iwasawa main conjecture[1] (see [BD05, Cor. 3]), it follows that Ш$(E/K)[p^\infty]$ is finite. Moreover, by [Rib90, Thm. 1.1] the residual representation $\bar{\rho}_{E,p}$ must ramify at $N^- = q$. Thus for every ring class character χ with $L(E/K, \chi^2, 1) \neq 0$ (as one can always find by virtue of [Vat03, Thm. 1.4], as extended in [CH18, Thm. D]), the examples below provide instances where Theorem 1 holds.

To explain these numerical examples, we prepare some notation. Let B be the definite quaternion algebra of discriminant q. Let R be an Eichler order of level N/q and $\mathrm{Cl}(R)$ be the class group of R. Let $f_E : \mathrm{Cl}(R) \to \mathbf{Z}$ be the (p-adically normalized) Hecke eigenfunction associated with f by the Jacquet–Langlands correspondence. Fix an optimal embedding $\mathcal{O}_K \hookrightarrow R$ and an isomorphism $i_p : R \otimes \mathbf{Z}_p \simeq \mathrm{M}_2(\mathbf{Z}_p)$ such that $i_p(K)$ lies in the subspace of diagonal matrices. For $a \in \mathbf{Z}_p^\times$ and an integer n, put

$$r_n(a) = i_p^{-1}(\begin{pmatrix} 1 & ap^{-n} \\ 0 & 1 \end{pmatrix}) \in \widehat{B}^\times, \quad \widehat{B} := B \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}.$$

Consider the sequence $\{P_n^a\}_{n=0,1,\ldots}$ of right R-ideals defined by $P_n^a := (r_n(a)\widehat{R}) \cap B$. The images of these ideals P_n^a in $\mathrm{Cl}(R)$ are usually referred to Gross points of level p^n. Letting $\mathbf{u} = 1 + p$, we define the n-th theta element $\Theta_{E/K,n}(T) \in \mathbf{Z}_p[T]$ by

$$\Theta_{E/K,n}(T) := \frac{1}{\alpha_p^{n+1}} \sum_{i=0}^{p^n - 1} \sum_{a \in \mu_{p-1}} \Big(\alpha_p \cdot f_E(P_n^{a\mathbf{u}^i}) - f_E(P_{n+1}^{a\mathbf{u}^i})\Big) (1 + T)^i.$$

[1] As extended by Pollack–Weston [PW11] to allow for weaker hypotheses.

By the definition of theta elements in [BD96, §2.7], if K has class number one, we then have

$$\Theta_{E/K}(T) \equiv \Theta_{E/K,n}(T) \pmod{(1+T)^{p^n}-1}.$$

Since $(p^n, (1+T)^{p^n}-1) \subset (p^n, T^p)$ and $p > 2$, to check the vanishing $\Theta_{E/K}(T)$ to exact order 2 at $T = 0$, it suffices to compute $\Theta_{E/K,n}(T)$ for sufficiently large n. The following examples are obtained by implementing the Brandt module package in SAGE.

E	p	$-\Delta$	$\Theta_{E/K,2}(T)$ mod (p^2, T^p)
389a1	11	-2	$120T^2+58T^3+78T^4+59T^5+40T^6$
433a1	11	-7	$88T^2+22T^3+86T^4+7T^5+10T^6+12T^7+29T^8+88T^9+48T^{10}$
446c1	7	-3	$22T^2+27T^3+3T^4+16T^5+11T^6$
563a1	5	-1	$18T^2+9T^3+5T^4$
643a1	5	-1	T^2+21T^4
709a1	11	-2	$27T^2+114T^3+3T^4+14T^5+36T^6+15T^7+42T^8+44T^9+91T^{10}$
718b1	5	-19	$3T^2+20T^3+12T^4$
794a1	7	-3	$47T^2+23T^3+8T^4+24T^5+7T^6$
997b1	11	-2	$71T^2+41T^3+83T^4+19T^5+114T^6+111T^7+101T^8+46T^9+102T^{10}$
997c1	11	-2	$54T^2+38T^3+36T^4+81T^5+82T^6+18T^7+72T^8+95T^9+4T^{10}$
1034a1	5	-19	$22T^2+4T^3+6T^4$
1171a1	5	-1	$6T^2+6T^3+20T^4$
1483a1	13	-1	$128T^2+148T^3+127T^4+162T^5+30T^6+149T^7+141T^8+97T^9+49T^{10}+13T^{11}+29T^{12}$
1531a1	5	-1	$16T^2+7T^3+21T^4$
1613a1	17	-2	$128T^2+165T^3+224T^4+287T^5+140T^6+211T^7+147T^8+160T^9+59T^{10}+122T^{11}+195T^{12}+43T^{13}+207T^{14}+214T^{15}+285T^{16}$
1627a1	13	-1	$101T^2+151T^3+58T^4+104T^5+3T^6+165T^7+128T^8+63T^9+17T^{10}+55T^{11}+166T^{12}$
1907a1	13	-1	$72T^2+131T^3+32T^4+142T^5+84T^6+104T^7+90T^8+105T^9+38T^{10}+92T^{11}+116T^{12}$
1913a1	7	-3	$41T^2+16T^3+28T^4+23T^5+14T^6$
2027a1	13	-1	$54T^2+128T^3+65T^4+93T^5+83T^6+161T^7+113T^8+133T^9+49T^{10}+151T^{11}+13T^{12}$

E	p	$-\Delta$	$\Theta_{E/K,3}(T)$ mod (p^3, T^p)
571b	5	-1	$100T^2+100T^3+15T^4$
1621a1	11	-2	$1089T^2+807T^4+986T^5+586T^6+1098T^7+772T^8+228T^9+1296T^{10}$

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