THE VANISHING OF μ-INVARIANT OF p-adic HECKE L-FUNCTIONS FOR CM FIELDS

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ABSTRACT. Let $p > 2$ be an ordinary prime for a CM field $\mathcal{K}$. Katz and Hida-Tilouine constructed the p-adic Hecke L-function attached to a p-ordinary CM type and a branch character. In this note, we prove that the μ-invariant of this p-adic Hecke L-function always vanishes when p is unramified in $\mathcal{K}$.

1. INTRODUCTION

The purpose of this note is to prove the vanishing of the μ-invariant of p-adic Hecke L-functions for CM fields constructed by Katz and Hida-Tilouine. We let $\mathcal{F}$ be a totally real field of degree d over $\mathbf{Q}$ and $\mathcal{K}$ be a totally imaginary quadratic extension of $\mathcal{F}$. Let $D_{\mathcal{F}}$ (resp. $\mathcal{D}_{\mathcal{F}}$) be the discriminant (resp. different) of $\mathcal{F}/\mathbf{Q}$. Let $p > 2$ be an odd rational prime. Fix two embeddings $\iota_\infty \colon \bar{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p \colon \bar{\mathbf{Q}} \to \mathbf{C}_p$ once and for all. Let $\bar{\mathbf{Z}}$ be the ring of algebraic integers and let $\bar{\mathbf{Z}}_p$ be the p-adic completion of $\iota_p(\bar{\mathbf{Z}})$ in $\mathbf{C}_p$. Denote by c the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(\mathcal{K}/\mathcal{F})$. We assume the following hypothesis throughout this article:

(ord) Every prime of $\mathcal{F}$ above p splits in $\mathcal{K}$.

Fix a p-ordinary CM type Σ, namely Σ is a CM type of $\mathcal{K}$ such that p-adic places induced by elements in Σ via ι_p are disjoint from those induced by elements in Σc. The existence of such Σ is assured by our assumption (ord). Let $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ be the relative different of $\mathcal{K}/\mathcal{F}$. Let $\mathfrak{C}$ be a prime-to-p integral ideal of $\mathcal{O}_{\mathcal{K}}$ and let $\vartheta \in \mathcal{K}$ such that

(d1) $c(\vartheta) = -\vartheta$ and $\operatorname{Im}\sigma(\vartheta) > 0$ for all $\sigma \in \Sigma$,
(d2) $\mathfrak{c}(\mathcal{O}_{\mathcal{K}}) := \mathcal{D}_{\mathcal{F}}^{-1}(2\vartheta\mathcal{D}_{\mathcal{K}/\mathcal{F}}^{-1})$ is prime to $p\mathfrak{C}\mathfrak{C}^c D_{\mathcal{K}/\mathcal{F}}$.

Let $\mathcal{K}_\infty^+$ and $\mathcal{K}_\infty^-$ be the cyclotomic $\mathbf{Z}_p$-extension and anticyclotomic $\mathbf{Z}_p^d$-extension of $\mathcal{K}$. Let $\mathcal{K}_\infty = \mathcal{K}_\infty^+\mathcal{K}_\infty^-$ be a $\mathbf{Z}_p^{d+1}$-extension of $\mathcal{K}$. If one assumes Leopoldt's conjecture for $\mathcal{K}$, then $\mathcal{K}_\infty$ is the maximal $\mathbf{Z}_p^{d+1}$-extension of $\mathcal{K}$. Let $\Gamma^\pm := \mathrm{Gal}(\mathcal{K}_\infty^\pm/\mathcal{K})$ and let $\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) \simeq \Gamma^+\times\Gamma^-$. Let $Z(\mathfrak{C})$ be the ray class group of $\mathcal{K}$ modulo $\mathfrak{C}p^\infty$. In [Kat78] and [HT93], a $\bar{\mathbf{Z}}_p$-valued p-adic measure $\mathcal{L}_{\mathfrak{C},\Sigma}$ on $Z(\mathfrak{C})$ is constructed such that

$$\begin{aligned}\frac{1}{\Omega_p^{k\Sigma+2\kappa}} \cdot \int_{Z(\mathfrak{C})} \widehat{\lambda} d\,\mathcal{L}_{\mathfrak{C},\Sigma} =& L^{(p\mathfrak{C})}(0,\lambda) \cdot Eul_p(\lambda) Eul_{\mathfrak{C}^+}(\lambda) \\ &\times \frac{\pi^\kappa \Gamma_\Sigma(k\Sigma+\kappa)}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}(\operatorname{Im}\vartheta)^\kappa \cdot \Omega_\infty^{k\Sigma+2\kappa}} \cdot [\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_{\mathcal{F}}^\times],\end{aligned}$$

where (i) λ is a Hecke character modulo $\mathfrak{C}p^\infty$ of infinity type $k\Sigma + \kappa(1-c)$ with either $k \geq 1$ and $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma]$ or $k \leq 1$ and $k\Sigma + \kappa \in \mathbf{Z}_{>0}[\Sigma]$, and $\widehat{\lambda}$ is the p-adic avatar of λ regarded as a p-adic Galois character via geometrically normalized reciprocity law, (ii) $Eul_p(\lambda)$ and $Eul_{\mathfrak{C}^+}(\lambda)$ are certain modified Euler factors (For the definitions, see [Hsi12, (4.16)]).

We fix a Hecke character λ of infinity type $k\Sigma$, $k \geq 1$. Let $\mathcal{L}_{\lambda,\Sigma}$ be the p-adic measure on Γ obtained by the pull-back of $\mathcal{L}_{\mathfrak{C},\Sigma}$ along λ. In other words, for every locally constant function φ on Γ, we have

$$\int_\Gamma \varphi d\mathcal{L}_{\lambda,\Sigma} = \int_{Z(\mathfrak{C})} \varphi\widehat{\lambda} d\,\mathcal{L}_{\mathfrak{C},\Sigma}.$$

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We call $\mathcal{L}_{\lambda,\Sigma}$ the p-adic L-function of the branch character λ with respect to the p-adic CM-type Σ. It is conjectured by Gillard [Gil91, Conj. (i), p.21] that the μ-invariant $\mu_{\lambda,\Sigma}$ of $\mathcal{L}_{\lambda,\Sigma}$ always vanishes. In this note, we prove this conjecture when $p \nmid D_{\mathcal{F}}$.

Theorem A. *Suppose that $p \nmid D_{\mathcal{F}}$. Then $\mu_{\lambda,\Sigma} = 0$.*

When $\mathcal{F} = \mathbf{Q}$ and λ arises from elliptic curves over $\mathbf{Q}$ with CM by $\mathcal{K}$, this theorem is an immediate consequence of the vanishing of the μ-invariant of Coates-Wiles p-adic L-functions due to Gillard [Gil87] and Schneps [Sch87] independently. When the conductor of the residual character $\widehat{\lambda} \pmod p$ is a product of primes split in $\mathcal{K}/\mathcal{F}$, the above theorem is due to Hida in [Hid11]. Note that since the branch character λ is of infinite order, this p-adic L-function $\mathcal{L}_{\lambda,\Sigma}$ indeed is a suitable twist of the p-adic L-functions considered by Hida.

To explain the idea of Hida, we need to introduce some notation. Let $\mathfrak{X}^+$ be the set consisting of finite order characters $\nu : \Gamma^+ \to \mu_{p^\infty}$. For every $\nu \in \mathfrak{X}^+$, we shall regard ν as a Hecke character of $\mathcal{K}^\times$ by the geometrically normalized reciprocity law $\mathrm{rec}_{\mathcal{K}} : \mathbf{A}_{\mathcal{K}}^\times \to \mathrm{Gal}(\bar{\mathbf{Q}}/\mathcal{K})^{ab} \to \Gamma$. Let $\mu^-_{\lambda\nu,\Sigma}$ denote the μ-invariant of the anticyclotomic projection $\mathcal{L}^-_{\lambda\nu,\Sigma}$ of Katz p-adic L-function $\mathcal{L}_{\lambda\nu,\Sigma}$ attached to the brach character $\lambda\nu$. When λ has split conductor, Hida in [Hid10] proves a precise formula of $\mu^-_{\lambda\nu,\Sigma}$ in terms of the p-adic valuation of Fourier coefficients of certain Eisenstein series. Based on this exact formula, Hida concludes the vanishing of $\mu_{\lambda,\Sigma}$ by showing directly that $\liminf_{\nu\in\mathfrak{X}^+} \mu^-_{\lambda\nu,\Sigma} = 0$.

Our proof of Theorem A follows the approach of Hida. It is shown in [Hsi12, Thm. 5.5] that $\mu^-_{\lambda\nu,\Sigma}$ in general can be written to be the p-adic valuation of Fourier coefficients of certain special *toric* Eisenstein series $\mathbb{E}^h_{\lambda\nu,u}$. We are not able to calculate the Fourier coefficients of these toric Eisenstein series in full generality, so we do not obtain a precise formula of $\mu^-_{\lambda\nu,\Sigma}$ in full generality. However, we can estimate an upper bound of the p-adic valuation of Fourier coefficients of $\mathbb{E}^h_{\lambda\nu,u}$, and obtain an upper bound of $\mu^-_{\lambda\nu,\Sigma}$. Following Hida, we show this upper bound is as small as possible when $\nu \in \mathfrak{X}^+$ has sufficiently deep conductor.

In virtue of [HT93, Thm. 8.2], Theorem A provides an alternative proof of the one-sided divisibility between anticyclotomic p-adic L-functions and the congruence ideals of CM forms, which was proved in [Hid09, Cor. 3.8] using the trick of base change. This divisibility result eventually leads to the solution of the anticyclotomic main conjure proved in [Hid09, Theorem, p.914] combined with results of Hida and Tilouine [HT94] and Hida [Hid06]. In addition, we remark that the μ-invariant $\mu_{\lambda,\Sigma}$ considered in this note is referred to as the analytic μ-invariant in Iwasawa theory. Iwasawa main conjecture for CM fields implies that $\mu_{\lambda,\Sigma}$ equals the algebraic μ-invariant attached to λ, i.e. the μ-invariant of characteristic power series of a certain Iwasawa module (*cf.* [HT94, Main conjecture, p.90]). In particular, we can consider an CM elliptic curve E over the totally real field $\mathcal{F}$ with complex multiplication by the ring of integers of an imaginary quadratic field $\mathcal{M}$. Assuming the validity of the main conjecture for the CM field $\mathcal{K} = \mathcal{F}\mathcal{M}$, our result would imply the algebraic μ-invariant for E over $\mathcal{K}_\infty$ vanishes as well. The arithmetic aspect of the vanishing of algebraic μ-invariants of elliptic curves in a more general setting is discussed in [Suj10].

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2. Eisenstein series and anticyclotomic μ-invariants

In this section, we recall without proofs the construction of certain special Eisenstein series, which are used to compute the anticyclotomic μ-invariant in [Hsi12].

2.1. **Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$.** Let χ be a Hecke character of infinity type $k\Sigma$, $k \geq 1$. Suppose that $\mathfrak{C}$ is the prime-to-p conductor of χ. We write $\mathfrak{C} = \mathfrak{C}^+\mathfrak{C}^-$ such that $\mathfrak{C}^+$ (resp. $\mathfrak{C}^-$) is a product of prime factors split (resp. non-split) over $\mathcal{F}$. We further decompose $\mathfrak{C}^+ = \mathfrak{F}\mathfrak{F}_c$ such that $(\mathfrak{F}, \mathfrak{F}_c) = 1$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. Let $D_{\mathcal{K}/\mathcal{F}}$ be the discriminant of $\mathcal{K}/\mathcal{F}$ and let

$$\mathfrak{D} = p\mathfrak{C}\mathfrak{C}^c D_{\mathcal{K}/\mathcal{F}}.$$

We will identify the CM-type $\Sigma \subset \mathrm{Hom}(\mathcal{K}, \mathbf{C})$ with the set $\mathrm{Hom}(\mathcal{F}, \mathbf{R})$ of archimedean places of $\mathcal{F}$ by the restriction map. Let $K^0_\infty := \prod_{\sigma\in\Sigma} \mathrm{SO}(2, \mathbf{R})$ be a maximal compact subgroup of $\mathrm{GL}_2(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$. We put

$$\chi^* = \chi|\cdot|^{-\frac{1}{2}}_{\mathbf{A}_{\mathcal{K}}} \text{ and } \chi_+ = \chi|_{\mathbf{A}^\times_{\mathcal{F}}}.$$

For $s \in \mathbf{C}$, we let $I(s,\chi_+)$ denote the space consisting of smooth and K_∞^0-finite functions $\phi : \mathrm{GL}_2(\mathbf{A}_\mathcal{F}) \to \mathbf{C}$ such that

$$\phi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g) = \chi_+^{-1}(d) \left|\frac{a}{d}\right|_{\mathbf{A}_\mathcal{F}}^s \phi(g).$$

Conventionally, functions in $I(s,\chi_+)$ are called *sections*. Let B be the upper triangular subgroup of GL_2. The adelic Eisenstein series associated to a section $\phi \in I(s,\chi_+)$ is defined by

$$E_\mathbf{A}(g,\phi) = \sum_{\gamma \in B(\mathcal{F}) \backslash \mathrm{GL}_2(\mathcal{F})} \phi(\gamma g).$$

It is known that the series $E_\mathbf{A}(g,\phi)$ is absolutely convergent for $\mathrm{Re} s \gg 0$.

2.2. **Fourier coefficients of Eisenstein series.** Let $\psi = \prod \psi_v : \mathbf{A}_\mathcal{F}/\mathcal{F} \to \mathbf{C}^\times$ be the standard additive character such that $\psi_\infty(x) = \exp(2\pi i \mathrm{T}_{\mathcal{F}/\mathbf{Q}}(x))$ for $x \in \mathcal{F} \otimes_\mathbf{Q} \mathbf{R}$. Put $\mathbf{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let v be a place of $\mathcal{F}$ and let $I_v(s,\chi_+)$ be the local constitute of $I(s,\chi_+)$ at v. For $\phi_v \in I_v(s,\chi_+)$ and $\beta \in \mathcal{F}_v$, we recall that the β-th local Whittaker integral $W_\beta(\phi_v, g_v)$ is defined by

$$W_\beta(\phi_v, g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w} \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix} g_v) \psi(-\beta x_v) dx_v,$$

and the intertwining operator $M_\mathbf{w}$ is defined by

$$M_\mathbf{w}\phi_v(g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w} \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix} g_v) dx_v.$$

Here dx_v is Lebesgue measure if $\mathcal{F}_v = \mathbf{R}$ and is the Haar measure on $\mathcal{F}_v$ normalized so that $\mathrm{vol}(\mathcal{O}_{\mathcal{F}_v}, dx_v) = 1$ if $\mathcal{F}_v$ is non-archimedean. By definition, $M_\mathbf{w}\phi_v(g_v)$ is the 0-th local Whittaker integral. It is well known that local Whittaker integrals converge absolutely for $\mathrm{Re} s \gg 0$, and have meromorphic continuation to all $s \in \mathbf{C}$.

If $\phi = \otimes_v \phi_v$ is a decomposable section, then $E_\mathbf{A}(g,\phi)$ has the following Fourier expansion:

$$\begin{aligned} E_\mathbf{A}(g,\phi) &= \phi(g) + M_\mathbf{w}\phi(g) + \sum_{\beta \in \mathcal{F}} W_\beta(E_\mathbf{A}, g), \text{ where} \\ M_\mathbf{w}\phi(g) &= \frac{1}{\sqrt{|D_\mathcal{F}|_\mathbf{R}}} \cdot \prod_v M_\mathbf{w}\phi_v(g_v)\,;\, W_\beta(E_\mathbf{A}, g) = \frac{1}{\sqrt{|D_\mathcal{F}|_\mathbf{R}}} \cdot \prod_v W_\beta(\phi_v, g_v). \end{aligned} \tag{2.1}$$

2.3. **The choice of the local sections.** We briefly recall the choice of local sections in [Hsi12, §4.3]. We begin with some notation. Let v be a place of $\mathcal{F}$. Let $F = \mathcal{F}_v$ (resp. $E = \mathcal{K} \otimes_\mathcal{F} \mathcal{F}_v$). Denote by $z \mapsto \bar{z}$ the complex conjugation. Let $|\cdot|$ be the standard absolute values on F and let $|\cdot|_E$ be the absolute value on E given by $|z|_E := |z\bar{z}|$. Let $d_F = d_{\mathcal{F}_v}$ be a fixed generator of the different $\mathcal{D}_\mathcal{F}$ of $\mathcal{F}/\mathbf{Q}$. Write χ (resp. χ_+) for χ_v (resp. $\chi_{+,v}$). If $v \in \mathbf{h}$, denote by ϖ_v a uniformizer of $\mathcal{F}_v$. For a set Y, denote by $\mathbb{I}_Y$ the characteristic function of Y.

Case I: $v \nmid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$. We first suppose that $v = \sigma \in \Sigma$ is archimedean and $F = \mathbf{R}$. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R})$, we put $J(g,i) := ci + d$. Define the sections $\phi_{k,s,\sigma}^h$ of weight k in $I_v(s,\chi_+)$ by

$$\phi_{k,s,\sigma}(g) = J(g,i)^{-k} |\det(g)|^s \cdot \left| J(g,i)\overline{J(g,i)} \right|^{-s}.$$

Suppose that v is non-archimedean. Denote by $\mathcal{S}(F)$ and (resp. $\mathcal{S}(F \oplus F)$) the space of Bruhat-Schwartz functions on F (resp. $F \oplus F$). Recall that the Fourier transform $\widehat{\varphi}$ for $\varphi \in \mathcal{S}(F)$ is defined by

$$\widehat{\varphi}(y) = \int_F \varphi(x)\psi(yx)dx.$$

For a character $\mu : F^\times \to \mathbf{C}^\times$, we define a function $\varphi_\mu \in \mathcal{S}(F)$ by

$$\varphi_\mu(x) = \mathbb{I}_{O_F^\times}(x)\mu(x).$$

If $v \mid p\mathfrak{F}\mathfrak{F}^c$ is split in $\mathcal{K}$, write $v = w\overline{w}$ with $w | \mathfrak{F}\Sigma_p$, and set

$$\varphi_w = \varphi_{\chi_w} \text{ and } \varphi_{\overline{w}} = \varphi_{\chi_{\overline{w}}^{-1}}.$$

To a Bruhat-Schwartz function $\Phi \in \mathcal{S}(F \oplus F)$, we can associate a Godement section $f_{\Phi,s} \in I_v(s,\chi_+)$ defined by

$$f_{\Phi,s}(g) := |\det g|^s \int_{F^\times} \Phi((0,x)g)\chi_+(x)\,|x|^{2s}\,d^\times x, \tag{2.2}$$

where $d^\times x$ is the Haar measure on $F^\times$ such that $\mathrm{vol}(\mathcal{O}_F^\times, d^\times x) = 1$. Define Godement sections by

$$\phi_{\chi,s,v} = f_{\Phi_v^0,s}, \text{ where } \Phi_v^0(x,y) = \begin{cases} \mathbb{I}_{O_F}(x)\mathbb{I}_{d_F^{-1}O_F}(y) & \cdots v \nmid \mathfrak{D}, \\ \varphi_{\overline{w}}(x)\widehat{\varphi}_w(y) & \cdots v \mid \mathfrak{F}\mathfrak{F}^c. \end{cases} \tag{2.3}$$

Let $u \in \mathcal{O}_F^\times$. Let $\varphi_{\overline{w}}^1$ and $\varphi_w^{[u]} \in \mathcal{S}(F)$ be the Bruhat-Schwartz functions defined by

$$\varphi_{\overline{w}}^1(x) = \mathbb{I}_{1+\varpi_v O_F}(x)\chi_{\overline{w}}^{-1}(x) \text{ and } \varphi_w^{[u]}(x) = \mathbb{I}_{u(1+\varpi_v O_F)}(x)\chi_w(x).$$

Define $\Phi_v^{[u]} \in \mathcal{S}(F \oplus F)$ by

$$\Phi_v^{[u]}(x,y) = \frac{1}{\mathrm{vol}(1+\varpi_v O_F, d^\times x)}\varphi_{\overline{w}}^1(x)\widehat{\varphi}_w^{[u]}(y) = (|\varpi_v|^{-1}-1)\varphi_{\overline{w}}^1(x)\widehat{\varphi}_w^{[u]}(y). \tag{2.4}$$

Case II: $v \mid D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}^-$. In this case, E is a field. We define an embedding $\rho : E \hookrightarrow M_2(F)$ by

$$a + b\vartheta \mapsto \rho(x + b\vartheta) = \begin{bmatrix} a & b\vartheta^2 \\ b & a \end{bmatrix}.$$

Then $\mathrm{GL}_2(F) = B(F)\rho(E^\times)$. We fix a O_F-basis $\{1, \boldsymbol{\theta}_v\}$ of $\mathcal{O}_E$ such that $\boldsymbol{\theta}_v$ is a uniformizer if v is ramified and $\overline{\boldsymbol{\theta}_v} = -\boldsymbol{\theta}_v$ if $v \nmid 2$. Let $t_v = \boldsymbol{\theta}_v + \overline{\boldsymbol{\theta}_v}$ and put

$$\varsigma_v = \begin{bmatrix} d_{\mathcal{F}_v} & -2^{-1}t_v \\ 0 & d_{\mathcal{F}_v}^{-1} \end{bmatrix}.$$

Let $\phi_{\chi,s,v}$ be the smooth section in $I_v(s,\chi_+)$ defined by

$$\phi_{\chi,s,v}(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\rho(z)\varsigma_v) = L(s,\chi_v)\cdot\chi_+^{-1}(d)\left|\frac{a}{d}\right|^s \cdot \chi^{-1}(z) \quad (b \in B(F),\, z \in E^\times). \tag{2.5}$$

Here $L(s,\chi_v)$ is the local Euler factor of χ_v.

2.4. **Fourier expansion of normalized Eisenstein series.** Let $\mathcal{U}_p$ be the torsion subgroup of $\mathcal{O}_{\mathcal{F}_p}^\times$. For $u = (u_v)_{v|p} \in \mathcal{U}_p$, let $\Phi_p^{[u]} = \otimes_{v|p}\Phi_v^{[u_v]}$ be the Bruhat-Schwartz function defined in (2.4). Define the section $\phi_{\chi,s}^h(\Phi_p^{[u]}) \in I(s,\chi_+)$ by

$$\phi_{\chi,s}^h(\Phi_p^{[u]}) = \bigotimes_{\sigma\in\Sigma}\phi_{k,s,\sigma}^h \bigotimes_{\substack{v\in\mathbf{h},\\ v\nmid p}}\phi_{\chi,s,v}\bigotimes_{v|p} f_{\Phi_v^{[u_v]},s}.$$

We put

$$X^+ = \left\{\tau = (\tau_\sigma)_{\sigma\in\Sigma} \in \mathbf{C}^\Sigma \mid \mathrm{Im}\,\tau_\sigma > 0 \text{ for all } \sigma \in \Sigma\right\}.$$

The holomorphic Eisenstein series $\mathbb{E}_{\chi,u}^h : X^+ \times \mathrm{GL}_2(\mathbf{A}_{\mathcal{F},f}) \to \mathbf{C}$ is defined by

$$\begin{gathered}\mathbb{E}_{\chi,u}^h(\tau,g_f) := \frac{\Gamma_\Sigma(k\Sigma)}{\sqrt{|D_\mathcal{F}|_\mathbf{R}}(2\pi i)^{k\Sigma}}\cdot E_\mathbf{A}\left((g_\infty,g_f),\phi_{\chi,s}^h(\Phi_p^{[u]})\right)|_{s=0}\cdot\prod_{\sigma\in\Sigma}J(g_\sigma,i)^k, \\ (g_\infty = (g_\sigma)_\sigma \in \mathrm{GL}_2(\mathcal{F}\otimes_\mathbf{Q}\mathbf{R}),\ (g_\sigma i)_{\sigma\in\Sigma} = (\tau_\sigma)_{\sigma\in\Sigma}).\end{gathered} \tag{2.6}$$

Let $\mathbf{c} = (\mathbf{c}_v) \in \mathbf{A}_{\mathcal{F},f}^\times$ such that $\mathbf{c}_v = 1$ at $v \mid \mathfrak{D}$ and let $\mathfrak{c} = \mathbf{c}(\mathcal{O}_\mathcal{F}\otimes_\mathbf{Z}\hat{\mathbf{Z}})\cap\mathcal{F}$. Define a function $\mathbb{E}_{\chi,u}^h|_\mathfrak{c} : X^+ \to \mathbf{C}$ by $\mathbb{E}_{\chi,u}^h|_\mathfrak{c}(\tau) := \mathbb{E}_{\chi,u}^h(\tau, \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{c}^{-1} \end{bmatrix})$. Then $\mathbb{E}_{\chi,u}^h|_\mathfrak{c}$ is a $\mathfrak{c}$-Hlbert modular form of weight $k\Sigma$ defined over $\mathbf{C}$ in the sense of [Kat78, p.211].

Proposition 2.1. *The q-expansion of $\mathbb{E}^h_{\chi,u}|_{\mathfrak{c}}$ at the cusp $(O,\mathfrak{c}^{-1})$ is given by*

$$\mathbb{E}^h_{\chi,u}|_{(O,\mathfrak{c}^{-1})}(q) = \sum_{\beta\in\mathcal{F}_+} \mathbf{a}_\beta(\mathbb{E}^h_{\chi,u},\mathfrak{c})\cdot q^\beta.$$

The β-th Fourier coefficient $\mathbf{a}_\beta(\mathbb{E}^h_{\chi,u},\mathfrak{c})$ is given by

$$\begin{aligned}\mathbf{a}_\beta(\mathbb{E}^h_{\chi,u},\mathfrak{c}) =&\beta^{(k-1)\Sigma}\prod_{w|\mathfrak{F}}\chi_w(\beta)\mathbb{I}_{O_F^\times}(\beta)\prod_{w\in\Sigma_p}\chi_w(\beta)\mathbb{I}_{u_v(1+\varpi_vO_F)}(\beta)\\ &\times\prod_{v\nmid\mathfrak{D}}\left(\sum_{i=0}^{v(\mathbf{c}_v\beta)}\chi^*(\varpi_v^i)\right)\cdot\prod_{v|\mathfrak{C}^-D_{\mathcal{K}/\mathcal{F}}}L(0,\chi_v)\widetilde{A}_\beta(\chi_v),\end{aligned}$$

where

$$\begin{aligned}\widetilde{A}_\beta(\chi_v) &:= \int_{\mathcal{F}_v}\chi_v^{-1}|\cdot|_E^s(x_v+\boldsymbol{\theta}_v)\psi(-d_{\mathcal{F}_v}^{-1}\beta x_v)dx_v|_{s=0}\\ &= \lim_{n\to\infty}\int_{\varpi_v^{-n}\mathcal{O}_{\mathcal{F}_v}}\chi_v^{-1}(x_v+\boldsymbol{\theta}_v)\psi(-d_{\mathcal{F}_v}^{-1}\beta x_v)dx_v.\end{aligned}\tag{2.7}$$

Proof. This follows from (2.1) and the calculations of local Whittaker integerals of special local sections in [Hsi11, §4.3] (*cf.* [Hsi12, Prop. 4.1 and Prop. 4.4]). □

2.5. **The μ-invariants of anticyclotomic p-adic L-functions.** Let $Z(\mathfrak{C})^-$ be the anticyclotomic quotient of $Z(\mathfrak{C})$. Let $\widehat{\mathcal{O}}_\mathcal{K} = \mathcal{O}_\mathcal{K}\otimes_\mathbf{Z}\hat{\mathbf{Z}}$ and $U(\mathfrak{C}p^n) := \left\{u\in\widehat{\mathcal{O}}_\mathcal{K}^\times \mid u\equiv 1 \pmod{\mathfrak{C}p^n}\right\}$. The reciprocity law $\mathrm{rec}_\mathcal{K}:\mathbf{A}^\times_{\mathcal{K},f}\to Z(\mathfrak{C})^-$ induces the isomorphism:

$$\mathrm{rec}_\mathcal{K}:\varprojlim_n \mathcal{K}^\times\mathbf{A}^\times_{\mathcal{F},f}\backslash\mathbf{A}^\times_{\mathcal{K},f}/U(\mathfrak{C}p^n)\xrightarrow{\sim} Z(\mathfrak{C})^-.$$

Let Γ^- be the maximal $\mathbf{Z}_p$-free quotient of $Z(\mathfrak{C})^-$. Each function ϕ on Γ^- will be regarded as a function on $Z(\mathfrak{C})$ by the natural projection $\pi_-: Z(\mathfrak{C})\to Z(\mathfrak{C})^-\to\Gamma^-$. The anticyclotomic projection $\mathcal{L}^-_{\chi,\Sigma}$ of the measure $\mathcal{L}_{\mathfrak{C},\Sigma}$ is defined by

$$\int_{\Gamma^-}\phi d\mathcal{L}^-_{\chi,\Sigma} := \int_{Z(\mathfrak{C})}\widehat{\chi}\phi d\,\mathcal{L}_{\mathfrak{C},\Sigma}.$$

Recall that the μ-invariant $\mu(\varphi)$ of a $\bar{\mathbf{Z}}_p$-valued p-adic measure φ on a p-adic group H is defined to be

$$\mu(\varphi) = \inf_{U\subset H\text{ open}} v_p(\varphi(U)).$$

We shall give a formula of the μ-invariant $\mu^-_{\chi,\Sigma}$ of $\mathcal{L}^-_{\chi,\Sigma}$ in terms of p-adic valuation of Fourier coefficients of $\mathbb{E}^h_{\chi,u}$. To state the formula precisely, we introduce some notation.

Let $Cl_- := \mathcal{K}^\times\mathbf{A}^\times_{\mathcal{F},f}\backslash\mathbf{A}_{\mathcal{K},f}/\widehat{\mathcal{O}}_\mathcal{K}^\times$ and let Cl_-^{alg} be the subgroup of Cl_- generated by ramified primes. Let $O_p := \mathcal{O}_\mathcal{F}\otimes\mathbf{Z}_p$. Let Γ' be the open subgroup of Γ^- generated by the image of $O_p^\times\times\prod_{v|D_{\mathcal{K}/\mathcal{F}}}\mathcal{K}_v^\times$ via $\mathrm{rec}_\mathcal{K}$. The reciprocity law $\mathrm{rec}_\mathcal{K}$ at Σ_p induces an injective map $\mathrm{rec}_{\Sigma_p}: 1+pO_p\hookrightarrow O_p^\times = \oplus_{w\in\Sigma_p}\mathcal{O}^\times_{\mathcal{K}_w}\xrightarrow{\mathrm{rec}_\mathcal{K}} Z(\mathfrak{C})^-$ with finite cokernel as $p\nmid D_\mathcal{F}$, and it is easy to see that rec_{Σ_p} induces an isomorphism $\mathrm{rec}_{\Sigma_p}: 1+pO_p\xrightarrow{\sim}\Gamma'$. We thus identify Γ' with the subgroup $\mathrm{rec}_{\Sigma_p}(1+pO_p)$ of $Z(\mathfrak{C})^-$. Let $Z' := \pi_-^{-1}(\Gamma')$ be the subgroup of $Z(\mathfrak{C})$ and let $Cl'_-\supset Cl_-^{\mathrm{alg}}$ be the image of Z' in Cl_- and let $\mathcal{D}'_1$ (resp. $\mathcal{D}''_1$) be a set of representatives of $Cl'_-/Cl_-^{\mathrm{alg}}$ (resp. Cl_-/Cl'_-) in $(\mathbf{A}^{(\mathfrak{D})}_{\mathcal{K},f})^\times$. Let $\mathcal{D}_1 := \mathcal{D}''_1\mathcal{D}'_1$ be a set of representatives of Cl_-/Cl_-^{alg}. Let $\mathcal{U}_p$ be the torsion subgroup of $(\mathcal{O}_\mathcal{F}\otimes_\mathbf{Z}\mathbf{Z}_p)^\times$ and let $\mathcal{U}^{\mathrm{alg}} := \widehat{\mathcal{O}}_\mathcal{K}^\times\cap(\mathcal{K}^\times)^{1-c}$. Let $\mathcal{D}_0$ be a set of representatives of $\mathcal{U}_p/\mathcal{U}^{\mathrm{alg}}$ in $\mathcal{U}_p$. For $a\in\mathbf{A}^\times_{\mathcal{K},f}$, let $\mathfrak{c}(a) := \mathfrak{c}(\mathcal{O}_\mathcal{K})\mathrm{N}_{\mathcal{K}/\mathcal{F}}(\mathfrak{a})$, where $\mathfrak{a} = a(\mathcal{O}_\mathcal{K}\otimes_\mathbf{Z}\hat{\mathbf{Z}})\cap\mathcal{K}$. The following theorem is proved by the ideas of Hida in [Hid10].

Theorem 2.2 (Thm. 5.5 [Hsi12])**.** *Suppose that $p\nmid D_\mathcal{F}$. Then we have*

$$\mu^-_{\chi,\Sigma} = \inf_{\substack{(u,a)\in\mathcal{D}_0\times\mathcal{D}_1\\ \beta\in\mathcal{F}_+}} v_p(\mathbf{a}_\beta(\mathbb{E}^h_{\chi,u},\mathfrak{c}(a))).$$

PROOF. For the convenience of the readers, we sketch the proof here. For each $b \in \mathcal{D}_1''$, we denote by $\mathcal{L}^b_{\chi,\Sigma}$ the p-adic measure on $1 + pO_p \simeq \Gamma'$ obtained by the restriction of $\mathcal{L}^-_{\chi,\Sigma}$ to $b.\Gamma' := \pi_-(\mathrm{rec}_{\mathcal{K}}(b))\Gamma^-$. To be precise, we have

$$\int_{\Gamma'} \phi d\mathcal{L}^b_{\chi,\Sigma} := \int_{\Gamma^-} \mathbb{I}_{b.\Gamma'} \cdot \phi|[b^{-1}] d\mathcal{L}^-_{\chi,\Sigma}$$

where $\mathbb{I}_{b.\Gamma'}$ is the characteristic functions of $b.\Gamma'$. Let $\mu^b_{\chi,\Sigma}$ be the μ-invariant of the p-adic measures $\mathcal{L}^b_{\chi,\Sigma}$. Note that $\Gamma^- = \bigsqcup_{b\in\mathcal{D}_1''} b.\Gamma'$, so it is clear that

$$\mu^-_{\chi,\Sigma} = \inf_{b\in\mathcal{D}_1''} \mu^b_{\chi,\Sigma}. \tag{2.8}$$

For $(u, a) \in \mathcal{D}_0 \times \mathcal{D}_1$, we let $\mathcal{E}_{u,a}$ be the p-adic avatar of $\mathbb{E}^h_{\chi,u}|_{\mathfrak{c}(a)}$ ([Hsi12, §2.5.5]). Let t be the Serre-Tate coordinate of the CM point $\mathbf{x}$ with the polarization ideal $\mathfrak{c}(\mathcal{O}_{\mathcal{K}})$ defined in [Hsi12, §5.2]. For $a \in \mathcal{D}_1'$, let $\langle a\rangle_\Sigma$ be the unique element in $1 + pO_p$ such that $\mathrm{rec}_{\Sigma_p}(\langle a\rangle_\Sigma) = \pi_-(\mathrm{rec}_{\mathcal{K}}(a)) \in \Gamma'$. For each $b \in \mathcal{D}_1''$, we define a t-expansion $\mathcal{E}^b(t)$ by

$$\mathcal{E}^b(t) := \#\mathcal{U}^{\mathrm{alg}} \cdot \sum_{(u,a)\in\mathcal{D}_0\times b\mathcal{D}_1'} \chi(ab^{-1})\mathcal{E}_{u,a}|[a](t^{\langle ab^{-1}\rangle_\Sigma u^{-1}}),$$

where $|[a]$ is the Hecke action induced by a (See [Hsi12, Remark 4.5]). With the help of an an explicit formula of toric period integral of Eisenstein series ([Hsi11, Prop. 5.1] and [Hsi12, Prop. 4.9]), it is shown in [Hsi12, Prop. 5.2] that $\mathcal{E}^b(t)$ essentially gives rise to the t-expansion of the measure $\mathcal{L}^b_{\chi,\Sigma}$, and hence we find that

$$\mu^b_{\chi,\Sigma} = \inf\left\{r \in \mathbf{Q}_{\geq 0} \mid p^{-r}\mathcal{E}^b(t) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_{\bar{\mathbf{Z}}_p})\right\}, \tag{2.9}$$

where $\mathfrak{m}_{\bar{\mathbf{Z}}_p}$ is the maximal ideal of $\bar{\mathbf{Z}}_p$. By the linear independence of p-adic modular forms modulo p [Hid10, Cor. 3.2], the q-expansion principle of p-adic modular forms combined with [Hsi12, Lemma 5.3], we can conclude from (2.8) and (2.9) that

$$\mu^-_{\chi,\Sigma} = \inf_{b\in\mathcal{D}_1''} \mu^b_{\chi,\Sigma} = \inf_{\substack{(u,a)\in\mathcal{D}_0\times\mathcal{D}_1,\\ \beta\in\mathcal{F}_+}} v_p(\mathbf{a}_\beta(\mathbb{E}^h_{\chi,u}, \mathfrak{c}(a))). \qquad \square$$

3. PROOF OF THEOREM A

We go back to our setting in the introduction. Let λ be a Hecke character of infinity type $k\Sigma$ with $k \geq 1$ and let $\lambda^* := \lambda|\cdot|^{-\frac{1}{2}}_{\mathbf{A}_{\mathcal{K}}}$. We may further assume that

$$\mathfrak{C} \text{ is the prime-to-}p\text{ conductor of } \lambda.$$

To prove Theorem A, we prepare two lemmas. The first lemma is taken from [Hid11].

Lemma 3.1. *Let $w \nmid p$ be a place of $\mathcal{K}$ and let ϖ_w be a uniformizer of $\mathcal{K}_w$. Let $a \in \bar{\mathbf{Z}}_p$. Given $e > 0$, we have*

$$v_p(a + \nu(\varpi_w)) < e \text{ for all but finitely many } \nu \in \mathfrak{X}^+.$$

PROOF. We note that $\nu(\varpi_w)$ is a primitive p^n-th root of unity for some $n \in \mathbf{Z}_{\geq 0}$, and for sufficiently large n, we have

$$v_p(a + \nu(\varpi_w)) \leq v_p(\nu(\varpi_w) - 1) = \frac{1}{p^n - p^{n-1}} < e.$$

The first equality holds precisely when $v_p(a + 1) > 0$. Therefore, it is not difficult to deduce the lemma from the fact that the image of ϖ_w in Γ^+ under $\mathrm{rec}_{\mathcal{K}} : \mathbf{A}^\times_{\mathcal{K}} \to \Gamma^+$ generates a subgroup of Γ^+ with finite index. $\square$

Lemma 3.2. *Let $v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$ and let $e_1 > 0$ be a positive number. Then there exists $\beta_v \in \mathcal{F}^\times_v$ such that for almost all $\nu \in \mathfrak{X}^+$, we have*

$$v_p(L(0, \lambda_v\nu_v)\widetilde{A}_{\beta_v}(\lambda_v\nu_v)) \leq e_1.$$

Here "for almost all" means "for all but finitely many".

PROOF. Let $E = \mathcal{K}_v$ and $F = \mathcal{F}_v$. Let $\mathfrak{m}_E$ be the maximal ideal of $\mathcal{O}_E$. For brevity, we drop the subscript v and simply write $\lambda = \lambda_v$, $\nu = \nu_v$. Let $a(\lambda) := \inf\{n \in \mathbf{Z}_{\geq 0} \mid \lambda(1+\mathfrak{m}_E^n) = 1\}$ be the conductor of λ. Suppose that $a(\lambda) > 1$. Then $\lambda(1+\mathfrak{m}) \neq 1$, and the invariant $\mu_p(\lambda) := \inf_{x \in E^\times} v_p(\lambda(x)-1) = 0$ as $v \nmid p$. It follows from [Hsi11, Lemma 6.4] that there exists β such that $v_p(\widetilde{A}_\beta(\lambda)) = 0$. Moreover, since $\widetilde{A}_\beta(\lambda\nu) \equiv \widetilde{A}_\beta(\lambda) \pmod{\mathfrak{m}_{\bar{\mathbf{Z}}_p}}$, we find that $v_p(\widetilde{A}_\beta(\lambda\nu)) = 0$ for all $\nu \in \mathfrak{X}^+$. To prove the remaining part, we assume the conductor $a(\lambda) = a(\lambda\nu) \leq 1$. In virtue of Lemma 3.1, it suffices to show that there exists β such that

$$v_p(\widetilde{A}_\beta(\lambda\nu)) = v_p(a + b \cdot \nu(\varpi_w)) \tag{3.1}$$

for some $a \in \bar{\mathbf{Z}}_p, b \in \bar{\mathbf{Z}}_p^\times$ independent of ν and a uniformizer ϖ_w of E.

Let ϖ be a uniformizer of F. Suppose that v is ramified. Recall that $\boldsymbol{\theta} = \boldsymbol{\theta}_v$ is chosen to be a uniformizer of E. Let $\beta \in \varpi^{-1}O_F^\times$, so $v(\beta) = -1$. If $v \nmid \mathfrak{C}^-$, then by [Hsi11, Lemma 4.1], we have

$$\widetilde{A}_\beta(\lambda\nu) = |\mathcal{D}_F|^{-1} \lambda^{-1}\nu^{-1}(\boldsymbol{\theta}) |\varpi| .$$

If $v \mid \mathfrak{C}^-$, then it follows from [Hsi11, Prop. 4.4 (1)] that

$$\begin{aligned}\widetilde{A}_\beta(\lambda\nu) =& \lambda^*(\boldsymbol{\theta}^{-1}) |\varpi|^{\frac{1}{2}} \nu(\boldsymbol{\theta}^{-1}) + \lambda^*(-\beta d_F^{-1})\nu(-\beta\varpi)\epsilon(1, \lambda_+|\cdot|^{-1}, \psi) \quad (\lambda_+ := \lambda|_{F^\times})\\ =& \lambda^*(\boldsymbol{\theta}^{-1}) |\varpi|^{\frac{1}{2}} \cdot \nu(\boldsymbol{\theta}^{-1}) + \lambda^*(-\beta d_F^{-1})\epsilon(1, \lambda_+|\cdot|^{-1}, \psi).\end{aligned}$$

Here $\epsilon(s, \lambda_+|\cdot|^{-1}, \psi)$ is the Tate's local epsilon factor attached to the additive character $\psi_v : F \to \mathbf{C}^\times$. In any case, it is clear that (3.1) holds for $\beta \in \varpi^{-1}\mathcal{O}_F^\times$ when v is ramified.

Suppose that v is inert. Then $a(\lambda\nu) = 1$. Let $\beta \in O_F^\times$ (so $v(\beta) = 0$). By [Hsi11, Prop. 4.5], if $\lambda|_{O_v^\times} = 1$, then

$$\widetilde{A}_\beta(\lambda\nu) = -|\varpi| (1 + \lambda^*\nu(\varpi)),$$

and if $\lambda|_{O_F^\times}$ is non-trivial, then

$$\widetilde{A}_\beta(\lambda\nu) = \mathcal{I}_{\lambda\nu}(0) + \lambda^*(-\beta d_F^{-1})\nu(\varpi)\epsilon(1, \lambda_+|\cdot|^{-1}, \psi),$$

where

$$\mathcal{I}_{\lambda\nu}(0) = \int_{O_F} \lambda^{-1}\nu^{-1}(x+\boldsymbol{\theta})dx.$$

Recall that $\boldsymbol{\theta}$ is chosen such that $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\boldsymbol{\theta}$. We have $x + \boldsymbol{\theta} \in \mathcal{O}_E^\times$ for $x \in \mathcal{O}_F$. As ν is unramified at v, we find that

$$\mathcal{I}_{\lambda\nu}(0) = \int_{O_F} \lambda^{-1}(x+\boldsymbol{\theta})dx$$

is independent of ν. Therefore, in either cases, (3.1) holds for $\beta \in \mathcal{O}_F^\times$. □

Theorem 3.3. *Suppose that $p \nmid D_{\mathcal{F}}$. Then*

$$\mu_{\lambda,\Sigma} = 0.$$

PROOF. Let S^- be the set of prime factors of $\mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$. By Lemma 3.2, for $e_1 > 0$, we choose $(\beta_v) \in \prod_{v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}} \mathcal{F}_v^\times$ such that

$$\sum_{v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}} v_p(L(0, \lambda_v\nu_v)\widetilde{A}_{\beta_v}(\lambda_v\nu_v)) \leq \#(S^-) \cdot e_1 \tag{3.2}$$

for almost $\nu \in \mathfrak{X}^-$. Let $\mathbf{c} \in \mathbf{A}_{\mathcal{F},f}^\times$ and $\mathfrak{c}$ be the associated ideal as in Prop. 2.1. We define an idele $\eta \in \mathbf{A}_{\mathcal{K},f}^\times$ such that

- $\eta_v = \beta_v^{-1}$ for all $v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$,
- $\eta_v = \mathbf{c}_v$ for all finite $v \nmid \mathfrak{D}$,
- $\eta_v = 1$ for the remaining places v.

Let $U = \prod U_v$ be an open subgroup of $\mathbf{A}_{\mathcal{F}}^{\times}$ such that $U_v = \mathcal{O}_{\mathcal{F}_v}^{\times}$ at all $v \nmid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$ and $U_\infty = (\mathcal{F} \otimes \mathbf{R})_+$. Moreover, it is not difficult to see from [Hsi11, (4.17)] that for $v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$, U_v can be chosen small enough, depending on λ_v and β_v, so that

$$\widetilde{A}_{\beta_v u}(\lambda_v \nu_v) = \widetilde{A}_{\beta_v}(\lambda_v \nu_v) \text{ for all } u \in U_v \text{ and } \nu \in \mathfrak{X}^+.$$

Consider the idele class $\mathcal{F}^\times \eta U$ in $\mathbf{A}_{\mathcal{F}}^{\times}$. We may choose a uniformizer $\varpi_{v_0} \in \mathcal{K}_{v_0}$ with a finite place $v_0 \nmid \mathfrak{D}$ such that ϖ_{v_0} lies in the class $\mathcal{F}^\times \eta U$. We can write

$$\varpi_{v_0} \in \beta \eta U \text{ for some } \beta \in \mathcal{F}^\times.$$

Since $\eta_v = 1$ when v is archimedean or $v|p$, we find that $\beta \in \mathcal{F}_+ \cap \mathcal{O}_{\mathcal{F},(p)}^{\times}$ by the choice of U. Let $u \in \mathcal{U}_p$ such that $\beta \equiv u \pmod{p}$. By Prop. 2.1 we have

$$\begin{aligned} v_p(\mathbf{a}_\beta(\mathbb{E}^h_{\lambda\nu,u}, \mathfrak{c})) &= \sum_{v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}} v_p(L(0, \lambda_v \nu_v) \widetilde{A}_\beta(\lambda_v \nu_v)) + \sum_{v \nmid \mathfrak{D}} v_p\Big(\sum_{i=0}^{v(\beta \mathbf{c}_v)} \lambda^* \nu(\varpi_v^i)\Big) \\ &= \sum_{v \mid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}} v_p(L(0, \lambda_v \nu_v) \widetilde{A}_{\beta_v}(\lambda_v \nu_v)) + v_p(\lambda^* \nu(\varpi_{v_0}) + 1). \end{aligned}$$

It follows that for almost all $\nu \in \mathfrak{X}^+$, we have

$$v_p(\mathbf{a}_\beta(\mathbb{E}^h_{\lambda\nu,u}, \mathfrak{c})) \leq \#(S^-) \cdot e_1 + v_p(\lambda^*(\varpi_{v_0}) \cdot \nu(\varpi_{v_0}) + 1).$$

Hence, from Theorem 2.2 and Lemma 3.1 we deduce that

$$0 \leq \mu_{\lambda,\Sigma} \leq \liminf_{\nu} \mu^-_{\lambda\nu,\Sigma} \leq \liminf_{\nu} v_p(\mathbf{a}_\beta(\mathbb{E}^h_{\lambda\nu,u}, \mathfrak{c})) \leq \#(S^-) \cdot e_1.$$

This inequality holds for all $e_1 > 0$, so $\mu_{\lambda,\Sigma} = 0$. □

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