THE VANISHING OF $\mu$-IN Variant OF $p$-adic HECKE $L$-FUNCTIONS FOR CM FIELDS

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Abstract. Let $p > 2$ be an ordinary prime for a CM field $K$. Katz and Hida-Tilouine constructed the $p$-adic Hecke $L$-function attached to a $p$-ordinary CM type and a branch character. In this note, we prove that the $\mu$-invariant of this $p$-adic Hecke $L$-function always vanishes when $p$ is unramified in $K$.

1. INTRODUCTION

The purpose of this note is to prove the vanishing of the $\mu$-invariant of $p$-adic Hecke $L$-functions for CM fields constructed by Katz and Hida-Tilouine. We let $F$ be a totally real field of degree $d$ over $Q$ and $K$ be a totally imaginary quadratic extension of $F$. Let $D_\infty$ (resp. $D_F$) be the discriminant (resp. different) of $F/Q$. Let $p > 2$ be an odd rational prime. Fix two embeddings $\iota_\infty: Q \rightarrow C$ and $\iota_p: Q \rightarrow C_p$ once and for all. Let $Z$ be the ring of algebraic integers and let $Z_p$ be the $p$-adic completion of $Z$ in $C_p$. Denote by $c$ the complex conjugation on $C$ which induces the unique non-trivial element of $\text{Gal}(K/F)$. We assume the following hypothesis throughout this article:

Every prime of $F$ above $p$ splits in $K$.

Fix a $p$-ordinary CM type $\Sigma$, namely $\Sigma$ is a CM type of $K$ such that $p$-adic places induced by elements in $\Sigma$ via $\iota_p$ are disjoint from those induced by elements in $\Sigma_c$. The existence of such $\Sigma$ is assured by our assumption (ord). Let $D_{K/F}$ be the relative different of $K/F$. Let $C$ be a prime-to-$p$ integral ideal of $O_K$ and let $\ell \in K$ such that

(d1) $c(\ell) = -\ell$ and $\text{Im}\sigma(\ell) > 0$ for all $\sigma \in \Sigma$,
(d2) $c(O_C) := D_F^{-1}(2\ell D_{K/F}^{-1})$ is prime to $p\mathcal{C}^C D_{K/F}$.

Let $K_\infty^+$ and $K_{\infty}^-$ be the cyclotomic $Z_p$-extension and anticyclotomic $Z_p^d$-extension of $K$. Let $K_\infty = K_\infty^+ K_{\infty}^-$. Let $K_\infty$ be a $Z_p^{d+1}$-extension of $K$. If one assumes Leopoldt’s conjecture for $K$, then $K_\infty$ is the maximal $Z_p^{d+1}$-extension of $K$. Let $\Gamma^\pm := \text{Gal}(K_\infty^\pm/K)$ and let $\Gamma = \text{Gal}(K_\infty/K) \simeq \Gamma^+ \times \Gamma^-$. Let $Z(C)$ be the ray class group of $K$ modulo $\mathcal{C}_p^{\infty}$. In [Kat78] and [HT93], a $Z_p$-valued $p$-adic measure $L_{\infty,\Sigma}$ on $Z(C)$ is constructed such that

\[ \int_{Z(C)} \hat{\lambda} dL_{\infty,\Sigma} = L^{(p\mathcal{C})}(0, \ell) \cdot \text{Eul}_p(\ell) \cdot \text{Eul}_{\mathcal{C}_p}(\ell), \]

where (i) $\lambda$ is a Hecke character modulo $\mathcal{C}_p^{\infty}$ of infinity type $k\Sigma + \kappa(1-c)$ with either $k \geq 1$ and $\kappa \in Z_{\geq 0}[\Sigma]$ or $k \leq 1$ and $k\Sigma + \kappa \in Z_{\leq 0}[\Sigma]$, and $\hat{\lambda}$ is the $p$-adic avatar of $\lambda$ regarded as a $p$-adic Galois character via geometrically normalized reciprocity law, (ii) $\text{Eul}_p(\lambda)$ and $\text{Eul}_{\mathcal{C}_p}(\lambda)$ are certain modified Euler factors (For the definitions, see [Hsi12, (4.16)]).

We fix a Hecke character $\lambda$ of infinity type $k\Sigma$, $k \geq 1$. Let $L_{\infty,\Sigma}$ be the $p$-adic measure on $\Gamma$ obtained by the pull-back of $L_{\infty,\Sigma}$ along $\lambda$. In other words, for every locally constant function $\varphi$ on $\Gamma$, we have

\[ \int_{\Gamma} \varphi dL_{\lambda,\Sigma} = \int_{Z(C)} \varphi \hat{\lambda} dL_{\infty,\Sigma}. \]

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We call $\mathcal{L}_{\lambda, \Sigma}$ the $p$-adic $L$-function of the branch character $\lambda$ with respect to the $p$-adic CM-type $\Sigma$. It is conjectured by Gillard [Gil91, Conj. (i), p.21] that the $\mu$-invariant $\mu_{\lambda, \Sigma}$ of $\mathcal{L}_{\lambda, \Sigma}$ always vanishes. In this note, we prove this conjecture when $p \nmid D_F$.

**Theorem A.** Suppose that $p \nmid D_F$. Then $\mu_{\lambda, \Sigma} = 0$.

When $\mathcal{F} = \mathbb{Q}$ and $\lambda$ arises from elliptic curves over $\mathbb{Q}$ with CM by $\mathcal{K}$, this theorem is an immediate consequence of the vanishing of the $\mu$-invariant of Coates-Wiles $p$-adic $L$-functions due to Gillard [Gil87] and Schneps [Sch87] independently. When the conductor of the residual character $\lambda (\bmod p)$ is a product of primes split in $\mathcal{K}/\mathcal{F}$, the above theorem is due to Hida in [Hid11]. Note that since the branch character $\lambda$ is of infinite order, this $p$-adic $L$-function $\mathcal{L}_{\lambda, \Sigma}$ indeed is a suitable twist of the $p$-adic $L$-functions considered by Hida.

To explain the idea of Hida, we need to introduce some notation. Let $\mathfrak{x}^+$ be the set consisting of finite order characters $\nu : \Gamma^+ \to \mu_{\infty}$. For every $\nu \in \mathfrak{x}^+$, we shall regard $\nu$ as a Hecke character of $\mathcal{K}^\chi$ by the geometrically normalized reciprocity law $\text{rec}_\mathcal{K} : A^1_\mathcal{K} \to \text{Gal}(\mathbb{Q}/\mathcal{K})^{ab} \to \Gamma$. Let $\mu_{\lambda, \Sigma}$ denote the $\mu$-invariant of the anticyclotomic projection $\mathcal{L}_{\lambda, \Sigma}$ of Katz $p$-adic $L$-function $\mathcal{L}_{\lambda, \Sigma}$ attached to the branch character $\lambda \nu$. When $\lambda$ has split conductor, Hida in [Hid10] proves a precise formula of $\mu_{\lambda, \Sigma}$ in terms of the $p$-adic valuation of Fourier coefficients of certain Eisenstein series. Based on this exact formula, Hida concludes the vanishing of Fourier coefficients of certain Eisenstein series. Based on this exact formula, Hida concludes the vanishing of $\mu_{\lambda, \Sigma}$ by showing directly that $\lim \inf_{\nu \in \mathfrak{x}^+} \mu_{\lambda, \Sigma} = 0$.

Our proof of Theorem A follows the approach of Hida. It is shown in [Hsi12, Thm. 5.5] that $\mu_{\lambda, \Sigma}$ in general can be written to be the $p$-adic valuation of Fourier coefficients of certain special toric Eisenstein series $E^{h}_{\lambda, \Sigma, u}$. We are not able to calculate the Fourier coefficients of these toric Eisenstein series in full generality, so we do not obtain a precise formula of $\mu_{\lambda, \Sigma}$ in full generality. However, we can estimate an upper bound of the $p$-adic valuation of Fourier coefficients of $E^{h}_{\lambda, \Sigma, u}$ and obtain an upper bound of $\mu_{\lambda, \Sigma}$. Following Hida, we show this upper bound is as small as possible when $\nu \in \mathfrak{x}^+$ has sufficiently deep conductor.

In virtue of [HT93, Thm. 8.2], Theorem A provides an alternative proof of the one-sided divisibility between anticyclotomic $p$-adic $L$-functions and the congruence ideals of CM forms, which was proved in [Hid09, Cor. 3.8] using the trick of base change. This divisibility result eventually leads to the solution of the anticyclotomic main conjecture for CM fields implies that $\sigma$-invariant attached to the branch character $\lambda \nu$, i.e. the $\mu$-invariant of characteristic power series of a certain Iwasawa module (cf. [HT94, Main conjecture, p.90]). In particular, we can consider an CM elliptic curve $E$ over the totally real field $\mathcal{F}$ with complex multiplication by the ring of integers of an imaginary quadratic field $\mathcal{M}$. Assuming the validity of the main conjecture for the CM field $\mathcal{K} = \mathcal{M}$, our result would imply the algebraic $\mu$-invariant for $E$ over $\mathcal{K}_{\infty}$ vanishes as well. The arithmetic aspect of the vanishing of algebraic $\mu$-invariants of elliptic curves in a more general setting is discussed in [Suj10].

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## 2. Eisenstein series and anticyclotomic $\mu$-invariants

In this section, we recall without proofs the construction of certain special Eisenstein series, which are used to compute the anticyclotomic $\mu$-invariant in [Hsi12].

### 2.1. Eisenstein series on $\text{GL}_2(A_F)$

Let $\chi$ be a Hecke character of infinite type $k\Sigma$, $k \geq 1$. Suppose that $\mathfrak{c}$ is the prime-to-$p$ conductor of $\chi$. We write $\mathfrak{c} = \mathfrak{c}^+ \mathfrak{c}^-$ such that $\mathfrak{c}^+$ (resp. $\mathfrak{c}^-$) is a product of prime factors split (resp. non-split) over $\mathcal{F}$. We further decompose $\mathfrak{c}^+ = \mathfrak{f}_1 \mathfrak{f}_c$ such that $(\mathfrak{f}_1, \mathfrak{f}_c) = 1$ and $\mathfrak{f}_c \subset \mathfrak{c}_c^+$. Let $D_{K/F}$ be the discriminant of $\mathcal{K}/\mathcal{F}$ and let

$$D = p\mathfrak{c}\mathfrak{c}^+ D_{K/F}.$$  

We will identify the CM-type $\Sigma \subset \text{Hom}(\mathcal{K}, \mathcal{C})$ with the set $\text{Hom}(\mathcal{F}, \mathbb{R})$ of archimedean places of $\mathcal{F}$ by the restriction map. Let $K^0_\Sigma := \prod_{\sigma \in \Sigma} \text{SO}(2, \mathbb{R})$ be a maximal compact subgroup of $\text{GL}_2(\mathcal{F} \otimes \mathbb{Q})$. We put

$$\chi^* = \chi|_{\mathfrak{A}_{\mathfrak{c}}^{-1}} \text{ and } \chi_+ = \chi|_{\mathfrak{A}_{\mathfrak{c}}^+}.$$
For $s \in \mathbb{C}$, we let $I(s, \chi_+)$ denote the space consisting of smooth and $K_{\infty}^0$-finite functions $\phi : GL_2(\mathbb{A}_F) \to \mathbb{C}$ such that

$$\phi \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) g = \chi_+^{-1}(d) \left| \frac{a}{d} \right|^s \phi(g).$$

Conventionally, functions in $I(s, \chi_+)$ are called sections. Let $B$ be the upper triangular subgroup of $GL_2$. The adelic Eisenstein series associated to a section $\phi \in I(s, \chi_+)$ is defined by

$$E_\mathcal{A}(g, \phi) = \sum_{\gamma \in B(\mathcal{F}) \backslash GL_2(\mathcal{F})} \phi(\gamma g).$$

It is known that the series $E_\mathcal{A}(g, \phi)$ is absolutely convergent for $Re(s) > 0$.

2.2. Fourier coefficients of Eisenstein series. Let $\psi = \prod_{v} \psi_v : A_F/F \to \mathbb{C}^\times$ be the standard additive character such that $\psi_v(x) = \exp(2\pi i T_{\mathcal{F}_v/Q}(x))$ for $x \in F \otimes_{\mathbb{Q}} \mathbb{R}$. Put $w = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$. Let $v$ be a place of $\mathcal{F}$ and let $I_v(s, \chi_+)$ be the local constituent of $I(s, \chi_+)$ at $v$. For $\phi_v \in I_v(s, \chi_+) = \beta \in F_v$, we recall that the $\beta$-th local Whittaker integral $W_\beta(\phi_v, g_v)$ is defined by

$$W_\beta(\phi_v, g_v) = \int_{F_v} \phi_v(w \left[ \begin{array}{cc} 1 & x_v \\ 0 & 1 \end{array} \right] g_v) \psi(-\beta x_v) dx_v,$$

and the intertwining operator $M_w$ is defined by

$$M_w(\phi_v(g_v)) = \int_{F_v} \phi_v(w \left[ \begin{array}{cc} 1 & x_v \\ 0 & 1 \end{array} \right] g_v) dx_v.$$

Here $dx_v$ is Lebesgue measure if $F_v = \mathbb{R}$ and is the Haar measure on $F_v$ normalized so that $\text{vol}(O_{F_v}, dx_v) = 1$ if $F_v$ is non-archimedean. By definition, $M_w(\phi_v(g_v))$ is the $0$-th local Whittaker integral. It is well known that local Whittaker integrals converge absolutely for $Re(s) > 0$, and have meromorphic continuation to all $s \in \mathbb{C}$.

If $\phi = \otimes_v \phi_v$ is a decomposable section, then $E_\mathcal{A}(g, \phi)$ has the following Fourier expansion:

$$E_\mathcal{A}(g, \phi) = \phi(g) + \sum_{\beta \in \mathcal{F}} W_\beta(E_\mathcal{A}, g),$$

where

$$E_\mathcal{A}(g, \phi) = \int_{\mathcal{F}} \phi(g, v) dx_v.$$

2.3. The choice of the local sections. We briefly recall the choice of local sections in [Hsi12, §4.3]. We begin with some notation. Let $v$ be a place of $\mathcal{F}$. Let $F = F_v$ (resp. $E = K \otimes_{\mathcal{F}} F_v$). Denote by $z \mapsto \bar{z}$ the complex conjugation. Let $| \cdot |$ be the standard absolute values on $F$ and let $| \cdot |_E$ be the absolute value on $E$ given by $|z|_E := |z\bar{z}|$. Let $d_F = d_{F_v}$ be a fixed generator of the different $D_{\mathcal{F}}$ of $\mathcal{F}/\mathcal{Q}$. Write $\chi$ (resp. $\chi_v$) for $\chi_v$ (resp. $\chi_{v, \nu}$). If $v \in \mathfrak{h}$, denote by $\varpi_v$ a uniformizer of $F_v$. For a set $Y$, denote by $1_Y$ the characteristic function of $Y$.

Case I: $v \nmid \mathbb{C}^* D_{\mathcal{F}}/\mathcal{F}$. We first suppose that $v = \sigma \in \Sigma$ is archimedean and $F = \mathbb{R}$. For $g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in GL_2(\mathbb{R})$, we put $J(g, i) := ci + d$. Define the sections $\phi_{k, s, \sigma}$ of weight $k$ in $I_v(s, \chi_+)$ by

$$\phi_{k, s, \sigma}(g) = J(g, i)^{-k} |\det(g)|^s \cdot |J(g, i)(g, i)|^{-s}.$$

Suppose that $v$ is non-archimedean. Denote by $S(F)$ (resp. $(S(F \oplus F)$) the space of Bruhat-Schwartz functions on $F$ (resp. $F \oplus F$). Recall that the Fourier transform $\widehat{\varphi}$ for $\varphi \in S(F)$ is defined by

$$\widehat{\varphi}(y) = \int_F \varphi(x) \psi(xy) dx.$$

For a character $\mu : F^\times \to \mathbb{C}^\times$, we define a function $\varphi_\mu \in S(F)$ by

$$\varphi_\mu(x) = \|D_F^\times(x)\mu(x).$$

If $v \mid p\mathfrak{p}_\nu^\infty$ is split in $K$, write $v = w\mathfrak{p}$ with $w|\mathfrak{p}\Sigma_p$, and set

$$\varphi_w = \varphi_{\chi_w} \text{ and } \varphi_\mathfrak{p} = \varphi_{\chi_\mathfrak{p}}^{-1}.$$
To a Bruhat-Schwartz function $\Phi \in \mathcal{S}(F \oplus F)$, we can associate a Godement section $f_{\Phi,s} \in I_v(s, \chi_+)$ defined by

$$f_{\Phi,s}(g) := |\det g|^s \int_{F^x} \Phi((0, x)g) \chi_+(x) |x|^{2s} d^x x,$$

where $d^x x$ is the Haar measure on $F^x$ such that $\text{vol}(O_F^x, d^x x) = 1$. Define Godement sections by

$$\phi_{\chi,s,v} = f_{\Phi_{\chi,s,v}},$$

where $\Phi_{\chi,s,v} = \begin{cases} \|O_F(x)\|_{d_F^{-1}O_F}(y) & \cdots \nu \in \mathcal{D}, \\ \phi_{\chi}(x)\tilde{\phi}_w(y) & \cdots \nu \in \mathcal{D}^c. \end{cases}$

Let $u \in O_F^x$. Let $\phi^\infty_w$ and $\phi^{[u]}_w \in \mathcal{S}(F)$ be the Bruhat-Schwartz functions defined by

$$\phi^\infty_w(x) = \mathbb{I}_{1 + \pi_v O_F}(x) \chi^{-1}_w(x)$$

and

$$\phi^{[u]}_w(x) = \mathbb{I}_{u(1 + \pi_v O_F)(x)} \chi_+(x).$$

Define $\Phi^{[u]}_w \in \mathcal{S}(F \oplus F)$ by

$$\phi^{[u]}_w(x, y) = \frac{1}{\mathcal{V}(1 + \pi_v O_F, d^x x)} \mathcal{V}^1_{\infty}(x) \tilde{\phi}^{[u]}_w(y) = (|\pi_v|^{-1} - 1) \phi^\infty_w(x) \tilde{\phi}^{[u]}_w(y).$$

Case II: $v | D_{\mathcal{K}/F} C\mathcal{F}$. In this case, $E$ is a field. We define an embedding $\rho : E \hookrightarrow M_2(F)$ by

$$a + b \sigma \mapsto \rho(a + b \sigma) = \begin{bmatrix} a & b \\ \overline{b} & a \end{bmatrix}.$$ 

Then $\text{GL}_2(F) = B(F) \rho(E^\times)$. We fix a $O_F$-basis $\{1, \theta_v\}$ of $O_F$ such that $\theta_v$ is a uniformizer if $v$ is ramified and $\overline{\theta}_v = -\theta_v$ if $v \mid 2$. Let $t_v = \theta_v + \overline{\theta}_v$ and put

$$s_v = \begin{bmatrix} d_{F_v} & -2^{-1} t_v \\ 0 & d_{F_v} \overline{t}_v \end{bmatrix}.$$ 

Let $\phi_{\chi,s,v}$ be the smooth section in $I_v(s, \chi_+)$ defined by

$$\phi_{\chi,s,v}(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) \rho(z) s_v) = L(s, \chi_v) \cdot \chi^{-1}_v(d) \left| \begin{bmatrix} a \\ d \end{bmatrix} \right|^s \cdot \chi^{-1}(z) \quad (b \in B(F), \ z \in E^\times).$$

Here $L(s, \chi_v)$ is the local Euler factor of $\chi_v$.

2.4. Fourier expansion of normalized Eisenstein series. Let $U_v$ be the torsion subgroup of $O_{F_v}^\times$. For $u \in \mathfrak{u}_v \in U_v$, let $\Phi^\infty_u = \otimes_{v \mid p} \Phi^{[u]}_w$ be the Bruhat-Schwartz function defined in (2.4). Define the section $\phi^\infty_{\chi,s,v} = I(s, \chi_+)$ by

$$\phi^\infty_{\chi,s,v} = \bigotimes_{v \mid p} \phi^\infty_{\chi,s,v} \otimes \phi^\infty_{\chi,s,v} f_{\Phi^{[u]}_w}.$$ 

We put

$$X^+ = \{ \tau = (\tau_\sigma)_{\sigma \in \Sigma} \in C^\Sigma \mid \text{Im} \tau_\sigma > 0 \text{ for all } \sigma \in \Sigma \}.$$ 

The holomorphic Eisenstein series $E^h_{\chi,u}(\tau, g_f) : X^+ \times \text{GL}_2(A_{F,f}) \to \mathbb{C}$ is defined by

$$E^h_{\chi,u}(\tau, g_f) := \Gamma_{\Sigma}(k_{\sigma}) \sqrt{|D_F|_{\mathbb{R}}(2\pi)^k |k_{\Sigma}|} \cdot A \left( (g_\infty, g_f), \phi^h_{\chi,s,v}(\Phi^{[u]}_w) \right) |u|_\mathfrak{u} \cdot \prod_{\sigma \in \Sigma} J(g_\sigma, i)^k,$$

where $(g_\infty, g_f) = \text{GL}_2(F \otimes \mathbb{Q})$, $(g_\sigma)_{\sigma \in \Sigma} = (\tau_\sigma)_{\sigma \in \Sigma}.$

Let $c = (c_v) \in A_{F,f}^\times$ such that $c_v = 1$ at $v \mid \mathcal{D}$ and let $c = c(O_F \otimes \mathbb{Z})c$. Define a function $E^h_{\chi,u,c} : X^+ \to \mathbb{C}$ by $E^h_{\chi,u,c}(\tau) := E^h_{\chi,u}(\tau, \begin{bmatrix} 1 & 0 \\ 0 & c^{-1} \end{bmatrix})$. Then $E^h_{\chi,u,c}$ is a $c$-Hilbert modular form of weight $k \Sigma$ defined over $\mathbb{C}$ in the sense of $[\text{Kat78}, \text{p.211}]$. 


Proposition 2.1. The $q$-expansion of $E^h_{\chi,\vec{u}}|_{c}$ at the cusp $(O, c^{-1})$ is given by

$$E^h_{\chi,\vec{u}}|_{c}(q) = \sum_{\beta \in \mathcal{F}_+} a_{\beta}(E^h_{\chi,\vec{u}}, \vec{c}) \cdot q^\beta.$$ 

The $\beta$-th Fourier coefficient $a_{\beta}(E^h_{\chi,\vec{u}}, \vec{c})$ is given by

$$a_{\beta}(E^h_{\chi,\vec{u}}, \vec{c}) = \frac{k-1}{\hat{\beta}} \prod_{\omega_1 \in \Sigma} \chi_{\omega_1}(\hat{\beta}) \prod_{\omega_2 \in \Pi} \chi_{\omega_2}(\hat{\beta}) \chi_{\omega}(\hat{\beta}) \chi_{\omega}(\hat{\beta})$$

where

$$\hat{\beta} = \lim_{n \to \infty} \int_{\mathcal{D}_\chi} \chi_{\omega}^{-1}(x) \psi(-d_{\mathcal{F}}^{-1}(\beta x_{\omega})) dx_{\omega}$$

(2.7)

Proof. This follows from (2.1) and the calculations of local Whittaker integrals of special local sections in [Hsi11, §4.3] (cf. [Hsi12, Prop. 4.1 and Prop. 4.4]).

2.5. The $\mu$-invariants of anticyclotomic $p$-adic $L$-functions. Let $Z(\mathcal{C})^{-}$ be the anticyclotomic quotient of $Z(\mathcal{C})$. Let $\hat{\mathcal{O}}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}} \otimes \mathcal{Z} \mathcal{Z}$ and $U(\mathcal{C}p^n) = \{ u \in \hat{\mathcal{O}}_{\mathcal{C}}^\times \mid u \equiv 1 \pmod{\mathcal{C}p^n} \}$. The reciprocity law $\text{rec}_\mathcal{C} : A^\times_{K,f} \to Z(\mathcal{C})^{-}$ induces the isomorphism:

$$\text{rec}_\mathcal{C} : \lim_{\pi \to \infty} \chi_{\mathcal{C}}^{-1}(x + \theta) \psi(-d_{\mathcal{F}}^{-1}(\beta x_{\omega})) dx_{\omega} = 0$$

Let $\Gamma^{-}$ be the maximal $\mathcal{Z}_p$-free quotient of $Z(\mathcal{C})^{-}$. Each function $\phi$ on $\Gamma^{-}$ will be regarded as a function on $Z(\mathcal{C})$ by the natural projection $\pi_- : Z(\mathcal{C}) \to Z(\mathcal{C})^{-} \to \Gamma^{-}$. The anticyclotomic projection $\mathcal{L}_\chi^{-} \mathcal{L}$ of the measure $\mathcal{L}_\psi$ is defined by

$$\mathcal{L}_\psi := \int_{\mathcal{D}_\chi} \hat{\chi}_{\psi} d\mathcal{L}_\psi.$$

Recall the $\mu$-invariant $\mu(\psi)$ of a $\mathcal{Z}_p$-valued $p$-adic measure $\psi$ on a $p$-adic group $H$ is defined to be

$$\mu(\psi) = \inf_{U \subseteq H \text{ open}} v_p(\psi(U)).$$

We shall give a formula of the $\mu$-invariant $\mu_\psi^{-}$ of $\mathcal{L}_\psi^{-}$ in terms of $p$-adic valuation of Fourier coefficients of $E_{\chi,\vec{u}}$. To state the formula precisely, we introduce some notation.

Let $\mathcal{C}_{\mathcal{F}} := \mathcal{C}_{\mathcal{F}}^\times / \mathcal{A}_{\mathcal{F},f} \otimes \hat{\mathcal{O}}_{\mathcal{C}}$ and let $\mathcal{C}_{\mathcal{F}}$ be the subgroup of $\mathcal{C}_{\mathcal{F}}$ generated by ramified primes. Let $O_{\mathcal{F}} := \mathcal{O}_{\mathcal{F}} \otimes \mathcal{Z}$. Let $\Gamma'$ be the open subgroup of $\Gamma^{-}$ generated by the image of $O_{\mathcal{F}}^{\times} \times \prod_{\mathcal{V}_{\mathcal{D}_{\mathcal{F}}}} \mathcal{O}^\times_{\mathcal{K}_u}$ via $\text{rec}_\mathcal{C}$. The reciprocity law $\text{rec}_\mathcal{C}$ at $\Sigma_p$ induces an injective map $\text{rec}_{\mathcal{C}} : 1 + \mathcal{P} O_{\mathcal{F}}^{-} \to O_{\mathcal{F}}^{\times} \otimes \Sigma_{\mathcal{D}_{\mathcal{F}}} \mathcal{O}^\times_{\mathcal{K}_u} \text{rec}_\mathcal{C} Z(\mathcal{C})^{-}$ with finite cokernel as $p \nmid D_{\mathcal{F}}$, and it is easy to see that $\text{rec}_{\mathcal{C}}$ induces an isomorphism $\text{rec}_{\mathcal{C}}^{-} : 1 + \mathcal{P} O_{\mathcal{F}}^{-} \to \Gamma'$. We thus identify $\Gamma'$ with the subgroup $\text{rec}_{\mathcal{C}}^{-} (1 + \mathcal{P} O_{\mathcal{F}}^{-})$ of $Z(\mathcal{C})^{-}$. Let $Z' := \pi^{-1} (\Gamma')$ be the subgroup of $Z(\mathcal{C})$ and let $\mathcal{C}_{\mathcal{F}} \cap \mathcal{C}_{\mathcal{F}}^{\mathcal{L}}$ be the image of $Z'$ in $\mathcal{C}_{\mathcal{F}}$ and let $\mathcal{D}_1 : \mathcal{D}_1^\prime$ be a set of representatives of $\mathcal{C}_{\mathcal{F}}^{\mathcal{L}} / \mathcal{C}_{\mathcal{F}}^{\mathcal{L}}$ (resp. $\mathcal{C}_{\mathcal{F}} / \mathcal{C}_{\mathcal{F}}^{\mathcal{L}}$) in $(\mathcal{A}_{\mathcal{K},f}^{\mathcal{D}})^{\times}$. Let $\mathcal{D}_1 := \mathcal{D}_1^\prime \mathcal{D}_1^\prime$ be a set of representatives of $\mathcal{C}_{\mathcal{F}} / \mathcal{C}_{\mathcal{F}}^{\mathcal{L}}$. Let $U_{\mathcal{F}}$ be the torsion subgroup of $(\mathcal{O}_{\mathcal{F}} \otimes \mathcal{Z})^{\times}$ and let $U_{\mathcal{F}} := \mathcal{K}_{\mathcal{F}} \otimes (\mathcal{K}^{\times})^{1-\epsilon}$. Let $D_0$ be a set of representatives of $U_{\mathcal{F}} / U_{\mathcal{F}}^{\mathcal{L}}$ in $U_{\mathcal{F}}$. For $a \in \mathcal{A}_{\mathcal{K},f}$, let $c(a) := c(\mathcal{O}_{\mathcal{K}}) N_{\mathcal{K}/K}(a)$, where $a = a(\mathcal{O}_{\mathcal{K}} \otimes \mathcal{Z}) \subset \mathcal{K}$. The following theorem is proved by the ideas of Hida in [Hid10].

Theorem 2.2 (Thm. 5.5[Hsi12]). Suppose that $p \nmid D_{\mathcal{F}}$. Then we have

$$\mu_\psi^{-} = \inf_{(u,a) \in \mathcal{D}_0 \times \mathcal{D}_1^\prime} v_p(a_{\beta}(E^h_{\chi,\vec{u}}, \vec{c}(a))).$$
Proof. For the convenience of the readers, we sketch the proof here. For each \( b \in D'_1 \), we denote by \( \mathcal{L}^b_{\chi, \Sigma} \) the \( p \)-adic measure on \( 1 + pO_p \cong \Gamma' \) obtained by the restriction of \( \mathcal{L}_{\chi, \Sigma}^{-} \) to \( b.\Gamma' := \pi_-(\text{rec}_K(b))\Gamma' \). To be precise, we have

\[
\int_{\Gamma'} \phi d\mathcal{L}_{\chi, \Sigma}^b := \int_{\Gamma'} \mathbb{I}_{b.\Gamma'}(\phi)[|b^{-1}|]d\mathcal{L}_{\chi, \Sigma}^{-}
\]

where \( \mathbb{I}_{b.\Gamma'} \) is the characteristic functions of \( b.\Gamma' \). Let \( \mu_{\chi, \Sigma}^b \) be the \( \mu \)-invariant of the \( p \)-adic measures \( \mathcal{L}_{\chi, \Sigma}^b \).

Note that \( \Gamma' = \bigsqcup_{b \in D'_1} b.\Gamma' \), so it is clear that

\[
\mathcal{L}_{\chi, \Sigma}^- = \inf_{b \in D'_1} \mu_{\chi, \Sigma}^b.
\]

For \( (u, a) \in D_0 \times D_1 \), we let \( \mathcal{E}_{u,a} \) be the \( p \)-adic avatar of \( E^b_{\chi, u}(\lambda(a)) \) ([Hsi12, §2.5.5]). Let \( t \) be the Serre-Tate coordinate of the CM point \( x \) with the polarization ideal \( \mathfrak{c}(O_K) \) defined in [Hsi12, §5.2]. For \( a \in D'_1 \), let \( \langle a \rangle_{\Sigma} \) be the unique element in \( 1 + pO_p \) such that \( \text{rec}_{E_p}(\langle a \rangle_{\Sigma}) = \pi_-(\text{rec}_K(a)) \in \Gamma' \). For each \( b \in D'_1 \), we define a \( t \)-expansion \( \mathcal{E}^b(t) \) by

\[
\mathcal{E}^b(t) := \#U_{\text{alg}}, \sum_{(u, a) \in D_0 \times D'_1} \chi(ab^{-1})\mathcal{E}_{u,a}[[a][t^{(v_0-1)}]_{u^{-1}}],
\]

where \( [[a]] \) is the Hecke action induced by \( a \) (See [Hsi12, Remark 4.5]). With the help of an an explicit formula of toric period integral of Eisenstein series ([Hsi11, Prop. 5.1] and [Hsi12, Prop. 4.9]), it is shown in [Hsi12, Prop. 5.2] that \( \mathcal{E}^b(t) \) essentially gives rise to the \( t \)-expansion of the measure \( \mathcal{L}_{\chi, \Sigma}^b \), and hence we find that

\[
\mu_{\chi, \Sigma}^b = \inf \left\{ r \in \mathbb{Q}_{\geq 0} \mid p^{-r}\mathcal{E}^b(t) \equiv 0 \pmod{m_{\mathbf{Z}_p}} \right\},
\]

where \( m_{\mathbf{Z}_p} \) is the maximal ideal of \( \mathbf{Z}_p \). By the linear independence of \( p \)-adic modular forms modulo \( p \) [Hid10, Cor. 3.2], the \( q \)-expansion principle of \( p \)-adic modular forms combined with [Hsi12, Lemma 5.3], we can conclude from (2.8) and (2.9) that

\[
\mu_{\chi, \Sigma}^- = \inf_{b \in D'_1} \mu_{\chi, \Sigma}^b = \inf_{(u, a) \in D_0 \times D'_1} v_p(\mathbf{a}_\beta(E_{\chi, u}^b, \lambda(a))). \tag{2.9}
\]

3. Proof of Theorem A

We go back to our setting in the introduction. Let \( \lambda \) be a Hecke character of infinity type \( k \Sigma \) with \( k \geq 1 \) and let \( \lambda^* := \lambda|_{A_{K}^\times}^{-\frac{1}{2}} \). We may further assume that

\( \mathcal{C} \) is the prime-to-\( p \) conductor of \( \lambda \).

To prove Theorem A, we prepare two lemmas. The first lemma is taken from [Hid11].

**Lemma 3.1.** Let \( w \mid p \) be a place of \( K \) and let \( \omega_w \) be a uniformizer of \( K_w \). Let \( a \in \mathbf{Z}_p \). Given \( e > 0 \), we have

\[ v_p(a + \nu(\omega_w)) < e \quad \text{for all but finitely many } \nu \in \mathbf{X}^+. \]

**Proof.** We note that \( \nu(\omega_w) \) is a primitive \( p^n \)-th root of unity for some \( n \in \mathbf{Z}_{\geq 0} \), and for sufficiently large \( n \), we have

\[ v_p(a + \nu(\omega_w)) \leq v_p(\nu(\omega_w) - 1) = \frac{1}{p^n - p^{n-1}} < e. \]

The first equality holds precisely when \( v_p(a + 1) > 0 \). Therefore, it is not difficult to deduce the lemma from the fact that the image of \( \omega_w \) in \( \Gamma^+ \) under \( \text{rec}_K : A_{K}^\times \rightarrow \Gamma^+ \) generates a subgroup of \( \Gamma^+ \) with finite index.

**Lemma 3.2.** Let \( v \mid \mathcal{C} \mid D_{K/F} \) and let \( e_1 > 0 \) be a positive number. Then there exists \( \beta_v \in \mathcal{F}^\times_v \) such that for almost all \( \nu \in \mathbf{X}^+ \), we have

\[ v_p(L(0, \lambda_v \nu_v) \hat{A}_{\beta_v}(\lambda_v \nu_v)) \leq e_1. \]

Here "for almost all" means "for all but finitely many".
Proof. Let $E = K_v$ and $F = F_v$. Let $m_E$ be the maximal ideal of $\mathcal{O}_E$. For brevity, we drop the subscript $v$ and simply write $\lambda = \lambda_v$, $\nu = \nu_v$. Let $a(\lambda) := \inf\{n \in \mathbb{Z}_{\geq 0} \mid \lambda(1 + m_E^n) = 1\}$ be the conductor of $\lambda$. Suppose that $a(\lambda) > 1$. Then $\lambda(1 + m) \neq 1$, and the invariant $\mu_p(\lambda) := \inf_{x \in E} v_p(\lambda(x) - 1) = 0$ as $v \nmid p$. It follows from [Hsi11, Lemma 6.4] that there exists $\beta$ such that $v_p(\tilde{A}_\beta(\lambda)) = 0$. Moreover, since $\tilde{A}_\beta(\lambda
u) \equiv \tilde{A}_\beta(\lambda) \pmod{m_E}$, we find that $v_p(\tilde{A}_\beta(\lambda
u)) = 0$ for all $\nu \in \mathbb{X}^+$. To prove the remaining part, we assume the conductor $a(\lambda) = a(\lambda\nu) \leq 1$. In virtue of Lemma 3.1, it suffices to show that there exists $\beta$ such that
\begin{equation}
 v_p(\tilde{A}_\beta(\lambda\nu)) = v_p(a + b \cdot \nu(\varpi_w))
\end{equation}
for some $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^*$ independent of $\nu$ and a uniformizer $\varpi_w$ of $E$.

Let $\varpi$ be a uniformizer of $F$. Suppose that $v$ ramified. Recall that $\theta = \theta_v$ is chosen to be a uniformizer of $E$. Let $\beta \in \varpi^{-1}\mathcal{O}_F^\times$, so $\nu(\beta) = -1$. If $v \nmid \mathfrak{c}^-$, then by [Hsi11, Lemma 4.1], we have
\[ \tilde{A}_\beta(\lambda\nu) = |\mathcal{D}_F|^{-1} \lambda^{-1}\nu^{-1}(\theta) |\varpi|. \]
If $v \mid \mathfrak{c}^-$, then it follows from [Hsi11, Prop. 4.4 (1)] that
\[ \tilde{A}_\beta(\lambda\nu) = \lambda^*(\theta^{-1}) |\varpi|^{1/2} \nu(\theta^{-1}) + \lambda^*(\beta d_F^{-1}) \nu(-\beta \varpi) \epsilon(1, \lambda_+ |\cdot|^{-1}, \psi) \quad (\lambda_+ := \lambda|_{\mathcal{O}_F^\times}) \]
\[ = \lambda^*(\theta^{-1}) |\varpi|^{1/2} \nu(\theta^{-1}) + \lambda^*(\beta d_F^{-1}) \epsilon(1, \lambda_+ |\cdot|^{-1}, \psi). \]
Here $\epsilon(s, \lambda_+ |\cdot|^{-1}, \psi)$ is the Tate’s epsilon factor attached to the additive character $\psi_v : F \rightarrow \mathbb{C}^\times$. In any case, it is clear that (3.1) holds for $\beta \in \varpi^{-1}\mathcal{O}_F$ when $v$ is ramified.

Suppose that $v$ is inert. Then $a(\lambda\nu) = 1$. Let $\beta \in \mathfrak{O}_F^\times$ (so $\nu(\beta) = 0$). By [Hsi11, Prop. 4.5], if $|\lambda|_{\mathcal{O}_F^\times} = 1$, then
\[ \tilde{A}_\beta(\lambda\nu) = -|\varpi|(-1 + \lambda^* \nu(\varpi)), \]
and if $|\lambda|_{\mathcal{O}_F^\times}$ is non-trivial, then
\[ \tilde{A}_\beta(\lambda\nu) = \mathcal{I}_{\lambda\nu}(0) + \lambda^*(-\beta d_F^{-1}) \nu(\varpi) \epsilon(1, \lambda_+ |\cdot|^{-1}, \psi), \]
where
\[ \mathcal{I}_{\lambda\nu}(0) = \int_{\mathcal{O}_F} \lambda^{-1}\nu^{-1}(x + \theta) dx. \]
Recall that $\theta$ is chosen such that $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F \theta$. We have $x + \theta \in \mathcal{O}_E^\times$ for $x \in \mathcal{O}_F$. As $\nu$ is unramified at $v$, we find that
\[ \mathcal{I}_{\lambda\nu}(0) = \int_{\mathcal{O}_F} \lambda^{-1}(x + \theta) dx \]
is independent of $\nu$. Therefore, in either cases, (3.1) holds for $\beta \in \mathcal{O}_F^\times$. \qed

Theorem 3.3. Suppose that $p \nmid D_F$. Then
\[ \mu_{\lambda, \Sigma} = 0. \]

Proof. Let $S^-$ be the set of prime factors of $\mathfrak{c}^- D_{K/F}$. By Lemma 3.2, for $e_1 > 0$, we choose $(\beta_v) \in \prod_{v | \mathfrak{c}^- D_{K/F}} \mathcal{F}_v^\times$ such that
\begin{equation}
\sum_{v | \mathfrak{c}^- D_{K/F}} v_p(L(0, \lambda_v \nu_v) \tilde{A}_{\beta_v}(\lambda_v \nu_v)) \leq \#(S^-) \cdot e_1
\end{equation}
for almost $\nu \in \mathbb{X}^-$. Let $c \in A_{K,v}^\times$ and $\mathfrak{c}$ be the associated ideal as in Prop. 2.1. We define an idele $\eta \in A_{K,v}^\times$ such that
- $\eta_v = \beta_v^{-1}$ for all $v | \mathfrak{c}^- D_{K/F}$,
- $\eta_v = c_v$ for all finite $v \nmid \mathcal{D}$,
- $\eta_v = 1$ for the remaining places $v$. 


Let $U = \prod U_v$ be an open subgroup of $\mathbb{A}_F^\times$ such that $U_v = \mathcal{O}_F^\times$ at all $v \mid \mathcal{C} - D_{K/F}$ and $U_{\infty} = (\mathcal{F} \otimes \mathbb{R})^+$. Moreover, it is not difficult to see from [Hsi11, (4.17)] that for $v \mid \mathcal{C} - D_{K/F}$, $U_v$ can be chosen small enough, depending on $\lambda_v$ and $\beta_v$, so that

\[ \tilde{A}_{\beta_v}(\lambda_v, \nu_v) = \tilde{A}_{\beta_v}(\nu_v) \quad \text{for all} \quad u \in U_v \quad \text{and} \quad \nu \in \mathfrak{X}^+. \]

Consider the idele class $\mathcal{F}^* \eta U$ in $\mathbb{A}_F^\times$. We may choose a uniformizer $\varpi_{v_0} \in \mathcal{K}_{v_0}$ with a finite place $v_0 \mid D$ such that $\varpi_{v_0}$ lies in the class $\mathcal{F}^* \eta U$. We can write

\[ \varpi_{v_0} \in \beta \eta U \quad \text{for some} \quad \beta \in \mathcal{F}^*. \]

Since $\eta = 1$ when $v$ is archimedean or $v | p$, we find that $\beta \in \mathcal{F}^* \cap \mathcal{O}_{\mathcal{F}, (p)}$ by the choice of $U$. Let $u \in U_p$ such that $\beta \equiv u \pmod{p}$. By Prop. 2.1 we have

\[
v_p(\mathfrak{a}_\beta (\mathbb{E}_{\lambda_v, u}, \mathfrak{c})) = \sum_{v \mid \mathcal{C} - D_{K/F}} v_p(L(0, \lambda_v, \nu_v) \tilde{A}_{\beta_v}(\lambda_v, \nu_v)) + \sum_{v \not\mid \mathcal{C} - D_{K/F}} v_p(\prod_{i=0}^{v(\beta_{i, v})} \lambda^i \nu(\varpi_{v_i})) \]

\[ = \sum_{v \mid \mathcal{C} - D_{K/F}} v_p(L(0, \lambda_v, \nu_v) \tilde{A}_{\beta_v}(\lambda_v, \nu_v)) + v_p(\lambda^* \nu(\varpi_{v_0}) + 1). \]

It follows that for almost all $\nu \in \mathfrak{X}^+$, we have

\[ v_p(\mathfrak{a}_\beta (\mathbb{E}_{\lambda_v, u}, \mathfrak{c})) \leq \#(S^-) \cdot e_1 + v_p(\lambda^* \nu(\varpi_{v_0}) \cdot \nu(\varpi_{v_0}) + 1). \]

Hence, from Theorem 2.2 and Lemma 3.1 we deduce that

\[ 0 \leq \mu_{\lambda, \Sigma} \leq \liminf \mathfrak{e}_{\lambda_v, \Sigma} \leq \liminf v_p(\mathfrak{a}_\beta (\mathbb{E}_{\lambda_v, u}, \mathfrak{c})) \leq \#(S^-) \cdot e_1. \]

This inequality holds for all $e_1 > 0$, so $\mu_{\lambda, \Sigma} = 0$.

\[ \square \]

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