

FOUR-VARIABLE p -ADIC TRIPLE PRODUCT L -FUNCTIONS AND THE TRIVIAL ZERO CONJECTURE

MING-LUN HSIEH AND SHUNSUKE YAMANA

ABSTRACT. We construct the four-variable primitive p -adic L -functions associated with the triple product of Hida families and prove the explicit interpolation formulae at all critical points in the balanced range. Our construction is to carry out the p -adic interpolation of Garrett's integral representation of triple product L -functions via the p -adic Rankin-Selberg convolution method. The main novelty in this paper is the construction and the patching of four p -adic families of the pull-back of nearly holomorphic vector-valued Siegel-Eisenstein series on $\mathrm{GSp}(6)$. As an application, we obtain the cyclotomic p -adic L -function for the motive associated with the triple product of p -ordinary elliptic curves and prove the trivial zero conjecture for this motive. In particular, this proves the first cases of the Greenberg-Benois trivial zero conjecture where multiple trivial zeros are present and the Galois representation is not of $\mathrm{GL}(2)$ -type.

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1. INTRODUCTION

The aim of this paper is to construct the four-variable p -adic triple product L -functions for the triple product of Hida families of elliptic newforms with explicit interpolation formulae at all critical specializations in the balanced region. This extends the construction of three-variable p -adic L -functions in [GS20] and [Hsi21] by incorporating the cyclotomic variable. As an application, we prove the trivial zero conjecture for triple product of elliptic curves with semi-stable reduction at p . Let $p > 2$ be a prime, \mathcal{O} a valuation ring finite flat over \mathbf{Z}_p and \mathbf{I} a normal domain finite flat over the Iwasawa algebra $\Lambda = \mathcal{O}[[\Gamma]]$ of the topological group $\Gamma = 1 + p\mathbf{Z}_p$. Let

$$\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h})$$

be a triplet of primitive Hida families of tame conductor (N_1, N_2, N_3) and nebentypus (χ_1, χ_2, χ_3) with coefficients in \mathbf{I} . We will construct a four-variable Iwasawa function that interpolates the algebraic part of critical values of the triple product L -function attached to \mathbf{F} at all balanced critical specializations twisted by Dirichlet characters. Our formulae completely comply with the conjectural form described in [CPR89], [Coa89a] and [Coa89b]. In order to state our result precisely, we begin with some notation in Hida theory for elliptic modular forms and technical items such as the modified Euler factors at p .

Date: April 14, 2021.

2010 *Mathematics Subject Classification.* 11F67, 11F33.

Hsieh was partially supported by a MOST grant 108-2628-M-001-009-MY4. Yamana is partially supported by JSPS Grant-in-Aid for Scientific Research (C) 18K03210. This work is partially supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

1.1. Galois representations attached to Hida families. Given a field F , we denote its separable closure by \bar{F} and put $G_F = \text{Gal}(\bar{F}/F)$. If $\mathcal{F} = \sum_{n=1}^{\infty} \mathbf{a}(n, \mathcal{F})q^n \in \mathbf{I}[[q]]$ is a primitive cuspidal Hida family of tame conductor $N_{\mathcal{F}}$ and nebentypus $\chi_{\mathcal{F}}$, let $\rho_{\mathcal{F}} : G_{\mathbf{Q}} \rightarrow \text{GL}_2(\text{Frac } \mathbf{I})$ be the associated big Galois representation such that $\text{Tr } \rho_{\mathcal{F}}(\text{Frob}_{\ell}) = \mathbf{a}(\ell, \mathcal{F})$ for primes $\ell \nmid N_{\mathcal{F}}p$, where Frob_{ℓ} is the geometric Frobenius at ℓ and $V_{\mathcal{F}}$ is the natural realization of $\rho_{\mathcal{F}}$ inside the étale cohomology groups of modular curves. Thus $V_{\mathcal{F}}$ is a lattice in $(\text{Frac } \mathbf{I})^2$ with the continuous Galois action via $\rho_{\mathcal{F}}$, and the $G_{\mathbf{Q}_p}$ -invariant subspace $\text{Fil}^0 V_{\mathcal{F}} := V_{\mathcal{F}}^{I_p}$ fixed by the inertia group I_p at p is free of rank one over \mathbf{I} ([Oht00, Corollary, page 558]). A point $Q \in \text{Spec } \mathbf{I}(\bar{\mathbf{Q}}_p)$ is called an arithmetic point if $Q|_{\Gamma} : \Gamma \hookrightarrow \Lambda^{\times} \xrightarrow{Q} \bar{\mathbf{Q}}_p^{\times}$ is given by $Q(x) = x^{k_Q} \epsilon_Q(x)$ for some integer $k_Q \geq 2$ and a finite order character $\epsilon_Q : \Gamma \rightarrow \bar{\mathbf{Q}}_p^{\times}$. Let $\mathfrak{X}_{\mathbf{I}}^+$ be the set of arithmetic points of \mathbf{I} . For each arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$, the specialization $V_{\mathcal{F}_Q} := V_{\mathcal{F}} \otimes_{\mathbf{I}, Q} \bar{\mathbf{Q}}_p$ is the geometric p -adic Galois representation associated with the p -stabilized newform $\mathcal{F}_Q = \sum_{n=1}^{\infty} Q(\mathbf{a}(n, \mathcal{F}))q^n$.

1.2. Triple product L -functions. The p -adic cyclotomic character $\varepsilon_{\text{cyc}} : G_{\mathbf{Q}} \rightarrow \mathbf{Z}_p^{\times}$ is defined by $\sigma(\zeta) = \zeta^{\varepsilon_{\text{cyc}}(\sigma)}$ for every p -power root ζ of unity and $\sigma \in G_{\mathbf{Q}}$. We denote by \mathbf{Q}_{∞} the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q} , by $\omega : G_{\mathbf{Q}} \rightarrow \mu_{p-1} \hookrightarrow \mathbf{Z}_p^{\times}$ the Teichmüller character, and by $\langle \varepsilon_{\text{cyc}} \rangle_T : G_{\mathbf{Q}} \twoheadrightarrow \text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q}) \hookrightarrow \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]]^{\times}$ the universal cyclotomic character. Let

$$\mathbf{I}_3 = \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I}, \quad \mathbf{I}_4 = \mathbf{I}_3[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]]$$

be finite extensions of the three and four-variable Iwasawa algebras.

Fix $a \in \mathbf{Z}/(p-1)\mathbf{Z}$. The main object of this paper is a construction of the p -adic L -function for the triple tensor product Galois representation

$$\mathcal{V} = V_{\mathcal{F}} \widehat{\otimes}_{\mathcal{O}} V_{\mathcal{G}} \widehat{\otimes}_{\mathcal{O}} V_{\mathcal{H}}, \quad \mathbf{V} = \mathcal{V} \widehat{\otimes}_{\mathcal{O}} \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T$$

of rank eight over \mathbf{I}_4 . If (k_1, k_2, k_3) is a triplet of positive integers, we say (k_1, k_2, k_3) is balanced if $k_1 + k_2 + k_3 > 2k^*$ with $k^* := \max\{k_1, k_2, k_3\}$. Let $\mathfrak{X}_{\mathbf{I}_3}^{\text{bal}}$ denote the set of balanced arithmetic points of $(\mathfrak{X}_{\mathbf{I}_3}^+)^3$. An integer k is said to be critical for (k_1, k_2, k_3) if

$$k^* \leq k \leq k_1 + k_2 + k_3 - k^* - 2.$$

We define the weight space $\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} \subset \text{Spec } \mathbf{I}_4(\bar{\mathbf{Q}}_p)$ to be the set of balanced critical points of \mathbf{I}_4 given by

$$\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} = \{(Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_3}^{\text{bal}} \times \mathfrak{X}_{\Lambda}^+ \mid k_P \text{ is critical for } (k_{Q_1}, k_{Q_2}, k_{Q_3})\}.$$

For each point $(Q, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$, the specialization $\mathbf{V}_{(Q,P)} = \mathcal{V}_Q \otimes \varepsilon_{\text{cyc}}^{k_P} \epsilon_P \omega^{a-k_P}$ is a p -adic geometric Galois representation, where $\mathcal{V}_Q = V_{\mathcal{F}_{Q_1}} \otimes V_{\mathcal{G}_{Q_2}} \otimes V_{\mathcal{H}_{Q_3}}$ and ϵ_P is regarded as a Galois character via $\epsilon_P \circ \varepsilon_{\text{cyc}}$.

Next we briefly recall the motivic L -function associated with the specialization $\mathbf{V}_{(Q,P)}$. To the geometric p -adic Galois representation $\mathbf{V}_{(Q,P)}$, we can associate the Weil-Deligne representation $\text{WD}_{\ell}(\mathbf{V}_{(Q,P)})$ of the Weil-Deligne group of \mathbf{Q}_{ℓ} over $\bar{\mathbf{Q}}_p$ (See [Tat79, (4.2.1)] for $\ell \neq p$ and [Fon94, (4.2.3)] for $\ell = p$). Fixing an isomorphism $\iota_p : \bar{\mathbf{Q}}_p \simeq \mathbf{C}$ once and for all, we define the motive L -function of $\mathbf{V}_{(Q,P)}$ by the Euler product

$$L(\mathbf{V}_{(Q,P)}, s) = \prod_{\ell < \infty} L_{\ell}(\mathbf{V}_{(Q,P)}, s)$$

of the local L -factors $L_{\ell}(\mathbf{V}_{(Q,P)}, s)$ attached to $\text{WD}_{\ell}(\mathbf{V}_{(Q,P)}) \otimes_{\bar{\mathbf{Q}}_p, \iota_p} \mathbf{C}$ (cf. [Del79, (1.2.2)], [Tay04, page 85]). On the other hand, we denote by $\pi_{\mathbf{f}_{Q_1}}$ (resp. $\pi_{\mathbf{g}_{Q_2}}, \pi_{\mathbf{h}_{Q_3}}$) the irreducible unitary cuspidal automorphic representation of $\text{GL}_2(\mathbf{A})$ associated with \mathbf{f}_{Q_1} (resp. $\mathbf{g}_{Q_2}, \mathbf{h}_{Q_3}$). Let $L(s, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P})$ be the automorphic L -function attached to the triple product of $\pi_{\mathbf{f}_{Q_1}}$, $\pi_{\mathbf{g}_{Q_2}}$, and $\pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P}$, as constructed by Garrett [Gar87] in the classical setting and by Piatetski-Shapiro and Rallis [PSR87] in the adèlic setting. The analytic theory of $L(s, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P})$ such as meromorphic continuation and a functional equation has been explored extensively in the literatures (cf. [PSR87, Ike89, Ike92]), and thanks to [Ram00, Theorem 4.4.1], we have

$$L(s + k_P - w_{\underline{Q}}/2, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P}) = \Gamma_{\mathbf{V}_{(Q,P)}}(s) \cdot L(\mathbf{V}_{(Q,P)}, s),$$

where $w_{\underline{Q}} := k_{Q_1} + k_{Q_2} + k_{Q_3} - 3$ and $\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(s)$ is the Gamma factor of $\mathbf{V}_{(\underline{Q},P)}$ as given by

$$\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(s) := \Gamma_{\mathbf{C}}(s + k_P) \Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_1}) \Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_2}) \Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_3}).$$

Here $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$. Hence we have a good understanding of the analytic properties of the motivic L -function $L(\mathbf{V}_{(\underline{Q},P)}, s)$. The rationality of its critical L -values in the balanced region was proved in [Orl87] and [GH93], where the authors verify that the Deligne's period for $\mathbf{V}_{(\underline{Q},P)}$ is the product of Petersson norms of \mathbf{f}_{Q_1} , \mathbf{g}_{Q_2} , \mathbf{h}_{Q_3} . In this article we shall investigate the arithmetic of critical values $L(\mathbf{V}_{(\underline{Q},P)}, 0)$ for $(\underline{Q}, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ and study the p -adic analytic behavior of its algebraic part viewed as a function on the weight space $\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$.

1.3. The modified Euler factors at p and ∞ . Let $G_{\mathbf{Q}_p}$ denote the decomposition group at p . Define the rank four $G_{\mathbf{Q}_p}$ -invariant subspace of \mathbf{V} by

$$\text{Fil}^+ \mathbf{V} := \text{Fil}^+ \mathcal{V} \otimes \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T,$$

where

$$\text{Fil}^+ \mathcal{V} := \text{Fil}^0 V_{\mathbf{f}} \otimes \text{Fil}^0 V_{\mathbf{g}} \otimes V_{\mathbf{h}} + V_{\mathbf{f}} \otimes \text{Fil}^0 V_{\mathbf{g}} \otimes \text{Fil}^0 V_{\mathbf{h}} + \text{Fil}^0 V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes \text{Fil}^0 V_{\mathbf{h}}.$$

The pair $(\text{Fil}^+ \mathbf{V}, \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}})$ satisfies the *Panchishkin condition* in [Gre94a, page 217] in the sense that for each arithmetic point $(\underline{Q}, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$, the Hodge-Tate numbers of $\text{Fil}^+ \mathbf{V}_{(\underline{Q},P)}$ are all positive, while none of the Hodge-Tate numbers of $\mathbf{V}_{(\underline{Q},P)}/\text{Fil}^+ \mathbf{V}_{(\underline{Q},P)}$ is positive. Here the Hodge-Tate number of $\mathbf{Q}_p(1)$ is one in our convention. Now we can define the modified p -Euler factor by

$$\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q},P)}) := \frac{L_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q},P)}, 0)}{\varepsilon(\text{WD}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q},P)})) \cdot L_p((\text{Fil}^+ \mathbf{V}_{(\underline{Q},P)})^\vee, 1)} \cdot \frac{1}{L_p(\mathbf{V}_{(\underline{Q},P)}, 0)}.$$

We note that this modified p -Euler factor is precisely the ratio between the factor $\mathcal{L}_p^{(\rho)}(\mathbf{V}_{(\underline{Q},P)})$ in [Coa89b, page 109, (18)] and the local L -factor $L_p(\mathbf{V}_{(\underline{Q},P)}, 0)$.

In the theory of p -adic L -functions, we also need the modified Euler factor $\mathcal{E}_\infty(\mathbf{V}_{(\underline{Q},P)})$ at the archimedean place observed by Deligne. It is defined to be the ratio between the factor $\mathcal{L}_\infty^{(\sqrt{-1})}(\mathbf{V}_{(\underline{Q},P)})$ in [Coa89b, page 103 (4)] and the Gamma factor $\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(0)$. In our current case it is explicitly given by

$$\mathcal{E}_\infty(\mathbf{V}_{(\underline{Q},P)}) = (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3}.$$

1.4. The p -modified periods. To give the precise definition of periods for the motive $\mathbf{V}_{(\underline{Q},P)}$, we introduce the p -modified period of an \mathbf{I} -adic primitive cuspidal Hida family \mathcal{F} of tame conductor $N_{\mathcal{F}}$. We denote by \mathcal{F}_Q° the normalized newform of weight k_Q , conductor $N_Q = N_{\mathcal{F}} p^{n_Q}$ with nebentypus χ_Q corresponding to \mathcal{F}_Q . There is a unique decomposition $\chi_Q = \chi'_Q \chi_{Q,(p)}$, where χ'_Q and $\chi_{Q,(p)}$ are Dirichlet characters modulo $N_{\mathcal{F}}$ and p^{n_Q} respectively. Let $\alpha_Q = \mathbf{a}(p, \mathcal{F}_Q)$. Define the modified Euler factor $\mathcal{E}_p(\mathcal{F}_Q, \text{Ad})$ for the adjoint motive of \mathcal{F}_Q by

$$\mathcal{E}_p(\mathcal{F}_Q, \text{Ad}) = \begin{cases} (1 - \alpha_Q^{-2} \chi_Q(p) p^{k_Q-1})(1 - \alpha_Q^{-2} \chi_Q(p) p^{k_Q-2}) & \text{if } n_Q = 0, \\ -1 & \text{if } n_Q = 1, \chi_{Q,(p)} = 1 \text{ (so } k_Q = 2), \\ (\alpha_Q^{-2} p^{(k_Q-2)})^{n_Q} \mathfrak{g}(\chi_{Q,(p)}) \chi_{Q,(p)}(-1) & \text{if } n_Q > 0, \chi_{Q,(p)} \neq 1. \end{cases}$$

Here $\mathfrak{g}(\chi_{Q,(p)})$ is the usual Gauss sum. Fixing the choice of a generator $\eta_{\mathcal{F}}$ and letting $\|\mathcal{F}_Q^\circ\|_{\Gamma_0(N_Q)}^2$ be the usual Petersson norm of \mathcal{F}_Q° , we define the p -modified period $\Omega_{\mathcal{F}_Q}^\dagger$ of \mathcal{F} at Q by

$$\Omega_{\mathcal{F}_Q}^\dagger := (-2\sqrt{-1})^{k_Q+1} \cdot \|\mathcal{F}_Q^\circ\|_{\Gamma_0(N_Q)}^2 \cdot \mathcal{E}_p(\mathcal{F}_Q, \text{Ad}) \in \mathbf{C}^\times.$$

By [Hid16, Corollary 6.24, Theorem 6.28], one can show that for each arithmetic point Q , up to a p -adic unit, the product of the normalized period $\Omega_{\mathcal{F}_Q}^\dagger$ and the congruence number of \mathcal{F}_Q is equal to the product of the plus/minus canonical periods $\Omega(+; \mathcal{F}_Q^\circ) \Omega(-; \mathcal{F}_Q^\circ)$ introduced in [Hid94, page 488].

1.5. **Statement of the main result.** We impose the following technical assumption:

(sf) N_i is square-free and $\chi_i = \omega^{a_i}$ is a power of the Teichmüller character for $i = 1, 2, 3$.

Our main result is a construction of the balanced p -adic triple product L -functions with the precise interpolation formulae at all critical points:

Theorem A. *Suppose that $p > 2$ and (sf) holds. Then for each $a \in \mathbf{Z}/(p-1)\mathbf{Z}$, there exists an element*

$$\mathcal{L}_{\mathbf{F},(a)} \in \mathbf{I}_3[[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \otimes_{\mathbf{I}_3} (\mathrm{Frac} \mathbf{I} \otimes \mathrm{Frac} \mathbf{I} \otimes \mathrm{Frac} \mathbf{I})$$

such that

- for each balanced critical $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\mathrm{bal}}$,

$$\mathcal{L}_{\mathbf{F},(a)}(\underline{Q}, P) = \frac{\Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(0) L(\mathbf{V}_{(\underline{Q}, P)}, 0)}{\Omega_{\mathbf{f}_{Q_1}}^\dagger \Omega_{\mathbf{g}_{Q_2}}^\dagger \Omega_{\mathbf{h}_{Q_3}}^\dagger} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{(\underline{Q}, P)});$$

- for any H_1, H_2 and H_3 in the congruence ideals of \mathbf{f}, \mathbf{g} and \mathbf{h} ,

$$H_1 H_2 H_3 \cdot \mathcal{L}_{\mathbf{F},(a)} \in \mathbf{I}_4.$$

In the literature, the three weight variable p -adic L -function for the triple product of Hida families in the balanced case has been extensively studied by Greenberg-Seveso [GS20], the first author [Hsi21] and so on. These works, based on Ichino's formula [Ich08], focus on the p -adic interpolation of central values and hence the cyclotomic variable is excluded. Our four-variable p -adic L -function $\mathcal{L}_{\mathbf{F},(a)}$ specializes to this three variable p -adic L -function along the central critical line (see Remark 7.8). The first attempt to construct the cyclotomic p -adic triple product L -functions was made by Böcherer and Panchishkin [BP06, BP09], where they constructed one-variable p -adic L -functions associated with three primitive elliptic newforms by using the pull-back of Siegel-Eisenstein series on $\mathrm{GSp}(6)$ and Garrett's integral representation of triple product L -functions. Their construction is not restricted to the ordinary case but the p -integrality of the p -adic L -function is not discussed, and the interpolation formula is less explicit and not complete for the interpolation at the trivial character is missing and not all of the critical points are covered. To obtain an explicit and complete interpolation formula, we introduce new p -adic sections and use vector-valued archimedean sections in the construction of Siegel-Eisenstein series on $\mathrm{GSp}(6)$.

Remark 1.1. We put a few words about our assumption.

- (1) The assumption (sf) is imposed due to some computational difficulty. To obtain the explicit interpolation formulae, we explicitly compute the local zeta integral for new vectors of unramified principal series and the Steinberg representation in §3. If one is able to choose a nice local test vector and compute the local zeta integral for other representations so that (i) it is non-vanishing or hopefully coincides with local L -factor and (ii) the degenerate Whittaker coefficients have a simple formula, then the assumption (sf) can be weakened. For example, one may choose the section in §2.3 for the generic subrepresentation of arbitrary principal series and apply Proposition 2.3 at the cost of having an incomplete local L -factor.
- (2) The p -adic L -function $\mathcal{L}_{\mathbf{F},(a)}$ may have some poles outside critical points. Under a hypothesis on the Gorensteiness of Hecke algebras, we can construct an optimal four-variable p -adic triple product L -function without denominators by multiplying generators of the congruence ideals of \mathbf{f}, \mathbf{g} , and \mathbf{h} (see Proposition 7.6).

1.6. **Application to the trivial zero conjecture.** Let E_i be a p -ordinary semi-stable elliptic curve over the rationals \mathbf{Q} for $i = 1, 2, 3$. We write $L(\mathbf{E}, s)$ for the degree eight motivic L -function for the triple product

$$(1.1) \quad \mathbf{V}_{\mathbf{E}} = \mathrm{H}_{\acute{e}t}^1(E_1/\overline{\mathbf{Q}}, \mathbf{Q}_p) \otimes \mathrm{H}_{\acute{e}t}^1(E_2/\overline{\mathbf{Q}}, \mathbf{Q}_p) \otimes \mathrm{H}_{\acute{e}t}^1(E_3/\overline{\mathbf{Q}}, \mathbf{Q}_p)$$

realized in the middle cohomology of the abelian variety $\mathbf{E} = E_1 \times E_2 \times E_3$ by the Künneth formula. Hence

$$L(\mathrm{H}_{\acute{e}t}^3(\mathbf{E}/\overline{\mathbf{Q}}, \mathbf{Q}_p), s) = L(\mathbf{E}, s) \prod_{i=1}^3 L(E_i, s-1)^2.$$

Our four-variable p -adic L -function yields a cyclotomic p -adic L -function

$$L_p(\mathbf{E}) \in \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \otimes \mathbf{Q}_p,$$

which roughly interpolates the algebraic part of central values $\frac{L(\mathbf{E} \otimes \chi, 2)}{\Omega}$ with a fixed period Ω for all finite order characters χ of $\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Define an analytic function $L_p(\mathbf{E}, s) := \varepsilon_{\text{cyc}}^{s-2}(L_p(\mathbf{E}))$ for $s \in \mathbf{Z}_p$ (See Proposition 8.2 for the precise statement). The Euler-like factor $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{\mathbf{E}}(2))$ can possibly vanish. In this case the interpolation formula forces $L_p(\mathbf{E}, 2)$ to be zero. Such a zero is called a trivial zero. For example, a trivial zero appears if all E_i have split multiplicative reduction at p (see Remark 8.3). In this particular case, the trivial zero conjecture predicts that the leading coefficient of $L_p(\mathbf{E}, s)$ is the product of the \mathcal{L} -invariants for E_i and the algebraic part of the complex central value $L(\mathbf{E}, 2)$ (cf. [Gre94b, (25), p. 166] and [Ben11, p. 1579]). We recall that if E/\mathbf{Q} is an elliptic curve with split multiplicative reduction at p , denote by $\mathcal{L}_p(E) = \frac{\log_p q_E}{\text{ord}_p q_E}$ the \mathcal{L} -invariant of E with Tate's p -adic period q_E attached to E . We construct several improved p -adic triple product L -functions and apply the idea of Greenberg-Stevens [GS93] and [BDJ] to establish the trivial zero conjecture for the triple product of elliptic curves. The following result is a special case of our more general result (see Theorem 8.4).

Theorem B. *Suppose that E_1 has split multiplicative reduction at p .*

(1) *If E_2 and E_3 are both split multiplicative at p , then $L_p(\mathbf{E}, s)$ has at least a triple zero at $s = 2$, and*

$$\lim_{s \rightarrow 2} \frac{L_p(\mathbf{E}, s)}{(s-2)^3} = -p \prod_{i=1}^3 \mathcal{L}_p(E_i) \cdot \frac{L(\mathbf{E}, 2)}{\Omega}.$$

(2) *If E_2 and E_3 are both good ordinary at p with $a_p(E_2) = a_p(E_3)$, where*

$$a_p(E_i) = 1 + p - \#E_i(\mathbb{F}_p),$$

then $L_p(\mathbf{E}, s)$ has at least a double zero at $s = 2$ and

$$\lim_{s \rightarrow 2} \frac{L_p(\mathbf{E}, s)}{(s-2)^2} = (-p\alpha_2^{-2})(1 - \alpha_2^{-2})^2 \mathcal{L}_p(E_1) \cdot \frac{L(\mathbf{E}, 2)}{\Omega},$$

where α_2 is the unit root of the Hecke polynomial $X^2 - a_p(E_2)X + p$ of E_2 .

In the case of the p -adic L -function $L_p(E, s)$ of an elliptic curve E over \mathbf{Q} the trivial zero arises if and only if E is split multiplicative at p . An analogous formula for $L'_p(E, 1)$ was experimentally discovered in [MTT86] and proved in [GS93], and for Hilbert modular forms in [Mok09], [Spi14] and [BDJ]. Later Greenberg formulated a more general trivial zero conjecture for ordinary motives [Gre94b], and Benois further extended this conjecture to semistable representations [Ben11]. The non-vanishing of \mathcal{L} -invariants $\mathcal{L}(E_i)$ is known thanks to [BSDGP96]. Our result thus proves the first cases of the Greenber-Benois trivial zero conjecture where multiple trivial zeros are present and the Galois representation is not of $\text{GL}(2)$ -type.

Remark 1.2. We only focus on the p -ordinary case in this paper. For the general case, one may expect to combine the construction of Böcherer and Panchishkin with the method in [AI21] to obtain a four-variable p -adic L -functions for the triple product of p -adic families of modular forms of finite slopes and investigate the trivial zeros for the triple product of modular forms of finite slopes. We hope to come back to this point in the future.

1.7. The construction of $\mathcal{L}_{\mathbf{F},(a)}$. We give a sketch of the construction of $\mathcal{L}_{\mathbf{F},(a)}$. Our method is the combination of Garrett's integral representation of the triple product L -function, an integrality result of critical L -values for triple products in [Miz90] and Hida's p -adic Rankin-Selberg method. We begin with a construction of the four-variable p -adic family of the pull-back of Siegel-Eisenstein series. For each point $x = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$, we reorder the weights $\{k_{Q_1}, k_{Q_2}, k_{Q_3}\} = \{k_x, l_x, m_x\}$ so that $k_x \geq l_x \geq m_x$. For each $\nu_1, \nu_2 \in \{0, 1\}$, we put

$$\mathfrak{X}_{(\nu_1, \nu_2)}^{\text{bal}} = \{x \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} \mid k_x \equiv l_x + \nu_1 \equiv m_x + \nu_2 \pmod{2}\}.$$

Hence we have the partition of the weight space

$$\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} = \coprod_{\nu_1, \nu_2 \in \{0, 1\}} \mathfrak{X}_{(\nu_1, \nu_2)}^{\text{bal}}.$$

Let $N = \text{lcm}(N_1, N_2, N_3)$. When χ is a Dirichlet character, we write $c(\chi)$ for the exponent of the p -part of its conductor. For each $x = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{(\nu_1, \nu_2)}^{\text{bal}}$ we shall construct a nearly holomorphic vector-valued Siegel-Eisenstein series $\mathbf{E}_x^{(\nu_1, \nu_2)}(Z, s)$ of degree three whose restriction to the three-fold product \mathfrak{H}_1^3 of the upper half-plane \mathfrak{H}_1 is a modular form of weight $(k_x, k_x - \nu_1, k_x - \nu_2)$ and level $\Gamma_1(Np^n)^3$, where $n = \max\{1, c(\epsilon_{Q_1}), c(\epsilon_{Q_2}), c(\epsilon_{Q_3}), c(\epsilon_P)\}$. We consider the pull-back given by

$$G_x^{(\nu_1, \nu_2)}(z_1, z_2, z_3) := e_{\text{ord}} \text{Hol} \left(\lambda_{z_2}^{\frac{k_x - l_x - \nu_1}{2}} \lambda_{z_3}^{\frac{k_x - m_x - \nu_2}{2}} \mathbf{E}_x^{(\nu_1, \nu_2)} \left(\text{diag}(z_1, z_2, z_3), k_P - \frac{w_Q + 1}{2} \right) \right),$$

where $\lambda_z := -\frac{1}{2\pi\sqrt{-1}}(\text{Im } z)^2 \frac{\partial}{\partial \bar{z}}$ is the weight-lowering differential operator, Hol is the holomorphic projection and e_{ord} is Hida's ordinary projector. Then we show that $G_x^{(\nu_1, \nu_2)}$ is an ordinary cusp form of weight (k_x, l_x, m_x) on \mathfrak{H}_1^3 the product of three copies of the upper half plane.

The most crucial and surprising point is that the four classes of vector-valued Siegel-Eisenstein series $\mathbf{E}_x^{(\nu_1, \nu_2)}$ can be constructed so that the pull-backs $G_x^{(\nu_1, \nu_2)}$ can be put into a single four-variable Hida family of triple product modular form due to a miraculous effect of the ordinary projector on the Fourier expansions of the pull-back of our vector-valued Siegel Eisenstein series (See the proof of Proposition 6.3). More precisely, let $\mathbf{S}^{\text{ord}}(N, \chi)$ denote the space of ordinary Λ -adic modular forms of tame level N and character χ . In the following we will associate to $a \in \mathbf{Z}/(p-1)\mathbf{Z}$ and $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$ an explicit triple product ordinary Λ -adic form

$$\mathcal{G}_{\underline{\chi}}^{(a)} \in \mathbf{S}^{\text{ord}}(N, \chi_1, \mathbf{Z}_p[[X_1]]) \hat{\otimes}_{\mathbf{Z}_p} \mathbf{S}^{\text{ord}}(N, \chi_2, \mathbf{Z}_p[[X_2]]) \hat{\otimes}_{\mathbf{Z}_p} \mathbf{S}^{\text{ord}}(N, \chi_3, \mathbf{Z}_p[[X_3]]) \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]].$$

By an explicit calculation of Fourier coefficients of $G_x^{(\nu_1, \nu_2)}$, we prove in Proposition 6.8 that the specialization $\mathcal{G}_{\underline{\chi}}^{(a)}(x)$ at every $x \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ is the q -expansion of $G_x^{(\nu_1, \nu_2)}$.

Let T_3^+ be the set of positive definite half-integral matrices of size 3. The Siegel series attached to $B \in T_3^+$ and a rational prime ℓ is defined by

$$b_{\ell}(B, s) = \sum_{z \in \text{Sym}_3(\mathbf{Q}_{\ell}) / \text{Sym}_3(\mathbf{Z}_{\ell})} \psi(-\text{tr}(Bz)) \nu[z]^{-s},$$

where ψ is an arbitrarily fixed additive character on \mathbf{Q}_{ℓ} of order 0 and $\nu[z]$ is the product of denominators of elementary divisors of z . There exists a polynomial $F_{B, \ell}(X) \in \mathbf{Z}[X]$ such that

$$b_{\ell}(T, s) = (1 - \ell^{-s})(1 - \ell^{2-2s})F_{B, \ell}(\ell^{-s}).$$

Let $z \mapsto [z]$ denote the inclusion of group-like elements $1 + p\mathbf{Z}_p \hookrightarrow \mathbf{Z}_p[[1 + p\mathbf{Z}_p]]^{\times}$. Fix a topological generator $\mathbf{u} \in 1 + p\mathbf{Z}_p$ and identify $\mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$ with $\mathbf{Z}_p[[X]]$, where $X = [\mathbf{u}] - 1$. Define a character $\langle \cdot \rangle : \mathbf{Z}_p^{\times} \rightarrow 1 + p\mathbf{Z}_p$ by $\langle x \rangle = x\omega(x)^{-1}$ and write $\langle x \rangle_X = [\langle x \rangle] = (1 + X)^{\log_p z / \log_p \mathbf{u}} \in \mathbf{Z}_p[[X]]$. Let Ξ_p be a set of symmetric matrices of size 3 over \mathbf{Z}_p whose off-diagonal entries are p -units but whose diagonal entries are not. Now the seven-variable formal power series is presented by

$$\mathcal{G}_{\underline{\chi}}^{(a)} = \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) \cdot \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where $\mathcal{Q}_B^{(a)}, \mathcal{F}_B^{(a)} \in \mathbf{Z}_p[[X_1, X_2, X_3, T]]$ are given by

$$\begin{aligned} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) &= \frac{\omega^a(8b_{23}b_{31}b_{12}) \langle 8b_{23}b_{31}b_{12} \rangle_T}{\chi_1(2b_{23})\chi_2(2b_{31})\chi_3(2b_{12}) \langle 2b_{23} \rangle_{X_1} \langle 2b_{31} \rangle_{X_2} \langle 2b_{12} \rangle_{X_3}}, \\ \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) &= \prod_{\ell \nmid pN} F_{B, \ell}(\langle \ell \rangle_T^{-2} (\omega^{-2a} \chi_1 \chi_2 \chi_3)(\ell) \langle \ell \rangle_{X_1} \langle \ell \rangle_{X_2} \langle \ell \rangle_{X_3} \ell^{-4}). \end{aligned}$$

Now we apply the p -adic Rankin-Selberg method to define the p -adic L -function. Denote by $\mathbf{T}(N, \mathbf{I})$ the \mathbf{I} -algebras generated by Hecke operators on the space of ordinary Λ -adic cusp forms of level N . For each $? \in \{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$ we write $\mathbf{1}_? \in \mathbf{T}(N, \mathbf{I}) \otimes_{\mathbf{I}} \text{Frac } \mathbf{I}$ for the idempotent corresponding to $?$. We define

$$\check{\mathcal{L}}_{\mathbf{F}, (a)} := \text{the first Fourier coefficient of } \mathbf{1}_{\mathbf{f}} \otimes \mathbf{1}_{\mathbf{g}} \otimes \mathbf{1}_{\mathbf{h}} (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\underline{\chi}}^{(a)})) \in \mathbf{I}_3[[T]],$$

where $\text{Tr}_{N/N_i} : \mathbf{S}^{\text{ord}}(N, \chi_i, \mathbf{I}) \rightarrow \mathbf{S}^{\text{ord}}(N_i, \chi_i, \mathbf{I})$ is the usual trace map, and then the p -adic triple product L -function is defined to be $\mathcal{L}_{\mathbf{F}, (a)} = \check{\mathcal{L}}_{\mathbf{F}, (a)} \cdot \mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}^{-1}$, where $\mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3} \in \mathbf{I}_4^{\times}$ is a fudge factor which is

essentially a product of epsilon factors at prime-to- p finite places. The p -adic Rankin-Selberg method tells us that the interpolation formula for the value $\check{\mathcal{L}}_{\mathbf{F},(a)}(x)$ at $x \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ is roughly given by

$$\lim_{s \rightarrow k_P - \frac{w_{Q_1} + 1}{2}} \eta_{\mathbf{f}_{Q_1}} \eta_{\mathbf{g}_{Q_2}} \eta_{\mathbf{h}_{Q_3}} \frac{\langle \mathbf{f}_{Q_1} \otimes \mathbf{g}_{Q_2} \otimes \mathbf{h}_{Q_3}, \mathbf{E}_x^{(\nu_1, \nu_2)}(s) \rangle}{\|\mathbf{f}_{Q_1}\|^2 \|\mathbf{g}_{Q_2}\|^2 \|\mathbf{h}_{Q_3}\|^2}$$

(cf. Lemma 7.3), where $\langle \cdot, \cdot \rangle$ is the Petersson pairing on \mathfrak{H}_1^3 and $\|\cdot\|$ is the Petersson norm on \mathfrak{H}_1 . The series $\mathbf{E}_x^{(\nu_1, \nu_2)}(Z, s)$ is constructed from a factorizable section of a certain family of induced representations. By means of the generalization of Garrett's work, carried out in [PSR87, Ike89] (see Lemma 7.1) the pairing can be unfolded and written as a product of $L(s + \frac{1}{2}, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P})$ and the normalized local zeta integrals at primes dividing pN . It turns out that these local zeta integrals are essentially given by the modified Euler factor $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)})$ at p and the local epsilon factors $\mathbf{f}_{\chi, a, N_1, N_2, N_3}$ at primes $\ell | N$. In both calculations the key ingredients are Lemma 2.1 and the local functional equations for GL_1 and GL_2 , by which we can generalize Proposition 4.2 of [GK92] without brute force calculations (see Remark 3.3).

This paper is organized as follows. In §2, §3 and §4, we make the choices of local datum for Siegel Eisenstein series $\mathbf{E}_x^{(\nu_1, \nu_2)}(Z, s)$ and carry out the explicit computation of local zeta integrals that appear in Garrett's integral representation of triple product L -functions. After preparing some notation in Hida theory in §5, we show that the Fourier expansion of $G_x^{(\nu_1, \nu_2)}$ can be p -adically interpolated by the power series $\mathcal{G}_x^{(a)}$ in §6. The key ingredient is Proposition 6.3 about the computation of Fourier coefficients of $G_x^{(\nu_1, \nu_2)}$. In §7, we put all the local computations in §2, 3, and 4 and prove the main interpolation formulae in Theorem 7.6. Finally, in §8 we construct some improved p -adic L -functions in Lemmas 8.5 and 8.6 and use them to prove the trivial zero conjecture for the triple product of elliptic curves in Theorem 8.4.

Notation. The following notations will be used frequently throughout the paper. For an associative ring R with identity element, we denote by R^\times the group of all its invertible elements, and by $M_{m,n}(R)$ the module of all $m \times n$ matrices with entries in R . Put $M_n(R) = M_{n,n}(R)$ and $\text{GL}_n(R) = M_n(R)^\times$ particularly when we view the set as a ring. The identity and zero elements of the ring $M_n(R)$ are denoted by $\mathbf{1}_n$ and $\mathbf{0}_n$ (when n needs to be stressed) respectively. The transpose of a matrix x is denoted by x^t . Let $\text{Sym}_n(R) = \{z \in M_n(R) \mid z^t = z\}$ be the space of symmetric matrices of size n over R . For any set X we denote by $\mathbb{1}_X$ the characteristic function of X . When X is a finite set, we denote by $\sharp X$ the number of elements in X . When X is a totally disconnected locally compact topological space or a smooth real manifold, we write $\mathcal{S}(X)$ for the space of Schwartz-Bruhat functions on X . If x is a real number, then we put $\lceil x \rceil = \max\{i \in \mathbf{Z} \mid i \leq x\}$.

If R is a commutative ring and $G = \text{GL}_2(R)$, we denote by ρ the right translation of G on the space of \mathbf{C} -valued functions on G . Thus $(\rho(g)f)(g') = f(g'g)$. We write $\mathbf{1} : G \rightarrow \mathbf{C}$ for the constant function $\mathbf{1}(g) = 1$. For a function $f : G \rightarrow \mathbf{C}$ and a character $\chi : R^\times \rightarrow \mathbf{C}^\times$, let $f \otimes \chi : G \rightarrow \mathbf{C}$ denote the function $f \otimes \chi(g) = f(g)\chi(\det g)$.

2. COMPUTATION OF THE LOCAL ZETA INTEGRAL: THE p -ADIC CASE

2.1. The local zeta integral. Let T_n be the subgroup of diagonal matrices in GL_n , U_n the subgroup of upper triangular unipotent matrices in GL_n , Z_n the subgroup of scalar matrices in GL_n and $B_n = T_n U_n$ the standard Borel subgroup of GL_n . The symplectic similitude group of degree n is defined by

$$\text{GSp}_{2n} = \{g \in \text{GL}_{2n} \mid gJ_n g^t = \nu_n(g)J_n, \nu_n(g) \in \text{GL}_1\}, \quad J_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

We define the homomorphisms

$$\mathbf{m} : \text{GL}_n \times \text{GL}_1 \rightarrow \text{GSp}_{2n}, \quad \mathbf{n}, \mathbf{n}^- : \text{Sym}_n \rightarrow \text{GSp}_{2n}$$

by

$$\mathbf{m}(A, \nu) = \begin{pmatrix} A & 0 \\ 0 & \nu(A^t)^{-1} \end{pmatrix}, \quad \mathbf{n}(z) = \begin{pmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{pmatrix}, \quad \mathbf{n}^-(z) = \begin{pmatrix} \mathbf{1}_n & 0 \\ z & \mathbf{1}_n \end{pmatrix}.$$

We write

$$\mathbf{m}(A) = \mathbf{m}(A, 1), \quad \mathbf{d}(\nu) = \mathbf{m}(\mathbf{1}_n, \nu).$$

A maximal parabolic subgroup $\mathcal{P}_n = \mathcal{M}_n N_n$ of GSp_{2n} is defined by

$$\mathcal{M}_n = \mathbf{m}(\mathrm{GL}_n \times \mathrm{GL}_1), \quad N_n = \mathbf{n}(\mathrm{Sym}_n).$$

Define algebraic groups of $U^0 \subset U \subset H$ by

$$\begin{aligned} H &= \{(g_1, g_2, g_3) \in (\mathrm{GL}_2)^3 \mid \det g_1 = \det g_2 = \det g_3\}, \\ U &= \{(\mathbf{n}(x_1), \mathbf{n}(x_2), \mathbf{n}(x_3)) \mid x_1, x_2, x_3 \in \mathbf{M}_1\}, \\ U^0 &= \{(\mathbf{n}(x_1), \mathbf{n}(x_2), \mathbf{n}(x_3)) \mid x_1 + x_2 + x_3 = 0\}. \end{aligned}$$

We define the embedding $\iota : H \hookrightarrow \mathrm{GSp}_6$ by

$$\iota \left(\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \right) \right) = \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ & & a_3 & b_3 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \\ & & c_3 & d_3 \end{array} \right).$$

We identify $Z = Z_6$ with the center of GSp_6 . Put

$$\eta = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right).$$

Let F be a local field of characteristic zero. In the nonarchimedean case F contains a ring \mathfrak{o} of integers having a single prime ideal \mathfrak{p} and the absolute value $\alpha_F = |\cdot|$ on F is normalized via $|\varpi| = q^{-1}$ for any generator ϖ of \mathfrak{p} , where q denotes the order of the residue field $\mathfrak{o}/\mathfrak{p}$. Fix an additive character ψ on F which is trivial on \mathfrak{o} but non-trivial on \mathfrak{p}^{-1} . When $F = \mathbf{R}$, we define $\psi(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbf{R}$.

Let K be a standard maximal compact subgroup of $\mathrm{GSp}_6(F)$. For quasi-characters $\hat{\omega}, \chi : F^\times \rightarrow \mathbf{C}^\times$ we let $I_3(\hat{\omega}, \chi) := \mathrm{Ind}_{\mathcal{P}_3}^{\mathrm{GSp}_6(F)} \chi^2 \hat{\omega} \boxtimes \chi^{-3} \hat{\omega}^{-1}$ be the space of all right K -finite functions f on $\mathrm{GSp}_6(F)$ which satisfy

$$f(\mathbf{m}(A, \lambda)\mathbf{n}(z)g) = \hat{\omega}(\lambda^{-2} \det A) \chi(\lambda^{-3} (\det A)^2) |\lambda^{-3} (\det A)^2| f(g)$$

for $A \in \mathrm{GL}_3(F)$, $\lambda \in F^\times$, $z \in \mathrm{Sym}_3(F)$ and $g \in \mathrm{GSp}_6(F)$. The group $\mathrm{GSp}_6(F)$ acts on $I_3(\hat{\omega}, \chi)$ by right translation ρ_3 . It is important to note that for $t = \mathrm{diag}(a, d) \in T_2$

$$(2.1) \quad f(\eta t(g_1, g_2, g_3)) = \hat{\omega}(d)^{-1} \chi(ad^{-1}) |ad^{-1}| f(\eta t(g_1, g_2, g_3)).$$

It is well worthy of notice that

$$(2.2) \quad I_3(\hat{\omega}, \chi) \otimes \mu \circ \nu_3 = I_3(\hat{\omega}\mu^{-2}, \chi\mu).$$

We call a K -finite function $(s, g) \mapsto f_s(g)$ on $\mathbf{C} \times \mathrm{GSp}_6(F)$ a holomorphic section of $I_3(\hat{\omega}, \chi\alpha_F^s)$ if $f_s(g)$ is holomorphic in s for each $g \in \mathrm{GSp}_6(F)$ and $f_s \in I_3(\hat{\omega}, \chi\alpha_F^s)$ for each $s \in \mathbf{C}$. We associate to a non-degenerate symmetric matrix B of size 3 the degenerate Whittaker functional

$$\mathcal{W}_B : I_3(\hat{\omega}, \chi\alpha_F^s) \rightarrow \mathbf{C}, \quad \mathcal{W}_B(f_s) = \int_{\mathrm{Sym}_3(F)} f_s(J_3 \mathbf{n}(z)) \psi(-\mathrm{tr}(Bz)) dz.$$

The integral converges if $\mathrm{Re} s$ is sufficiently large and can be continued to an entire function.

Given an irreducible admissible infinite dimensional representation π of $\mathrm{GL}_2(F)$, we denote by $\mathscr{W}(\pi)$ the Whittaker model of π with respect to ψ . Let π_1, π_2, π_3 be a triplet of irreducible admissible infinite dimensional representations of $\mathrm{GL}_2(F)$. We denote the central character of π_i by ω_i . Set $\hat{\omega} = \omega_1 \omega_2 \omega_3$. We associate to a holomorphic section f_s of $I(\hat{\omega}, \chi\alpha_F^s)$ and Whittaker functions $W_i \in \mathscr{W}(\pi_i)$ the local zeta integral

$$Z(W_1, W_2, W_3, f_s) = \int_{U^0 Z \backslash H} W_1(g_1) W_2(g_2) W_3(g_3) f_s(\eta t(g_1, g_2, g_3)) dg_1 dg_2 dg_3,$$

which converges absolutely if $\mathrm{Re} s$ is sufficiently large.

We define a map $\iota_0 : H \hookrightarrow \mathrm{GSp}_6$ by

$$\iota_0(g_1, g_2, g_3) = \boldsymbol{\eta}^\iota(g_1, g_2 J_1, g_3 J_1).$$

As a preliminary step, we choose a coordinate system on an open dense subset of $U^0 Z \backslash H$.

Lemma 2.1. *If $(x_1, u_1, u_2, u_3, a_2, a_3) \in F^4 \oplus F^{\times 2}$, then*

$$\iota_0(\mathbf{n}^-(u_1)\mathbf{n}(x_1), \mathbf{m}(a_2)\mathbf{n}^-(u_2), \mathbf{m}(a_3)\mathbf{n}^-(u_3)) = \begin{pmatrix} A & B \\ \mathbf{0}_3 & (A^t)^{-1} \end{pmatrix} J_3 \mathbf{n}(-z),$$

where

$$A = \begin{pmatrix} 1 & a_2 u_1 & a_3 u_1 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} -u_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} -x_1 & a_2 & a_3 \\ a_2 & u_2 + a_2^2 u_1 & a_2 a_3 u_1 \\ a_3 & a_2 a_3 u_1 & u_3 + a_3^2 u_1 \end{pmatrix}.$$

PROOF. We can prove Lemma 2.1 by the matrix expression of ι_0 . \square

2.2. The unramified case. When π_i is unramified, we write $W_i^0 \in \mathscr{W}(\pi_i)$ for the unique Whittaker function which takes the value 1 on $\mathrm{GL}_2(\mathfrak{o})$. Assume that $\hat{\omega}$ and χ are unramified. Then we define the holomorphic section $f_s^0(\chi)$ of $I_3(\hat{\omega}, \chi \boldsymbol{\alpha}_F^s)$ by the condition that $f_s^0(k, \chi) = 1$ for $k \in \mathrm{GSp}_6(\mathfrak{o})$. Garrett has proved that

$$Z(W_1^0, W_2^0, W_3^0, f_s^0(\chi)) = \frac{L\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi\right)}{L(2s + 2, \chi^2 \hat{\omega}) L(4s + 2, \chi^4 \hat{\omega}^2)}.$$

We associate to a half-integral symmetric matrix B the series defined by

$$b(B, s) = \sum_{z \in \mathrm{Sym}_3(F)/\mathrm{Sym}_3(\mathfrak{o})} \boldsymbol{\psi}(-\mathrm{tr}(Bz)) \nu[z]^{-s},$$

where $\boldsymbol{\psi}$ is an arbitrarily fixed additive character on F of order 0 and $\nu[z] = [z\mathfrak{o}^3 + \mathfrak{o}^3 : \mathfrak{o}^3]$. If $\det B \neq 0$, then there exists a polynomial $F_B(X) \in \mathbf{Z}[X]$ such that

$$b(B, s) = (1 - q^{-s})(1 - q^{2-2s})F_B(q^{-s}).$$

The following relation is well-known (cf. [Shi97, Proposition 19.2, page 158]):

$$(2.3) \quad \mathcal{W}_B(f_s^0(\chi)) = \frac{F_B(\chi^2 \hat{\omega}(\varpi) q^{-2s-2})}{L(2s + 2, \chi^2 \hat{\omega}) L(4s + 2, \chi^4 \hat{\omega}^2)}.$$

2.3. The p -adic case. Let St stand for the Steinberg representation of $\mathrm{GL}_2(F)$. For quasi-characters μ, ν of F^\times the representation $I(\mu, \nu)$ is realized on the space of functions $f : \mathrm{GL}_2(F) \rightarrow \mathbf{C}$ which satisfy

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu(a)\nu(d) \left|\frac{a}{d}\right|^{1/2} f(g)$$

for $a, d \in F^\times$, $b \in F$ and $g \in \mathrm{GL}_2(F)$, where $\mathrm{GL}_2(F)$ acts by right translation ρ . Hereafter we assume that π_i are not supercuspidal and are infinite dimensional. Then π_i is a quotient of a principal series representation $I(\mu_i, \nu_i)$ with quasi-characters μ_i, ν_i . If $\mu_i \nu_i^{-1} \neq \boldsymbol{\alpha}_F^{-1}$, then $\pi_i \simeq I(\mu_i, \nu_i)$. If $\mu_i \nu_i^{-1} = \boldsymbol{\alpha}_F^{-1}$, then $\pi_i \simeq \mathrm{St} \otimes \mu_i \boldsymbol{\alpha}_F^{1/2}$. Let $W_i^{\mathrm{ord}} \in \mathscr{W}(\pi_i)$ be the unique Whittaker function characterized by

$$W_i^{\mathrm{ord}}(\mathbf{t}(a)) = \nu_i(a) |a|^{1/2} \mathbb{I}_{\mathfrak{o}}(a)$$

for $a \in F^\times$, where $\mathbf{t}(a) = \mathrm{diag}(a, 1)$. Fix a prime element ϖ of \mathfrak{o} . For each non-negative integer n we put

$$m_n = \mathbf{m}(\varpi^n), \quad t_n = J_1^{-1} m_n, \quad W_i^{(n)} = \pi_i(t_n) W_i^{\mathrm{ord}}.$$

Given a character μ of \mathfrak{o}^\times , we define $\varphi_\mu \in \mathcal{S}(F)$ by

$$\varphi_\mu(x) = \mu(x) \mathbb{I}_{\mathfrak{o}^\times}(x).$$

We write $c(\mu)$ for the smallest integer n such that μ is trivial on $\mathfrak{o}^\times \cap (1 + \mathfrak{p}^n)$. Define the open compact subgroup $\mathcal{K}_0^{(g)}(\mathfrak{p}^n)$ of $\mathrm{GSp}_{2g}(F)$ by

$$\mathcal{K}_0^{(g)}(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathfrak{o}) \mid c \in \mathbf{M}_g(\mathfrak{p}^n) \right\}.$$

We can define characters μ^\uparrow and μ^\downarrow of $K_0^{(1)}(\mathfrak{p}^n)$ by

$$(2.4) \quad \mu^\uparrow \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mu(a), \quad \mu^\downarrow \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mu(d),$$

provided that $n \geq c(\mu)$. We define the Fourier transform of $\Phi \in \mathcal{S}(\text{Sym}_g(F))$ with respect to ψ by

$$\widehat{\Phi}(w) = \int_{\text{Sym}_g(F)} \Phi(z) \psi(\text{tr}(zw)) dz.$$

Given a Schwartz function $\Phi \in \mathcal{S}(\text{Sym}_3(F))$, we can define a section $f_\Phi(\chi)$ of $I_3(\widehat{\omega}, \chi)$ by requiring that

$$(2.5) \quad f_\Phi(J_3 \mathbf{n}(z), \chi) = \Phi(z)$$

for $z \in \text{Sym}_3(F)$. Lemma 2.1 gives

$$(2.6) \quad f_\Phi(\iota_0(\mathbf{n}^-(u_1)\mathbf{n}(x_1), \mathbf{m}(a_2)\mathbf{n}^-(u_2), \mathbf{m}(a_3)\mathbf{n}^-(u_3)), \chi) \\ = \widehat{\omega}(a_2a_3)\chi(a_2a_3)^2|a_2a_3|^2\Phi \left(\begin{pmatrix} x_1 & -a_2 & -a_3 \\ -a_2 & -u_2 - a_2^2u_1 & -a_2a_3u_1 \\ -a_3 & -a_2a_3u_1 & -u_3 - a_3^2u_1 \end{pmatrix} \right).$$

Now we define $\Phi \in \mathcal{S}(\text{Sym}_3(F))$ by

$$(2.7) \quad \Phi \left(\begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix} \right) = \prod_{i=1}^3 \phi_i(u_i) \varphi_i(x_i),$$

where we define $\phi_1, \phi_2, \phi_3, \varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}(F)$ by

$$\phi_1 = \phi_2 = \phi_3 = \widehat{\mathbb{I}}_{\mathfrak{p}}, \quad \varphi_1 = \widehat{\varphi}_{\chi\mu_1\nu_2\nu_3}, \quad \varphi_2 = \widehat{\varphi}_{\chi\nu_1\mu_2\nu_3}, \quad \varphi_3 = \widehat{\varphi}_{\chi\nu_1\nu_2\mu_3}.$$

Lemma 2.2. *If $n \geq \max\{1, c(\chi), c(\mu_i), c(\nu_i) \mid i = 1, 2, 3\}$, then*

$$\rho_3(\iota(g_1, g_2, g_3))f_\Phi(\chi) = f_\Phi(\chi) \prod_{i=1}^3 \mu_i^\uparrow(g_i)^{-1} \nu_i^\downarrow(g_i)^{-1}$$

for $g_1, g_2, g_3 \in K_0^{(1)}(\mathfrak{p}^{2n})$ with $\det g_1 = \det g_2 = \det g_3$.

PROOF. One can easily check that

$$(2.8) \quad \widehat{\varphi}_\mu \in \mathcal{S}(\mathfrak{p}^{-c(\mu)}), \quad \widehat{\varphi}_\mu(ax) = \mu(a)^{-1} \widehat{\varphi}_\mu(x), \quad \widehat{\varphi}_\mu(x+b) = \widehat{\varphi}_\mu(x)$$

for $a \in \mathfrak{o}^\times$, $b \in \mathfrak{o}$ and $x \in F$. Simply because $\phi_i = \widehat{\mathbb{I}}_{\mathfrak{p}^{-1}}$, we see that $\Phi(z+c) = \Phi(z)$ for $c \in \text{Sym}_3(\mathfrak{o})$, which means that $f_\Phi(\chi)$ is fixed by the action of $\mathbf{n}(\text{Sym}_3(\mathfrak{o}))$. Put

$$\chi_1 = \chi\mu_1\nu_2\nu_3, \quad \chi_2 = \chi\nu_1\mu_2\nu_3, \quad \chi_3 = \chi\nu_1\nu_2\mu_3.$$

Since

$$\begin{pmatrix} \frac{1}{a_1} & & \\ & \frac{1}{a_2} & \\ & & \frac{1}{a_3} \end{pmatrix} \begin{pmatrix} u_1 & x_2 & x_3 \\ x_2 & u_2 & x_1 \\ x_3 & x_1 & u_3 \end{pmatrix} \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} = \begin{pmatrix} \frac{d_1 u_1}{a_1} & \frac{d_2 x_2}{a_1} & \frac{d_3 x_3}{a_1} \\ \frac{d_1 x_2}{a_2} & \frac{d_2 u_2}{a_2} & \frac{d_3 x_1}{a_2} \\ \frac{d_1 x_3}{a_3} & \frac{d_2 x_1}{a_3} & \frac{d_3 u_3}{a_3} \end{pmatrix},$$

if $a_i, d_i \in \mathfrak{o}^\times$ and $\lambda = a_1 d_1 = a_2 d_2 = a_3 d_3$, then by (2.8)

$$\begin{aligned}
& \rho_3(\iota(\text{diag}(a_1, d_1), \text{diag}(a_2, d_2), \text{diag}(a_3, d_3))) f_\Phi(\chi) \\
&= \hat{\omega} \left(\frac{d_1 d_2 d_3}{\lambda^2} \right) \chi \left(\frac{(d_1 d_2 d_3)^2}{\lambda^3} \right) \chi_1 \left(\frac{a_2}{d_3} \right) \chi_2 \left(\frac{a_1}{d_3} \right) \chi_3 \left(\frac{a_1}{d_2} \right) f_\Phi(\chi) \\
&= \hat{\omega} \left(\frac{d_1 d_2 d_3}{\lambda^2} \right) (\mu_1 \nu_2 \nu_3) \left(\frac{a_2}{d_3} \right) (\nu_1 \mu_2 \nu_3) \left(\frac{a_1}{d_3} \right) (\nu_1 \nu_2 \mu_3) \left(\frac{a_1}{d_2} \right) f_\Phi(\chi) \\
&= \hat{\omega} \left(\frac{d_1 d_2 d_3}{\lambda^2} \right) \omega_1 \left(\frac{a_2}{d_3} \right) \omega_2 \left(\frac{a_1}{d_3} \right) \omega_3 \left(\frac{a_1}{d_2} \right) f_\Phi(\chi) \prod_{i=1}^3 \nu_i \left(\frac{a_i}{d_i} \right) \\
&= f_\Phi(\chi) \prod_{i=1}^3 \omega_i(a_i)^{-1} \nu_i \left(\frac{a_i}{d_i} \right) = f_\Phi(\chi) \prod_{i=1}^3 \mu_i(a_i)^{-1} \nu_i(d_i)^{-1}.
\end{aligned}$$

Let $w \in \text{Sym}_3(\mathfrak{p}^{2n})$. If $f_\Phi(\mathfrak{gn}^-(w), \chi) \neq 0$, then since $\mathfrak{gn}^-(w) \in \mathcal{P}_3 J_3 \mathfrak{n}(z)$ with $z \in \text{Sym}_3(\mathfrak{p}^{-n})$ and since

$$\mathfrak{n}(z) \mathfrak{n}^-(-w) = \begin{pmatrix} \mathbf{1}_3 - zw & \mathbf{0}_3 \\ -w & (\mathbf{1}_3 - wz)^{-1} \end{pmatrix} \mathfrak{n}((\mathbf{1}_3 - zw)^{-1} z),$$

we have $g \in \mathcal{P}_3 J_3 \mathfrak{n}(\text{Sym}_3(\mathfrak{p}^{-n}))$. We see by the identity above that

$$f_\Phi(J_3 \mathfrak{n}(z) \mathfrak{n}^-(w), \chi) = f_\Phi(J_3 \mathfrak{n}((\mathbf{1}_3 + zw)^{-1} z), \chi) = f_\Phi(J_3 \mathfrak{n}(z), \chi)$$

for $z \in \text{Sym}_3(\mathfrak{p}^{-n})$ and $w \in \text{Sym}_3(\mathfrak{p}^{2n})$. We conclude that $f_\Phi(\chi)$ is fixed by right translation by $\mathfrak{n}^-(\text{Sym}_3(\mathfrak{p}^{2n}))$. The proof is complete by $K_0^{(1)}(\mathfrak{p}^m) = \mathfrak{n}(\mathfrak{o}) \mathfrak{d}(\mathfrak{o}^\times) \mathfrak{m}(\mathfrak{o}^\times) \mathfrak{n}^-(\mathfrak{p}^m)$. \square

2.4. The p -adic zeta integral.

Proposition 2.3. *If $n \geq \max\{1, c(\chi), c(\mu_i), c(\nu_i) \mid i = 1, 2, 3\}$, then*

$$\begin{aligned}
Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) &= (1 + q^{-1})^{-3} \prod_{j=1}^3 \left(\frac{\beta_j}{q \alpha_j} \right)^n \\
&\quad \times (\chi \nu_1 \nu_2 \nu_3)(-1) \gamma \left(\frac{1}{2}, \pi_1 \otimes \chi \nu_2 \nu_3, \psi \right)^{-1} \prod_{i=2,3} \gamma \left(\frac{1}{2}, \chi \nu_1 \nu_i \mu_{5-i}, \psi \right)^{-1},
\end{aligned}$$

where $\alpha_i = \mu_i(\varpi)$ and $\beta_i = \nu_i(\varpi)$.

PROOF. We associate to $f_i \in I(\mu_i, \nu_i)$ a function $W(f_i) \in \mathcal{W}(\pi_i)$ by

$$W(g, f_i) = \int_F f_i(J_1 \mathfrak{n}(u) g) \psi(-u) du = \lim_{k \rightarrow \infty} \int_{\mathfrak{p}^{-k}} f_i(J_1 \mathfrak{n}(u) g) \psi(-u) du.$$

Here the limit stabilizes and the integral makes sense for any $f_i \in \pi_i$. The integral W factors through the quotient $I(\mu_i, \nu_i) \rightarrow \text{St} \otimes \mu_i \alpha_F^{1/2}$ when $\mu_i \nu_i^{-1} = \alpha_F^{-1}$. Let $f_i^{\text{ord}} \in I(\mu_i, \nu_i)$ be such that $f_i^{\text{ord}}(g) = 0$ unless $g \in B_2 J_1 U_2$ and such that $f_i^{\text{ord}}(J_1 \mathfrak{n}(x)) = \mathbb{1}_{\mathfrak{o}}(x)$ for $x \in F$. One can easily check

$$W_i^{\text{ord}} = W(f_i^{\text{ord}}).$$

For $i = 2, 3$ we put

$$f'_i = \rho(m_n) f_i^{\text{ord}} \in I(\mu_i, \nu_i), \quad W'_i = W(f'_i).$$

Recall that $m_n = \text{diag}(\varpi^n, \varpi^{-n})$. Then

$$Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f) = \int_{U^0 Z \backslash H} W_1^{(n)}(g_1) W'_2(g_2) W'_3(g_3) f(\iota_0(g_1, g_2, g_3)) dg_1 dg_2 dg_3.$$

Observe that

$$W_1^{(n)}(g_1) W'_2(g_2) W'_3(g_3) = \int_{U^0} \mathbf{W}_1(u_0(g_1, g_2, g_3); f'_2, f'_3) du_0,$$

where

$$\mathbf{W}_1(g_1, g_2, g_3; f'_2, f'_3) = W_1^{(n)}(g_1) f'_2(J_1 g_2) f'_3(J_1 g_3).$$

Substituting this expression, we are led to

$$Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) = \int_{Z \setminus H} \mathbf{W}_1(g; f'_2, f'_3) f_\Phi(\iota_0(g), \chi) dg.$$

Define a function \mathcal{F} on $\mathrm{SL}_2(F)$ by

$$\mathcal{F}(g) = \int_{\mathrm{SL}_2(F)^2} f'_2(J_1 g_2) f'_3(J_1 g_3) f_\Phi(\iota_0(g, g_2, g_3), \chi) dg_2 dg_3.$$

Let $T' = \mathbf{m}(F^\times)$ be the diagonal torus of $\mathrm{SL}_2(F)$. Then

$$\begin{aligned} Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) &= \int_{F^\times} d^\times a \int_{T' \setminus \mathrm{SL}_2(F)} W_1^{(n)}(\mathbf{t}(a)g) \int_{\mathrm{SL}_2(F)^2} \\ &\quad f'_2(J_1 \mathbf{t}(a)g_2) f'_3(J_1 \mathbf{t}(a)g_3) f_\Phi(\iota_0(\mathbf{t}(a)g, \mathbf{t}(a)g_2, \mathbf{t}(a)g_3), \chi) dg_2 dg_3 dg \\ (2.9) \qquad \qquad \qquad &= \int_{F^\times} \int_{T' \setminus \mathrm{SL}_2(F)} W_1^{(n)}(\mathbf{t}(a)g) \chi(a) \nu_2(a) \nu_3(a) \mathcal{F}(g) dg d^\times a \end{aligned}$$

by (2.1). To justify the manipulations we show that the integral

$$\int_{F^\times} \int_{\mathrm{SL}_2(\mathfrak{o})} \int_F |W_1^{(n)}(\mathbf{t}(a)k) \chi(a) \nu_2(a) \nu_3(a) \mathcal{F}(\mathbf{n}(x)k)| dx dk d^\times a$$

is convergent for $\mathrm{Re} \chi \gg 0$. The integral

$$\int_{F^\times} |W_1^{(n)}(\mathbf{t}(a)k) \chi(a) \nu_2(a) \nu_3(a)| d^\times a$$

is absolutely convergent. We frequently use the integration formula

$$\int_{\mathrm{SL}_2(F)} h(g) dg = \frac{\zeta(2)}{\zeta(1)} \int_F \int_F \int_{F^\times} h(\mathbf{m}(a) \mathbf{n}^-(u) \mathbf{n}(x)) d^\times a du dx$$

for an integrable function h on $\mathrm{SL}_2(F)$. Observe that

$$\begin{aligned} \mathcal{F}(g) &= \frac{\zeta(2)^2}{\zeta(1)^2} \int_{F^2} dx_2 dx_3 f'_2(J_1 \mathbf{n}(x_2)) f'_3(J_1 \mathbf{n}(x_3)) \int_{F^{\times 2}} \prod_{i=2,3} (\nu_i \mu_i^{-1})(a_i) \frac{d^\times a_i}{|a_i|} \\ &\quad \times \int_{F^2} f_\Phi(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2) \mathbf{n}(x_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3) \mathbf{n}(x_3)), \chi) du_2 du_3. \end{aligned}$$

Recall that

$$\mathbf{t}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{m}(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \mathbf{n}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \mathbf{n}^-(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

Observe that

$$f'_i(J_1 \mathbf{n}(x)) = f_i^{\mathrm{ord}}(J_1 m_n \mathbf{n}(x \varpi^{-2n})) = \beta_i^n \alpha_i^{-n} q^n \mathbb{I}_{\mathfrak{p}^{2n}}(x).$$

Lemma 2.2 shows that

$$\rho_3(\iota(\mathbf{1}_2, \mathbf{n}(x_2), \mathbf{n}(x_3)) J_3) f_\Phi(\chi) = \rho_3(J_3) f_\Phi(\chi) \quad (x_2, x_3 \in \mathfrak{p}^{2n}).$$

It follows that $\mathcal{F}(g)$ equals the product of $\frac{f'_2(J_1) f'_3(J_1)}{q^{4n(1+q^{-1})^2}}$ and

$$\int_{F^{\times 2} \oplus F^2} f_\Phi(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3)), \chi) \prod_{i=2,3} \frac{\nu_i(a_i) d^\times a_i}{\mu_i(a_i) |a_i|} du_i.$$

In particular, $\mathcal{F}(\mathbf{n}^-(u) \mathbf{n}(x))$ equals the product of $\frac{f'_2(J_1) f'_3(J_1)}{q^{4n(1+q^{-1})^2}}$ and

$$\begin{aligned} &\int_{F^{\times 2} \oplus F^2} \Phi \left(\begin{pmatrix} x & -a_2 & -a_3 \\ -a_2 & -u_2 & -a_2 a_3 u \\ -a_3 & -a_2 a_3 u & -u_3 \end{pmatrix} \right) \prod_{i=2,3} (\hat{\omega} \chi^2 \nu_i \mu_i^{-1})(a_i) |a_i| d^\times a_i du_i \\ &= \int_{F^{\times 2} \oplus F^2} \varphi_1(-a_2 a_3 u) \varphi_2(-a_2) \varphi_3(-a_3) \phi_1(x) \phi_3(-u_2 - a_2^2 u) \phi_2(-u_3 - a_3^2 u) \prod_{i=2,3} (\hat{\omega} \chi^2 \nu_i \mu_i^{-1})(a_i) |a_i| d^\times a_i du_i \end{aligned}$$

by (2.6). Its integral over $x, u \in F$ converges absolutely if $\mathrm{Re} \chi$ is large.

Recall the functional equations

$$(2.10) \quad \begin{aligned} \gamma\left(\frac{1}{2}, \pi_1 \otimes \chi, \boldsymbol{\psi}\right) \int_{F^\times} W_1(\mathbf{t}(a)g)\chi(a)d^\times a &= \int_{F^\times} W_1(\mathbf{t}(a)J_1^{-1}g)(\chi\omega_1)^{-1}(a)d^\times a, \\ \gamma(s, \chi, \boldsymbol{\psi}) \int_{F^\times} \varphi(a)\chi(a)|a|^s d^\times a &= \int_{F^\times} \widehat{\varphi}(a)\chi(a)^{-1}|a|^{1-s} d^\times a \end{aligned}$$

for every $W_1 \in \mathscr{W}(\pi_1)$ and $\varphi \in \mathcal{S}(F)$. It follows from (2.9) that

$$(2.11) \quad \begin{aligned} &\gamma\left(\frac{1}{2}, \pi_1 \otimes \chi\nu_2\nu_3, \boldsymbol{\psi}\right) Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) \\ &= \int_{F^\times} \int_{T' \setminus \mathrm{SL}_2(F)} W_1^{(n)}(\mathbf{t}(a)J_1^{-1}g)(\chi\nu_2\nu_3\omega_1)(a)^{-1} \mathcal{F}(g) dg d^\times a \\ &= \int_{F^\times} \int_F W_1^{(n)}(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))(\chi\nu_2\nu_3\omega_1)(a)^{-1} \mathcal{F}_\psi(a, x) dx d^\times a, \end{aligned}$$

where

$$\mathcal{F}_\psi(a, x) = (1 + q^{-1})^{-1} \int_F \mathcal{F}(\mathbf{n}^-(u)\mathbf{n}(x))\psi(-au) du.$$

We have seen that

$$\begin{aligned} \frac{q^{4n}(1+q^{-1})^3}{f_2'(J_1)f_3'(J_1)} \mathcal{F}_\psi(a, x) &= \int_{F^{\times 2}} \widehat{\omega}(a_2a_3)\chi(a_2a_3)^2 |a_2a_3|^2 \varphi_3(-a_2)\varphi_2(-a_3) \\ &\quad \times \int_F \varphi_1(-a_2a_3u)\overline{\psi}(au) du \phi_1(x) \prod_{i=2,3} \frac{\nu_i(a_i)d^\times a_i}{\mu_i(a_i)|a_i|} \int_F \phi_i(u_i) du_i \\ &= \phi_1(x) \int_{F^{\times 2}} \widehat{\varphi}_1\left(\frac{a}{a_2a_3}\right) \prod_{i=2,3} \widehat{\phi}_i(0)(\widehat{\omega}\chi^2\nu_i\mu_i^{-1})(a_i)\varphi_{5-i}(-a_i) d^\times a_i. \end{aligned}$$

If $xu \neq -1$, then

$$J_1\mathbf{n}(x)\mathbf{n}^-(u) = \mathbf{m}((1+ux)^{-1})\mathbf{n}(-(1+ux)u)J_1\mathbf{n}((1+ux)^{-1}x),$$

which implies that

$$\rho(\mathbf{n}^-(u))f_1^{\mathrm{ord}} = f_1^{\mathrm{ord}}, \quad \pi_1(\mathbf{n}^-(u))W_1^{\mathrm{ord}} = W_1^{\mathrm{ord}}$$

for $u \in \mathfrak{p}^n$. If $\phi_1(x) \neq 0$, then since $x \in \mathfrak{p}^{-1}$,

$$W_1^{(n)}(\mathbf{t}(a)J_1\mathbf{n}(x)) = W_1^{\mathrm{ord}}(\mathbf{t}(a)m_n\mathbf{n}^-(\varpi^{2n}x)) = q^{-n}\beta_1^n\alpha_1^{-n}\nu_1(a)|a|^{1/2}\mathbb{I}_\circ(a\varpi^{2n}).$$

We conclude by (2.11) that

$$\begin{aligned} &\gamma\left(\frac{1}{2}, \pi_1 \otimes \chi\nu_2\nu_3, \boldsymbol{\psi}\right) Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) \\ &= \int_{F^\times} \int_F W_1^{(n)}(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))(\chi\nu_2\nu_3\omega_1)(a)^{-1} \frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^3} \mathcal{F}_\psi(a, x) dx d^\times a \\ &= \int_{F^\times} W_1^{(n)}(\mathbf{t}(a)J_1^{-1})(\chi\nu_2\nu_3\omega_1)(a)^{-1} \frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^3} \int_F \mathcal{F}_\psi(a, x) dx d^\times a \\ &= \frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^3} \widehat{\phi}_1(0)\widehat{\phi}_2(0)\widehat{\phi}_3(0) \int_{F^{\times 3}} \frac{W_1^{(n)}(\mathbf{t}(a)J_1^{-1})}{(\chi\nu_2\nu_3\omega_1)(a)} \widehat{\varphi}_1\left(\frac{a}{a_2a_3}\right) d^\times a \prod_{i=2,3} (\widehat{\omega}\chi^2\nu_i\mu_i^{-1})(a_i)\varphi_{5-i}(-a_i) d^\times a_i. \end{aligned}$$

The last integral is equal to

$$\begin{aligned}
& \hat{\omega}(-1) \int_{F^{\times 3}} \frac{W_1^{(n)}(\mathfrak{t}(aa_2a_3)J_1)}{(\chi\nu_2\nu_3\omega_1)(a)} \widehat{\varphi}_1(a) d^\times a \prod_{i=2,3} (\chi\nu_i\mu_{5-i})(a_i) \varphi_{5-i}(a_i) d^\times a_i \\
&= \hat{\omega}(-1) W_1^{(n)}(J_1) \int_{F^{\times 3}} \frac{\nu_1(aa_2a_3)}{(\chi\nu_2\nu_3\omega_1)(a)} |aa_2a_3|^{1/2} \mathbb{1}_{\mathfrak{o}}(aa_1a_2\varpi^{2n}) \varphi_{\chi\mu_1\nu_2\nu_3}(-a) d^\times a \prod_{i=2,3} (\chi\nu_i\mu_{5-i})(a_i) \varphi_{5-i}(a_i) d^\times a_i \\
&= \hat{\omega}(-1) W_1^{(n)}(J_1) (\chi\mu_1\nu_2\nu_3)(-1) \prod_{i=2,3} \int_{F^\times} \varphi_{5-i}(a_i) (\chi\nu_1\nu_i\mu_{5-i})(a_i) |a_i|^{1/2} d^\times a_i.
\end{aligned}$$

In the last line we employ the fact that if $\varphi_{5-i}(a_i) \neq 0$, then $a_i \in \mathfrak{p}^{-n}$. The proof is now complete by $f'_i(J_1) = \beta_i^n \alpha_i^{-n} q^n$, $W_1^{(n)}(J_1) = \beta_1^n \alpha_1^{-n} q^{-n}$ and the functional equation (2.10). \square

2.5. Degenerate Whittaker functions at \mathfrak{p} . Let $\Xi_{\mathfrak{p}}$ be a subset of $\text{Sym}_3(F)$ which consists of symmetric matrices whose the diagonal entries belong to \mathfrak{p} and whose off-diagonal entries belong to $\frac{1}{2}\mathfrak{o}^\times$.

Proposition 2.4. *Let $B = (b_{ij}) \in \text{Sym}_3(F)$. Put $y_i = b_{jk}$ whenever $\{i, j, k\} = \{1, 2, 3\}$. Then*

$$\mathcal{W}_B(f_\Phi(\chi)) = \chi(8y_1y_2y_3) \prod_{i=1}^3 \mu_i(2y_i) \mathbb{1}_{\mathfrak{o}^\times}(2y_i) \mathbb{1}_{\mathfrak{p}}(b_{ii}) \prod_{j \in \{1,2,3\} \setminus \{i\}} \nu_j(2y_i).$$

In particular, $\mathcal{W}_B(f_\Phi(\chi)) \neq 0$ if and only if $B \in \Xi_{\mathfrak{p}}$.

PROOF. Observe that

$$(2.12) \quad \mathcal{W}_B(f_\Phi(\chi)) = \int_{\text{Sym}_3(F)} f_\Phi(J_3 \mathbf{n}(z), \chi) \psi(-\text{tr}(Bz)) dz = \widehat{\Phi}(-B)$$

for any $\Phi \in \mathcal{S}(\text{Sym}_3(F))$. We have

$$\widehat{\Phi} \left(- \begin{pmatrix} b_{11} & y_3 & y_2 \\ y_3 & b_{22} & y_1 \\ y_2 & y_1 & b_{33} \end{pmatrix} \right) = \prod_{i=1}^3 \widehat{\varphi}_i(-2y_i) \widehat{\varphi}_i(-b_{ii}) = \varphi_{\chi\mu_1\nu_2\nu_3}(2y_1) \varphi_{\chi\nu_1\mu_2\nu_3}(2y_2) \varphi_{\chi\nu_1\nu_2\mu_3}(2y_3) \prod_{i=1}^3 \mathbb{1}_{\mathfrak{p}}(b_{ii})$$

by definition. \square

2.6. Restatements. We rewrite Propositions 2.3 and 2.4 in a form which is suitable for our later discussion. Suppose that π_i is a subrepresentation of $I(\mu_i, \nu_i)$ with μ_i unramified. Thus $\omega_i = \mu_i\nu_i$ coincides with ν_i on \mathfrak{o}^\times . Let

$$\check{W}_i(\text{diag}(a, 1)) = \nu_i(a)^{-1} |a|^{1/2} \mathbb{1}_{\mathfrak{o}}(a).$$

Definition 2.5. We associate to the quadruplet of characters of \mathfrak{o}^\times

$$\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3)$$

a holomorphic section $f_{\mathcal{D},s} = f_{\Phi_{\mathcal{D}}}(\chi \hat{\omega} \alpha_F^s)$ of $I_3(\hat{\omega}^{-1}, \chi \hat{\omega} \alpha_F^s)$ by

$$\Phi_{\mathcal{D}} \left(\begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix} \right) = \prod_{i=1}^3 \widehat{\mathbb{1}}_{\mathfrak{p}}(u_i) \widehat{\varphi}_{\chi\omega_i}(x_i).$$

For each quadruplet $(\chi_0, \chi_1, \chi_2, \chi_3)$ of characters of \mathfrak{o}^\times , valued in a commutative ring R we set

$$\mathcal{Q}_B(\chi_0, \chi_1, \chi_2, \chi_3) := \chi_0(8b_{12}b_{23}b_{13}) \cdot \chi_1(2b_{23}) \chi_2(2b_{13}) \chi_3(2b_{12}) \mathbb{1}_{\Xi_{\mathfrak{p}}}(B).$$

Given a section f_s of $I_3(\hat{\omega}^{-1}, \chi \hat{\omega} \alpha_F^s)$, we are interested in the quantity

$$(2.13) \quad Z_{\mathfrak{p}}^*(f_s) = \frac{Z(\rho(t_n) \check{W}_1, \rho(t_n) \check{W}_2, \rho(t_n) \check{W}_3, f_s)}{L(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)} \prod_{i=1}^3 \frac{\zeta(1)}{\zeta(2)} \left(\frac{\omega_i(\varpi)q}{\mu_i(\varpi)^2} \right)^n.$$

Proposition 2.6. *Notations and assumptions being as above, we have*

$$\rho_3(\nu(g_1, g_2, g_3))f_{\mathcal{D},s} = f_{\mathcal{D},s} \prod_{i=1}^3 \omega_i^\dagger(g_i), \quad g_1, g_2, g_3 \in K_0^{(1)}(\mathfrak{p}^{2n})$$

if $\det g_1 = \det g_2 = \det g_3$ and $n \geq \max\{1, c(\chi), c(\omega_i)\}$. Moreover,

$$\mathcal{W}_B(f_{\mathcal{D},s}) = \mathcal{Q}_B(\mathcal{D}), \quad Z_{\mathfrak{p}}^*(f_{\mathcal{D},s}) = \chi(-1)E_{\mathfrak{p}}\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi\right),$$

where

$$E_{\mathfrak{p}}(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)^{-1} = L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi) \gamma(s, \pi_1 \otimes \chi \mu_2 \mu_3, \psi) \prod_{i=2,3} \gamma(s, \chi \mu_1 \mu_i \nu_{5-i}, \psi).$$

PROOF. Since ω_i coincides with ν_i on \mathfrak{o}^\times , we apply Proposition 2.4 and get the formula for $\mathcal{W}_B(f_{\mathcal{D},s})$ by replacing $\pi_i, \mu_i, \omega_i, \nu_i, \chi$ by $\pi_i^\vee \simeq \pi_i \otimes \omega_i^{-1}, \omega_i^{-1}, \mu_i^{-1}, \nu_i^{-1}, \chi \hat{\omega}$, respectively. Proposition 2.3 applied to \check{W}_i^{ord} and $I_3(\hat{\omega}^{-1}, \chi \hat{\omega})$ gives

$$\begin{aligned} Z(\rho(t_n)\check{W}_1, \rho(t_n)\check{W}_2, \rho(t_n)\check{W}_3, f_{\mathcal{D},s}) &= (1+q^{-1})^{-3} \prod_{i=1}^3 \left(\frac{\nu_i(\varpi)^{-1}}{q\mu_i(\varpi)^{-1}} \right)^n \\ &\quad \times \chi(-1) \gamma\left(s + \frac{1}{2}, \pi_1^\vee \otimes (\chi \hat{\omega}) \nu_2^{-1} \nu_3^{-1}, \psi\right)^{-1} \prod_{i=2,3} \gamma\left(s + \frac{1}{2}, (\chi \hat{\omega}) \nu_1^{-1} \nu_i^{-1} \mu_{5-i}^{-1}, \psi\right)^{-1}, \end{aligned}$$

from which the formula for $Z_{\mathfrak{p}}^*(s)$ readily follows. \square

We will use the following lemma to achieve the functional equation of the p -adic L -function in §7.7.

Lemma 2.7. *Put $\check{\chi} = \chi^{-1} \hat{\omega}^{-1}$. Then*

$$E_{\mathfrak{p}}(1-s, \pi_1 \times \pi_2 \times \pi_3 \otimes \check{\chi}) = \hat{\omega}(-1)E_{\mathfrak{p}}(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi) \varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi).$$

PROOF. Since $\pi_i \otimes \omega_i^{-1} \simeq \pi_i^\vee$, we get

$$E_{\mathfrak{p}}(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \check{\chi})^{-1} = L(s, \pi_1^\vee \times \pi_2^\vee \times \pi_3^\vee \otimes \bar{\chi}) \gamma(s, \pi_1^\vee \otimes \bar{\chi} \nu_2 \nu_3, \psi) \prod_{i=2,3} \gamma(s, \bar{\chi} \nu_1 \nu_i \mu_{5-i}, \psi),$$

where $\pi_i \simeq I(\mu_i, \nu_i)$. By definition we arrive at

$$\begin{aligned} &\varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi) L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)^{-1} E_{\mathfrak{p}}(1-s, \pi_1 \times \pi_2 \times \pi_3 \otimes \check{\chi})^{-1} \\ &= \gamma(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi) \gamma(1-s, \pi_1^\vee \otimes \bar{\chi} \mu_2 \mu_3, \psi) \prod_{i=2,3} \gamma(1-s, \bar{\chi} \mu_1 \mu_i \nu_{5-i}, \psi). \end{aligned}$$

The statement can now be deduced from multiplicativity and the functional equation of gamma factors. \square

3. COMPUTATION OF THE LOCAL ZETA INTEGRAL: THE RAMIFIED CASE

Recall that St denotes the Steinberg representation of $\text{GL}_2(F)$. Let π_i be either an irreducible unramified principal series representation or the Steinberg representation. Since

$$(3.1) \quad Z(W_1 \otimes \chi_1, W_2 \otimes \chi_2, W_3 \otimes \chi_3, f) = Z(W_1, W_2, W_3, f \otimes (\chi_1 \chi_2 \chi_3) \circ \nu_3)$$

for characters χ_1, χ_2, χ_3 of F^\times , where

$$(W_i \otimes \chi_i)(g_i) = W_i(g_i) \chi_i(\det g_i), \quad (f \otimes \chi \circ \nu_3)(g) = f(g) \chi(\nu_3(g)),$$

there is no harm in assuming that $\pi_i \simeq I(\alpha_F^{-t_i}, \alpha_F^{t_i})$ with $t_i \in \mathbf{C}$ or $\pi_i \simeq \text{St}$. When $\pi_i \simeq I(\alpha_F^{-t_i}, \alpha_F^{t_i})$, we denote the unique Whittaker function which takes the value 1 on $\text{GL}_2(\mathfrak{o})$ by $W_i^0 \in \mathscr{W}(\pi_i)$ and let $W_i^\pm \in \mathscr{W}(\pi_i)$ be the unique Whittaker function characterized by

$$W_i^\pm(\mathfrak{t}(a)) = |a|^{(\pm 2t_i + 1)/2} \mathbb{I}_{\mathfrak{o}}(a)$$

for $a \in F^\times$. When $\pi_i \simeq \text{St} \otimes \alpha_F^{s_i}$, we define $W_i^+ \in \mathscr{W}(\text{St} \otimes \alpha_F^{s_i})$ by

$$W_i^+(\mathfrak{t}(a)) = |a|^{s_i + 1} \mathbb{I}_{\mathfrak{o}}(a)$$

and set $t_i = s_i + \frac{1}{2}$ to be uniform. We define $f_i^{\text{ord}} \in I(\alpha_F^{-t_i}, \alpha_F^{t_i})$ as before. Recall that $W_i^+ = W(f_i^{\text{ord}})$. Put

$$\eta_1 = \begin{pmatrix} 0 & -1 \\ \varpi & 0 \end{pmatrix}, \quad \mathcal{W}_i^\pm = \pi_i(\eta_1)W_i^\pm, \quad \mathcal{W}_i^0 = \pi_i(\eta_1)W_i^0.$$

Lemma 3.1. *If π_i is an irreducible unramified principal series, then*

$$W_i^0 = q^{1/2} \frac{\mathcal{W}_i^+ - \mathcal{W}_i^-}{q^{-t_i} - q^{t_i}}.$$

PROOF. The relation $W_i^\pm = W_i^0 - q^{(\pm 2t_i - 1)/2} \mathcal{W}_i^0$ implies the stated identity in view of $\pi_i(\eta_1)\mathcal{W}_i^0 = W_i^0$. \square

Fix an unramified character $\chi = \alpha_F^s$ of F^\times . We will abbreviate $I_3(\chi) = I_3(1, \chi)$. Take $\Phi = \mathbb{1}_{\text{Sym}_3(\mathfrak{o})}$ and put $h^0(\chi) = f_\Phi(\chi)$. Since

$$\mathcal{P}_3 J_3 \mathbf{n}(\text{Sym}_3(\mathfrak{o})) = \mathcal{P}_3 J_3 K_0^{(3)}(\mathfrak{p}) = \mathcal{P}_3 K_0^{(3)}(\mathfrak{p}) J_3 K_0^{(3)}(\mathfrak{p}),$$

the restriction of the section $h^0(\chi)$ to $\text{GSp}_6(\mathfrak{o})$ is the characteristic function of $K_0^{(3)}(\mathfrak{p}) J_3 K_0^{(3)}(\mathfrak{p})$. In particular,

$$\rho_3(k)h^0(\chi) = h^0(\chi)$$

for $k \in K_0^{(3)}(\mathfrak{p})$ (cf. Lemma 2.2).

Lemma 3.2. *Assume that $\pi_1 \simeq \text{St}$. Then*

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = -\frac{q^{s-2}}{(1+q^{-1})^3} \zeta(s+1+t_2+t_3) \prod_{i=2,3} \zeta(s+1+t_i-t_{5-i}).$$

Remark 3.3. Lemma 3.2 is compatible with the computation [GK92]. Let $\Phi^0(\chi) \in I_3(\chi)$ be the function whose restriction to $\text{GSp}_6(\mathfrak{o})$ is the characteristic function of $K_0^{(3)}(\mathfrak{p})$. Put $\eta_3 = \iota(\eta_1, \eta_1, \eta_1)$. Then

$$\eta_3 K_0^{(3)}(\mathfrak{p}) \eta_3^{-1} = K_0^{(3)}(\mathfrak{p}), \quad h^0(\chi) = q^{3+3s} \rho_3(\eta_3) \Phi^0(\chi)$$

by Lemma 3.1 of [GK92]. We obtain

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = q^{3+3s} Z(W_1^+, W_2^+, W_3^+, \Phi^0).$$

When $\pi_1 \simeq \pi_2 \simeq \pi_3 \simeq \text{St}$, Proposition 4.2 of [GK92] gives

$$Z(W_1^+, W_2^+, W_3^+, \Phi^0) = -(q+1)^{-3} q^{-2s-2} L\left(s + \frac{1}{2}, \text{St} \times \text{St} \times \text{St}\right).$$

PROOF. On account of (2.9) we have

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = \int_{F^\times} \int_{T' \setminus \text{SL}_2(F)} \mathcal{W}_1^+(\mathbf{t}(a)g) |a|^{s+t_2+t_3} \mathcal{F}'(g) dg d^\times a.$$

Put $f_i'' = \pi_i(\mathbf{t}(\varpi)) f_i^{\text{ord}}$. Since

$$f_i''(J_1 \mathbf{n}(x)) = f_i^{\text{ord}}(J_1 \mathbf{t}(\varpi) \mathbf{n}(x/\varpi)) = q^{(1-2t_i)/2} \mathbb{1}_{\mathfrak{p}}(x),$$

we get

$$\begin{aligned} \mathcal{F}'(g) &= (1+q^{-1})^{-2} \int_{F^2} dx_2 dx_3 f_2''(J_1 \mathbf{n}(x_2)) f_3''(J_1 \mathbf{n}(x_3)) \int_{F^{\times 2}} \prod_{i=2,3} |a_i|^{2t_i} \frac{d^\times a_i}{|a_i|} \\ &\quad \times \int_{F^2} h^0(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2) \mathbf{n}(x_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3) \mathbf{n}(x_3)), \chi) du_2 du_3 \\ &= \int_{F^{\times 2} \oplus F^2} h^0(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3)), \chi) \frac{\prod_{i=2,3} |a_i|^{2t_i-1} d^\times a_i du_i}{q^{1+t_2+t_3} (1+q^{-1})^2}. \end{aligned}$$

In view of (2.6)

$$\begin{aligned} \mathcal{F}'(\mathbf{n}^-(u)\mathbf{n}(x)) &= \int_{F^{\times 2} \oplus F^2} \Phi \left(\begin{pmatrix} x & -a_2 & -a_3 \\ -a_2 & -u_2 & -a_2 a_3 u \\ -a_3 & -a_2 a_3 u & -u_3 \end{pmatrix} \right) \frac{\prod_{i=2,3} |a_i|^{1+2s+2t_i} d^\times a_i du_i}{q^{1+t_2+t_3}(1+q^{-1})^2} \\ &= q^{-1-t_2-t_3}(1+q^{-1})^{-2} \mathbb{I}_0(x) \int_{\mathfrak{o}^2} \mathbb{I}_0(a_2 a_3 u) \prod_{i=2,3} |a_i|^{1+2s+2t_i} d^\times a_i. \end{aligned}$$

Owing to (2.11) we arrive at

$$\begin{aligned} &\gamma \left(s + \frac{1}{2}, \pi_1 \otimes \alpha_F^{t_2+t_3}, \psi \right) Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) \\ &= \int_{F^\times} \int_F \mathcal{W}_1^+(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x)) |a|^{-s-t_2-t_3} \mathcal{F}'_\psi(a, x) dx d^\times a, \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}'_\psi(a, x) &= (1+q^{-1})^{-1} \int_F \mathcal{F}'(\mathbf{n}^-(u)\mathbf{n}(x)) \psi(-au) du \\ &= q^{-1-t_2-t_3}(1+q^{-1})^{-3} \mathbb{I}_0(x) \int_{\mathfrak{o}^2} \mathbb{I}_0 \left(\frac{a}{a_2 a_3} \right) \prod_{i=2,3} |a_i|^{2s+2t_i} d^\times a_i. \end{aligned}$$

We conclude that

$$\begin{aligned} &q^{1+t_2+t_3}(1+q^{-1})^3 \gamma \left(s + \frac{1}{2}, \pi_1 \otimes \alpha_F^{t_2+t_3}, \psi \right) Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) \\ &= \int_{F^\times} \int_F dx d^\times a \frac{\mathcal{W}_1^+(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))}{|a|^{s+t_2+t_3}} \mathbb{I}_0(x) \int_{\mathfrak{o}^2} \mathbb{I}_0 \left(\frac{a}{a_2 a_3} \right) \prod_{i=2,3} |a_i|^{2s+2t_i} d^\times a_i \\ &= \int_{F^\times} d^\times a \frac{\mathcal{W}_1^+(\mathbf{t}(a_2 a_3 a)J_1^{-1})}{|a_2 a_3 a|^{s+t_2+t_3}} \int_{\mathfrak{o}^2} \mathbb{I}_0(a) \prod_{i=2,3} |a_i|^{2s+2t_i} d^\times a_i \\ &= \int_{\mathfrak{o}} d^\times a \frac{W_1^+(\mathbf{t}(a_2 a_3 a \varpi))}{|a|^{s+t_2+t_3}} \prod_{i=2,3} \int_{\mathfrak{o}} |a_i|^{s+t_i-t_5-i} d^\times a_i \\ &= \frac{\zeta \left(\frac{1}{2} - s + t_1 - t_2 - t_3 \right)}{q^{(2t_1+1)/2}} \prod_{i=2,3} \zeta \left(s + \frac{1}{2} + t_1 + t_i - t_5 - i \right). \end{aligned}$$

Assume that $\pi_1 \simeq \text{St}$. Then $t_1 = \frac{1}{2}$ and

$$\gamma \left(s + \frac{1}{2}, \pi_1 \otimes \alpha_F^{t_2+t_3}, \psi \right) = -q^{-s-t_2-t_3} \frac{\zeta(1-s-t_2-t_3)}{\zeta(s+1+t_2+t_3)},$$

from which we complete our proof. \square

Proposition 3.4. *Let π_i be either an unramified principal series representation or the Steinberg representation twisted by an unramified character. Set $W_i = W_i^0$ in the former case and $W_i = W_i^+$ in the latter case. Put $\check{W}_i = W_i \otimes \omega_i^{-1}$. If not all π_i are principal series, then for an unramified character χ of F^\times*

$$\begin{aligned} Z(W_1, W_2, W_3, h^0(\chi)) &= Z(\check{W}_1, \check{W}_2, \check{W}_3, h^0(\chi \hat{\omega})) \\ &= -(\hat{\omega}^2 \chi^4)(\varpi) q(1+q)^{-3} \frac{L \left(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi \right)}{\varepsilon \left(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi \right)}. \end{aligned}$$

Remark 3.5. If π_1 and π_2 are irreducible unramified principal series representations, then

$$\begin{aligned} L(s, \pi_1 \times \pi_2 \times \text{St}) &= L \left(s + \frac{1}{2}, \pi_1 \times \pi_2 \right), & \varepsilon(s, \pi_1 \times \pi_2 \times \text{St}, \psi) &= q^{-4s+2} \omega_1(\varpi)^2 \omega_2(\varpi)^2, \\ L(s, \pi_1 \times \text{St} \times \text{St}) &= L(s, \pi_1) L(s+1, \pi_1), & \varepsilon(s, \pi_1 \times \text{St} \times \text{St}, \psi) &= q^{-4s+2} \omega_1(\varpi)^2, \\ L(s, \text{St} \times \text{St} \times \text{St}) &= \zeta \left(s + \frac{3}{2} \right) \zeta \left(s + \frac{1}{2} \right)^2, & \varepsilon(s, \text{St} \times \text{St} \times \text{St}, \psi) &= -q^{-(10s-5)/2}. \end{aligned}$$

PROOF. In view of [Ike89, Lemma 3.1] and (3.1) we may assume that $\pi_1 \simeq \text{St}$ and π_i is a quotient of $I(\alpha_F^{-t_i}, \alpha_F^{t_i})$ for $i = 2, 3$. If all π_i are discrete series representations, then since $\mathcal{W}_1^+ = -W_1$, the result follows from Lemma 3.2. Let $\chi = \alpha_F^s$ and $\pi_3 \simeq I(\alpha_F^{-t_3}, \alpha_F^{t_3})$. Lemma 3.2 gives

$$Z(W_1^+, \mathcal{W}_2^+, \mathcal{W}_3^\pm, h^0(\chi)) = \frac{q^{s-2}}{(1+q^{-1})^3} L(s+1+t_2, \pi_3) \zeta(s+1-t_2 \pm t_3).$$

Thanks to Lemma 3.1 we obtain

$$\begin{aligned} \frac{Z(W_1^+, \mathcal{W}_2^+, W_3^0, h^0(\chi))}{q^{s-2} L(s+1+t_2, \pi_3)} &= q^{1/2} \frac{\zeta(s+1-t_2+t_3) - \zeta(s+1-t_2-t_3)}{(1+q^{-1})^3 (q^{-t_3} - q^{t_3})} \\ &= (1+q^{-1})^{-3} q^{1/2} q^{-s-1+t_2} L(s+1-t_2, \pi_3). \end{aligned}$$

If $\pi_2 \simeq I(\alpha_F^{-t_2}, \alpha_F^{t_2})$, then

$$Z(W_1^+, \mathcal{W}_2^+, W_3^0, h^0(\chi)) = (1+q^{-1})^{-3} q^{(2t_2-5)/2} L(s+1, \pi_2 \times \pi_3),$$

and so again by Lemma 3.1,

$$Z(W_1^+, W_2^0, W_3^0, h^0(\chi)) = (1+q^{-1})^{-3} L(s+1, \pi_2 \times \pi_3) \frac{q^{t_2-2} - q^{-t_2-2}}{q^{-t_2} - q^{t_2}} = -(1+q^{-1})^{-3} q^{-2} L(s+1, \pi_2 \times \pi_3).$$

If $\pi_2 \simeq \text{St}$, we obtain the claimed result by letting $t_2 = \frac{1}{2}$. \square

4. COMPUTATION OF THE LOCAL ZETA INTEGRAL: THE ARCHIMEDEAN CASE

4.1. Archimedean sections. We define the sign character $\text{sgn} : \mathbf{R}^\times \rightarrow \{\pm 1\}$ by $\text{sgn}(x) = \frac{x}{|x|}$. Let $\text{Sym}_n^+(\mathbf{R})$ denote the set of positive definite symmetric matrices of rank n . The Siegel upper half-space \mathfrak{H}_n of degree n consists of complex symmetric matrices of size n with positive definite imaginary part. The Lie group $\text{GSp}_n^+(\mathbf{R}) = \{g \in \text{GSp}_n(\mathbf{R}) \mid \nu_n(g) > 0\}$ acts on the space \mathfrak{H}_n by $gZ = (AZ+B)(CZ+D)^{-1}$, where $Z \in \mathfrak{H}_n$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with matrices A, B, C, D of size n . Let $C^\infty(\mathfrak{H}_n)$ be the space of \mathbf{C} -valued smooth functions on the upper half complex plane \mathfrak{H}_n . For an integer k and $f \in C^\infty(\mathfrak{H}_n)$ we define

$$(4.1) \quad f|_k g(Z) = f(gZ) J(g, Z)^{-k}, \quad J(g, Z) = \nu_n(g)^{-n/2} \det(CZ+D).$$

Put $\mathbf{i} = \sqrt{-1}\mathbf{1}_n$. We will identify the compact unitary group $U(n) = \{u \in \text{GL}_n(\mathbf{C}) \mid \bar{u}^t u = \mathbf{1}_n\}$ with the fixator $\{g \in \text{Sp}_n(\mathbf{R}) \mid g(\mathbf{i}) = \mathbf{i}\}$ via the map $g \mapsto J(g, \mathbf{i})$.

For $1 \leq i, j \leq 3$ and $u \in U(3)$ we define $H_{ij}(u)$ to be the (i, j) -entry of the matrix $u^t u$. By definition, H_{ij} is a function on $O(3) \backslash U(3)$, and hence we can extend it to a unique function on $\text{GSp}_6(\mathbf{R})$ such that

$$H_{ij}(\mathbf{n}(z)\mathbf{m}(A, \nu)u) = H_{ij}(u) \quad (z \in \text{Sym}_3(\mathbf{R}), A \in \text{GL}_3(\mathbf{R}), \nu \in \mathbf{R}^\times, u \in U(3)).$$

A *parity type* is a triplet $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of integers which belongs to one of the following triplets

$$\lambda \in \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 2)\}.$$

Fix a parity type λ and a character χ_∞ of \mathbf{R}^\times . Put

$$H_\lambda := \begin{cases} 1 & \text{if } \lambda = (0, 0, 0), \\ \overline{H_{23}} & \text{if } \lambda = (0, 1, 1), \\ H_{12} & \text{if } \lambda = (1, 0, 1), \\ H_{12} \overline{H_{23}} & \text{if } \lambda = (1, 1, 2). \end{cases}$$

For each integer k we define $f_{\infty, s}^{[k, \lambda]} \in I_3(\text{sgn}^{k-\lambda_1}, \chi_\infty \text{sgn}^{k-\lambda_1} \alpha_{\mathbf{R}}^s)$ by

$$f_{s, \infty}^{[k, \lambda]}(g) := H_\lambda(g) \chi_\infty(\nu_3(g)) \cdot J(g, \mathbf{i})^{-k+\lambda_1} |J(g, \mathbf{i})|^{k-\lambda_1-2s-2}.$$

Since

$$H_{ij}(g\nu(\kappa_{\theta_1}, \kappa_{\theta_2}, \kappa_{\theta_3})) = e^{\sqrt{-1}(\theta_i + \theta_j)} H_{ij}(g), \quad \kappa_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

we have

$$(4.2) \quad f_{s, \infty}^{[k, \lambda]}(g\nu(\kappa_{\theta_1}, \kappa_{\theta_2}, \kappa_{\theta_3})) = f_{s, \infty}^{[k, \lambda]}(g) e^{\sqrt{-1}\{k\theta_1 + (k-\lambda_2)\theta_2 + (k-\lambda_3)\theta_3\}}.$$

4.2. **Archimedean degenerate Whittaker functions.** For a positive integer m we put

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma\left(s - \frac{j}{2}\right).$$

If h is positive definite and $\alpha, \beta \in \mathbf{C}$, then the integral

$$\omega(h; \alpha, \beta) = \frac{\det(h)^\beta}{\Gamma_3(\beta)} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(uh)} \det(u + \mathbf{1}_3)^{\alpha-2} (\det u)^{\beta-2} du$$

is absolutely convergent for $\text{Re } \beta > 2$ and can be continued to a holomorphic function on $\mathbf{C} \times \mathbf{C}$ by Theorem 3.1 of [Shi82]. It is convenient to introduce the function $\omega^*(h; \alpha, \beta)$ given by

$$(4.3) \quad \begin{aligned} \omega^*(h; \alpha, \beta) &:= \det(4\pi h)^{\alpha-2} \cdot \omega(4\pi h; \alpha, \beta) \\ &= \frac{1}{\Gamma_3(\beta)} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} \det(u + 4\pi h)^{\alpha-2} (\det u)^{\beta-2} du. \end{aligned}$$

It follows from this expression that if $\alpha \in \mathbf{Z}$ and $\alpha \geq 2$, then $\omega^*(h; \alpha, \beta)$ is a polynomial function in h of degree at most $\alpha - 2$ and makes sense for an arbitrary symmetric matrix h .

Lemma 4.1. *For $x \in \text{Sym}_3(\mathbf{R})$ we have*

$$\begin{aligned} H_{23}(J_3 \mathbf{n}(x)) &= 2\sqrt{-1}(x_{11}x_{23} - x_{12}x_{13} + \sqrt{-1}x_{23})/\det(x + \mathbf{i}), \\ H_{12}(J_3 \mathbf{n}(x)) &= 2\sqrt{-1}(x_{12}x_{33} - x_{23}x_{13} + \sqrt{-1}x_{12})/\det(x + \mathbf{i}). \end{aligned}$$

PROOF. The Iwasawa decomposition of $J_3 \mathbf{n}(x)$ can be written as

$$J_3 \mathbf{n}(x) = \begin{pmatrix} z^\dagger & * \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} zx & -z \\ z & zx \end{pmatrix}, \quad z \in \text{GL}_3(\mathbf{R})$$

with $z^\dagger z = (\mathbf{1}_3 + x^2)^{-1}$. Let $u = z(x - \mathbf{i}) \in \text{U}(3)$. Then $u^\dagger u = (x - \mathbf{i})(x + \mathbf{i})^{-1}$. We denote the adjugate of a matrix $A \in \text{M}_3(\mathbf{R})$ by $\text{adj}(A)$. Since $A \cdot \text{adj}(A) = (\det A)\mathbf{1}_3$, we have

$$u^\dagger u = \det(x + \mathbf{i})^{-1}(x - \mathbf{i})\text{adj}(x + \mathbf{i}) = -2\sqrt{-1}\det(x + \mathbf{i})^{-1}\text{adj}(x + \mathbf{i}) + \mathbf{1}_3.$$

By definition we find that

$$\begin{aligned} H_{23}(J_3 \mathbf{n}(x)) &= H_{23}(u) = \det(x + \mathbf{i})^{-1} \cdot 2\sqrt{-1} \det \begin{pmatrix} x_{11} + \sqrt{-1} & x_{12} \\ x_{13} & x_{23} \end{pmatrix} \\ &= \det(x + \mathbf{i})^{-1} \cdot 2\sqrt{-1}(x_{11}x_{23} - x_{12}x_{13} + \sqrt{-1}x_{23}). \end{aligned}$$

One can compute $H_{12}(J_3 \mathbf{n}(x))$ in the same way. □

Definition 4.2. We associate to a parity type λ the differential operator \mathcal{D}_λ on $T = (T_{ij}) \in \text{Sym}_3(\mathbf{R})$ by

$$\begin{aligned} \mathcal{D}_{(0,1,1)} &:= \frac{1}{2\pi^2\sqrt{-1}} \{\partial_{13}\partial_{12} - \partial_{23}(\partial_{11} - 4\pi)\}, & \mathcal{D}_{(0,0,0)} &= \text{id}, \\ \mathcal{D}_{(1,0,1)} &:= \frac{1}{2\pi^2\sqrt{-1}} \{\partial_{12}\partial_{33} - \partial_{23}\partial_{13}\}, & \mathcal{D}_{(1,1,2)} &:= \mathcal{D}_{(0,1,1)}\mathcal{D}_{(1,0,1)}. \end{aligned}$$

Here

$$\partial_{ij} := \frac{\partial}{\partial T_{ij}} \cdot \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{2} & \text{if } i \neq j. \end{cases}$$

Definition 4.3. For each parity type λ and an integer $\lambda_2 \leq r \leq k - 2$ we put $M = k - r - 2$ and define

$$\begin{aligned} \mathbf{K}_{\mathcal{D}_\lambda}^M(T; u) &:= \mathcal{D}_\lambda \{\det(4\pi T + u)^M\}, \\ \omega_{\mathcal{D}_\lambda}^M(T, s) &:= \frac{1}{\Gamma_3(s)} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} \mathbf{K}_{\mathcal{D}_\lambda}^M(T; u) (\det u)^{s-2} du = \mathcal{D}_\lambda \omega^*(T; M + 2, s). \end{aligned}$$

Lemma 4.4. *Let $A \in \mathrm{GL}_3(\mathbf{R})^+$ and $B \in \mathrm{Sym}_3(\mathbf{Q})$ with $\det B \neq 0$. If B is positive definite, then*

$$\lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} \mathcal{W}_B(\mathbf{m}(A), f_{s,\infty}^{[k,\lambda]}) = C_1^{[k,r,\lambda]} e^{-2\pi \mathrm{tr}(A^t B A)} \frac{\omega_{\mathcal{D}_\lambda}^M(A^t B A; \lambda_2 - r)}{(\det A)^{k-\lambda_1-2r-4}},$$

where

$$C_1^{[k,r,\lambda]} = (\sqrt{-1})^{k-\lambda_2} \frac{2^{3(3+2r-k-\lambda_2)} \pi^6}{\Gamma_3(k-r)}.$$

If B is not positive definite, then for any integer $0 \leq r < k-1$,

$$\lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} \mathcal{W}_B(\mathbf{m}(A), f_{s,\infty}^{[k,\lambda]}) = 0.$$

PROOF. For each parity type λ we define another differential operator \mathcal{D}_λ on $\mathrm{Sym}_3(\mathbf{R})$ by

$$\begin{aligned} \mathcal{D}_{(0,1,1)} &:= \frac{1}{2\pi^2 \sqrt{-1}} \{ \partial_{13} \partial_{12} - \partial_{23} (\partial_{11} - 2\pi) \}, & \mathcal{D}_{(0,0,0)} &:= \mathrm{id}, \\ \mathcal{D}_{(1,0,1)} &:= \frac{1}{2\pi^2 \sqrt{-1}} \{ \partial_{12} (\partial_{33} + 2\pi) - \partial_{23} \partial_{13} \}, & \mathcal{D}_{(1,1,2)} &:= \mathcal{D}_{(0,1,1)} \mathcal{D}_{(1,0,1)}. \end{aligned}$$

It should be remarked that by Lemma 4.1

$$\mathcal{D}_\lambda(e^{-2\pi \sqrt{-1} \mathrm{tr}(Tx)}) = \det(x + \mathbf{i})^{\lambda_1} \det(x - \mathbf{i})^{\lambda_2} H_\lambda(J_3 \mathbf{n}(x)) e^{-2\pi \sqrt{-1} \mathrm{tr}(Tx)}.$$

Recall that

$$\mathcal{W}_B(\mathbf{m}(A), f_{s,\infty}^{[k,\lambda]}) = (\det A)^{-2s-2} \mathcal{W}_{A^t B A}(\mathbf{1}_6, f_{s,\infty}^{[k,\lambda]}),$$

which reduces our computation to the case $A = \mathbf{1}_3$. We see that

$$\begin{aligned} \mathcal{W}_B(\mathbf{1}_6, f_{s,\infty}^{[k,\lambda]}) &= \int_{\mathrm{Sym}_3(\mathbf{R})} \det(x + \mathbf{i})^{-\alpha_0} \det(x - \mathbf{i})^{-\beta_0} H_\lambda(J_3 \mathbf{n}(x)) e^{-2\pi \sqrt{-1} \mathrm{tr}(Bx)} \mathrm{d}x \\ &= \mathcal{D}_\lambda(\xi(\mathbf{1}_3, T; \alpha_0 + \lambda_1, \beta_0 + \lambda_2))|_{T=B} \end{aligned}$$

with $\alpha_0 = s + 1 + \frac{k-\lambda_1}{2}$ and $\beta_0 = s + 1 - \frac{k-\lambda_1}{2}$. On the other hand, for any $h \in \mathrm{Sym}_3(\mathbf{R})$, we have

$$\xi(\mathbf{1}_3, h; \alpha, \beta) = (\sqrt{-1})^{3(\beta-\alpha)} \frac{(2\pi)^6 e^{-2\pi \mathrm{tr}(h)}}{2^3 \Gamma_3(\alpha) \Gamma_3(\beta)} \int_{u>0, u>-2\pi h} e^{-2\pi \mathrm{tr}(u)} \det(u + 2\pi h)^{\alpha-2} (\det u)^{\beta-2} \mathrm{d}u$$

by [Shi82, (1.29)]. If h is positive definite, then the last integral equals $2^{3(2-\alpha-\beta)} \omega^*(h; \alpha, \beta) \cdot \Gamma_3(\beta)$. Observe that for every polynomial P on $\mathrm{Sym}_3(\mathbf{R})$

$$\mathcal{D}_\lambda(e^{-2\pi \mathrm{tr}(T)} P(T)) = e^{-2\pi \mathrm{tr}(T)} \mathcal{D}_\lambda P(T).$$

This proves the case where B is positive definite. If the signature of B is $(3-q, q)$, then Theorem 4.2 of [Shi82] gives that a holomorphic function $\tilde{\omega}(\alpha, \beta)$ such that

$$\xi(\mathbf{1}_3, B; \alpha, \beta) = \frac{\Gamma_p(\beta - \frac{q}{2}) \Gamma_q(\alpha - \frac{p}{2})}{\Gamma_3(\alpha) \Gamma_3(\beta)} \cdot \tilde{\omega}(\alpha, \beta).$$

Thus $\xi(\mathbf{1}_3, B; k-r, -r) = 0$ for $0 \leq r \leq k-2$ unless B is positive definite. \square

4.3. The constant term of $\mathcal{W}_B(\mathbf{m}(\mathrm{diag}(\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3})), f_{s,\infty}^{[k,\lambda]})$ as a polynomial of y_1^{-1} . Given $y = \mathrm{diag}(y_1, y_2, y_3) \in \mathbf{R}_+^3$, we put $A = \mathrm{diag}(\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3})$ and define

$$\mathbf{W}_B^{[k,r,\lambda]}(y) := (y_1 y_2 y_3)^{r-k+2} \sqrt{y_1}^{\lambda_1} \sqrt{y_2}^{-\lambda_1+\lambda_2} \sqrt{y_3}^{-2\lambda_1+\lambda_2} \cdot \omega_{\mathcal{D}_\lambda}^M(A^t B A, \lambda_2 - r).$$

Now we write

$$\omega^*(T; M+2, s) = \sum_{0 \leq j_1, j_2, j_3 \leq M} c_{j_1 j_2 j_3} T_{12}^{j_3} T_{23}^{j_1} T_{13}^{j_2}, \quad c_{j_1 j_2 j_3} \in \mathbf{C}[T_{11}, T_{22}, T_{33}],$$

where $T = (T_{ij}) \in \mathrm{Sym}_3^+(\mathbf{R})$. Since

$$\omega^*(\varepsilon^t T \varepsilon; M+2, s) = \omega^*(T; M+2, s), \quad \varepsilon = \mathrm{diag}(-1, 1, 1)$$

in view of the expression (4.3), we get $c_{j_1 j_2 j_3} = (-1)^{j_2 + j_3} c_{j_1 j_2 j_3}$. Thus $c_{j_1 j_2 j_3} = 0$ unless $j_2 \equiv j_3 \pmod{2}$. By symmetry we conclude that $c_{j_1 j_2 j_3} = 0$ unless $j_1 \equiv j_2 \equiv j_3 \pmod{2}$. Moreover, we can write

$$T_{23}^{\lambda_1} T_{12}^{\lambda_2} T_{13}^{\lambda_1 + \lambda_2} \omega_{\mathcal{D}_\lambda}^*(T, s) = \sum_{0 \leq j_1, j_2, j_3 \leq M, j_1 \equiv j_2 \equiv j_3 \pmod{2}} a_{j_1 j_2 j_3} T_{12}^{j_3} T_{23}^{j_1} T_{13}^{j_2}, \quad a_{j_1 j_2 j_3} \in \mathbf{C}[T_{11}, T_{22}, T_{33}].$$

Thus we can write

$$(4.4) \quad \mathbf{W}_B^{[k, r, \lambda]}(\text{diag}(y_1, y_2, y_3)) = \sum_{0 \leq a, b, c \leq M} Q_{a, b, c}^{[k, \lambda]}(B, r) y_1^{-a} y_2^{-b} y_3^{-c}.$$

We shall determine the coefficient $Q_{0, b, c}^{[k, \lambda]}(B, r)$ of $\mathbf{W}_B^{[k, r, \lambda]}(y)$ for matrices B with zero diagonal entries

$$B = \begin{pmatrix} 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \\ b_2 & b_1 & 0 \end{pmatrix}.$$

Let \mathcal{Y} be the matrix with variables Y_1, Y_2, Y_3 given by

$$\mathcal{Y} = \begin{pmatrix} 0 & \sqrt{Y_1 Y_2} & \sqrt{Y_1 Y_3} \\ \sqrt{Y_1 Y_2} & 0 & \sqrt{Y_2 Y_3} \\ \sqrt{Y_1 Y_3} & \sqrt{Y_2 Y_3} & 0 \end{pmatrix}.$$

For two functions $f, g : \mathbf{R}_+ \rightarrow \mathbf{C}$ and $c \in \mathbf{R}$ we say that $f(y) = g(y) + o(y^c)$ if $\lim_{y \rightarrow \infty} \frac{f(y) - g(y)}{y^c} = 0$.

Lemma 4.5. *The polynomial $\mathbf{K}_{\mathcal{D}_\lambda}^M(\frac{\mathcal{Y}}{4\pi}; u) \in \mathbf{C}[\sqrt{Y_1}, \sqrt{Y_2}, \sqrt{Y_3}, u]$ in Definition 4.3 has the form*

$$\mathbf{K}_{\mathcal{D}_\lambda}^M((4\pi)^{-1} \mathcal{Y}; u) = C_2^{[k, r, \lambda]} \mathbf{c}_\lambda(Y_2, Y_3; u) \cdot Y_1^{M - \frac{\lambda_1}{2}} + o(Y_1^{M - \frac{\lambda_1}{2}})$$

with $C_2^{[k, r, \lambda]} \in \mathbf{C}$ and $\mathbf{c}_\lambda(Y_2, Y_3; u) \in \mathbf{C}[\sqrt{Y_2}, \sqrt{Y_3}, u]$ give by

$$C_2^{[k, r, \lambda]} = \frac{(2M + \lambda_1)!}{(2M)!} \cdot \frac{2^{3(\lambda_1 + \lambda_2) - \lambda_1} M!}{(\sqrt{-1})^{\lambda_2 - \lambda_1} (M - \lambda_1 - \lambda_2)!},$$

$$\mathbf{c}_\lambda(Y_2, Y_3; u) = \left(-u_{22} Y_3 - u_{33} Y_2 + 2Y_2 Y_3 + 2u_{23} \sqrt{Y_2 Y_3} \right)^{M - \lambda_1 - \lambda_2} \cdot \sqrt{Y_2}^{\lambda_1 + \lambda_2} \sqrt{Y_3}^{\lambda_2}.$$

PROOF. This is proved by a direct computation. Note that

$$\begin{aligned} \partial_{11} \det(T + u) &= (T_{22} + u_{22})(T_{33} + u_{33}) - (T_{23} + u_{23})^2, \\ \partial_{12} \det(T + u) &= -(T_{12} + u_{12})(T_{33} + u_{33}) + (T_{23} + u_{23})(T_{13} + u_{13}), \\ \partial_{13} \det(T + u) &= -(T_{13} + u_{13})(T_{22} + u_{22}) + (T_{12} + u_{12})(T_{23} + u_{23}), \\ \partial_{23} \det(T + u) &= -(T_{23} + u_{23})(T_{11} + u_{11}) + (T_{12} + u_{12})(T_{13} + u_{13}), \\ \partial_{33} \det(T + u) &= (T_{11} + u_{11})(T_{22} + u_{22}) - (T_{12} + u_{12})^2. \end{aligned}$$

Put

$$\Delta = \det(T + u), \quad R = (-u_{22} Y_3 - u_{33} Y_2 + 2Y_2 Y_3 + 2u_{23} \sqrt{Y_2 Y_3}) Y_1.$$

Since $\Delta|_{T=\mathcal{Y}} = R + o(Y_1)$, we have

$$\begin{aligned} & \mathbf{K}_{\mathcal{D}_{(0,1,1)}}^M((4\pi)^{-1} \mathcal{Y}; u) \\ &= (2\pi^2 \sqrt{-1})^{-1} \cdot (4\pi)^2 \{ \partial_{13} \partial_{12} - \partial_{23} (\partial_{11} - 1) \} \Delta^M|_{T=\mathcal{Y}} \\ &\equiv -8\sqrt{-1} [M(M-1)R^{M-2} (\partial_{13} \Delta \partial_{12} \Delta - \partial_{23} \Delta \partial_{11} \Delta) + MR^{M-1} \{ \partial_{13} \partial_{12} - \partial_{23} (\partial_{11} - 1) \} \Delta]|_{T=\mathcal{Y}} \\ &\equiv -8\sqrt{-1} MR^{M-1} \partial_{23} \Delta|_{T=\mathcal{Y}} \\ &\equiv -8\sqrt{-1} MR^{M-1} \sqrt{Y_2 Y_3} Y_1 \pmod{o(Y_1^M)}, \end{aligned}$$

which verifies the case $\lambda = (0, 1, 1)$. When $\lambda = (1, 0, 1)$, we have

$$\begin{aligned} & \mathbf{K}_{\mathcal{D}(1,0,1)}^M((4\pi)^{-1}\mathcal{Y}; u) \\ & \equiv -8\sqrt{-1}\{M(M-1)R^{M-2}(\partial_{12}\Delta\partial_{33}\Delta - \partial_{13}\Delta\partial_{23}\Delta) + MR^{M-1}(\partial_{12}\partial_{33}\Delta - \partial_{13}\partial_{23}\Delta)\}|_{T=\mathcal{Y}} \\ & \equiv -8\sqrt{-1}\left\{M(M-1)R^{M-2}(-R\sqrt{Y_1Y_2}) + MR^{M-1}\left(-\frac{3}{2}\sqrt{Y_1Y_2}\right)\right\} \\ & \equiv 4\sqrt{-1}M(2M+1)R^{M-1}\sqrt{Y_1Y_2} \pmod{o(Y_1^{M-\frac{1}{2}})} \end{aligned}$$

as claimed. Since

$$\mathcal{D}_{(0,1,1)}\Delta^M|_{T=\mathcal{Y}} = -8\sqrt{-1}M\Delta^{M-1}T_{12}T_{13}|_{T=\mathcal{Y}} + o(Y_1^M),$$

we have

$$\begin{aligned} \mathbf{K}_{\mathcal{D}(1,1,2)}^M((4\pi)^{-1}\mathcal{Y}; u) & \equiv 32M(M-1)(2M-1)R^{M-2}\sqrt{Y_1Y_2}Y_1\sqrt{Y_2Y_3} \\ & \quad - 64M(M-1)\Delta^{M-2}(T_{13}\partial_{33}\Delta - T_{12}\partial_{23}\Delta)|_{T=\mathcal{Y}} \pmod{o(Y_1^{M-\frac{1}{2}})}, \end{aligned}$$

which proves the case $\lambda = (1, 1, 2)$. \square

Lemma 4.6. *Let $F(T)$ be a polynomial in $T = (T_{ij}) \in \text{Sym}_3(\mathbf{R})$. Then we have*

$$\int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} F(u) \frac{(\det u)^{s-2}}{\Gamma_3(s)} du = F(-\partial_{ij})(\det T)^{-s}|_{T=\mathbf{1}_3}.$$

PROOF. If T is positive definite and $\text{Re } s > 2$, then

$$\int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(Tu)} \frac{(\det u)^{s-2}}{\Gamma_3(s)} du = (\det T)^{-s}$$

by [Shi81, (1.14)]. The declared formula follows immediately from the fact that

$$F(-\partial_{ij})(e^{-\text{tr}(Tu)}) = F(u)e^{-\text{tr}(Tu)}. \quad \square$$

Now let $k \geq l \geq m$ be a set of balanced integers. We say that (k, l, m) has the parity type λ if

$$\lambda_1, \lambda_2 \in \{0, 1\}, \quad \lambda_1 \equiv l - m \pmod{2}, \quad \lambda_2 \equiv k - l \pmod{2}, \quad \lambda_3 = \lambda_1 + \lambda_2.$$

Lemma 4.7. *Let λ be the parity type of $k \geq l \geq m$ and r an integer such that $k - \frac{l+m+\lambda_1}{2} \leq r \leq \frac{l+m}{2} - 2$. Put*

$$M = k - r - 2, \quad b = \frac{1}{2}(k - l - \lambda_2), \quad c = \frac{1}{2}(k - m - \lambda_3), \quad n = M + \frac{1}{2}(l + m - \lambda_1).$$

Then we have

$$Q_{0,b,c}^{[k,\lambda]}(B, r) = w_{0,b,c} \cdot (b_1 b_2 b_3)^n b_1^{-k} b_2^{-l} b_3^{-m},$$

where

$$w_{0,b,c} = (4\pi)^{3M-b-c-2\lambda_1-\lambda_2} 2^{M+\lambda_1+2\lambda_2-b-c} \frac{(\sqrt{-1})^{\lambda_1-\lambda_2} (2M+\lambda_1)! M!}{(2M)!(M-\lambda_1-\lambda_2-b-c)! b! c! (r-\lambda_2-b-c)!} \frac{(r-\lambda_2)!}{(r-\lambda_2-b-c)!}.$$

PROOF. Substitute $Y_i = 4\pi \frac{b_1 b_2 b_3}{b_i^2} y_i$ into the matrix \mathcal{Y} . Then $\mathcal{Y} = 4\pi A^t B A$ and

$$\mathbf{W}_B^{[k,r,\lambda]}(y) = \sum_{a,b,c} Q_{a,b,c}^{[k,\lambda]}(B, r) (4\pi)^{a+b+c} Y_1^{-a} Y_2^{-b} Y_3^{-c} \cdot b_1^{b+c-a} b_2^{a+c-b} b_3^{a+b-c}.$$

On the other hand,

$$\mathbf{W}_B^{[k,r,\lambda]}(y) = \left(\frac{(4\pi)^3 b_1 b_2 b_3}{Y_1 Y_2 Y_3} \right)^M \frac{\sqrt{Y_1}^{\lambda_1} \sqrt{Y_2}^{\lambda_1+\lambda_2} \sqrt{Y_3}^{2\lambda_1+\lambda_2}}{(4\pi)^{2\lambda_1+\lambda_2} b_1^{\lambda_1+\lambda_2} b_2^{\lambda_1}} \omega_{\mathcal{D},\lambda}^*((4\pi)^{-1}\mathcal{Y}, \lambda_2 - r)$$

by definition. The equations above give a complex number $w_{0,b,c}$ such that

$$Q_{0,b,c}^{[k,\lambda]}(B, r) = w_{0,b,c} \cdot (b_1 b_2 b_3)^n b_1^{-k} b_2^{-l} b_3^{-m}.$$

Our task is to determine $w_{0,b,c}$. It is the coefficient of $Y_2^{M-\lambda_1-\lambda_2-b}Y_3^{M-\lambda_1-\lambda_2-c}$ in the polynomial

$$\begin{aligned} & \frac{(4\pi)^{3M-b-c-2\lambda_1-\lambda_2}}{\sqrt{Y_2^{\lambda_1+\lambda_2}}\sqrt{Y_3^{\lambda_2}}} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} C_2^{[k,r,\lambda]} \mathbf{c}_\lambda(Y_2, Y_3; u) \frac{(\det u)^{s-2}}{\Gamma_3(s)} du \Big|_{s=\lambda_2-r} \\ &= (4\pi)^{3M-b-c-2\lambda_1-\lambda_2} C_2^{[k,r,\lambda]} \\ & \quad \times \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} \left(-u_{22}Y_3 - u_{33}Y_2 + 2Y_2Y_3 + 2u_{23}\sqrt{Y_2Y_3} \right)^{M-\lambda_1-\lambda_2} \frac{(\det u)^{s-2}}{\Gamma_3(s)} du \Big|_{s=\lambda_2-r} \end{aligned}$$

by Lemma 4.5. Put

$$L = M - \lambda_1 - \lambda_2, \quad r_1 = r - \lambda_2.$$

Notice that $b \leq c$ by assumption. The coefficient of $Y_2^{L-b}Y_3^{L-c}$ in the last integral is given by

$$\begin{aligned} & \sum_{i=0}^b \frac{2^{L-b-c}(-1)^{b+c} \cdot L!}{(b-i)!(c-i)!(L-b-c)!(2i)!} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} u_{33}^{b-i} u_{22}^{c-i} (2u_{23})^{2i} \frac{(\det u)^{s-2}}{\Gamma_3(s)} du \Big|_{s=-r_1} \\ &= \sum_{i=0}^b \frac{2^{L-b-c} \cdot L! 2^{2i}}{(b-i)!(c-i)!(L-b-c)!(2i)!} \partial_{33}^{b-i} \partial_{22}^{c-i} \partial_{23}^{2i} (T_{22}T_{33} - T_{23}^2)^{r_1} \Big|_{T_{22}=T_{33}=1, T_{23}=0} \\ &= \sum_{i=0}^b \frac{2^{L-b-c} \cdot L! 2^{2i}}{(b-i)!(c-i)!(L-b-c)!(2i)!} \sum_{j=0}^{r_1} \binom{r_1}{j} \partial_{33}^{b-i} \partial_{22}^{c-i} \partial_{23}^{2i} T_{22}^{r_1-j} T_{33}^{r_1-j} (-T_{23}^2)^j \Big|_{T_{22}=T_{33}=1, T_{23}=0} \\ &= \sum_{i=0}^b \frac{2^{L-b-c} \cdot L!}{(b-i)!(c-i)!(L-b-c)!} \binom{r_1}{i} (-1)^i \partial_{33}^{b-i} \partial_{22}^{c-i} (T_{22}^{r_1-i} T_{33}^{r_1-i}) \Big|_{T_{22}=T_{33}=1} \\ &= \frac{2^{L-b-c} \cdot L!}{(L-b-c)!} \sum_{i=0}^b \binom{r_1}{i} \binom{r_1-i}{b-i} \binom{r_1-i}{c-i} (-1)^i \end{aligned}$$

in view of Lemma 4.6. The last summation equals

$$\frac{r_1!}{(r_1-b)!b!} \sum_{i=0}^b \binom{b}{i} \binom{r_1-i}{r_1-c} (-1)^i = \frac{r_1!}{(r_1-b)!b!} \cdot \binom{r_1-b}{r_1-b-c} = \frac{r_1!}{b!c!(r_1-b-c)!},$$

where we can deduce this equality by equating the terms of degree $r_1 - c$ of the identity

$$\sum_{i=0}^b \binom{b}{i} (1+X)^{r_1-i} (-1)^i = (1+X)^{r_1} \left(1 - \frac{1}{1+X}\right)^b = (1+X)^{r_1-b} X^b,$$

Finally, we see that $w_{0,b,c}$ equals

$$(4\pi)^{3M-b-c-2\lambda_1-\lambda_2} C_2^{[k,r,\lambda]} \frac{2^{M-\lambda_1-\lambda_2-b-c} \cdot (M-\lambda_1-\lambda_2)!}{(M-\lambda_1-\lambda_2-b-c)!} \frac{(r-\lambda_2)!}{b!c!(r-\lambda_2-b-c)!}$$

by putting together the above computations, which completes our proof. \square

4.4. The archimedean zeta integral. Let V_\pm be the weight raising/lowering operator given by

$$V_\pm := \frac{1}{(-8\pi)} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1} \right) \in \text{Lie}(\text{GL}_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C}.$$

For each integer k we denote by σ_k the (limit of) discrete series of $\text{GL}_2(\mathbf{R})$ of the minimal weight $\pm k$ and by W_k the Whittaker function of σ_k characterized by

$$W_k(\text{diag}(y, 1)) = y^{k/2} e^{-2\pi y} \mathbb{I}_{\mathbf{R}_+}(y).$$

Set $W_k^{[t]} = V_+^t W_k$. It follows from (6.2) below that

$$(4.5) \quad W_k^{[t]}(\text{diag}(y, 1)) = \sum_{j=0}^t (-4\pi)^{j-t} \binom{t}{j} \frac{\Gamma(t+k)}{\Gamma(j+k)} \cdot y^{\frac{k}{2}+j} e^{-2\pi y} \mathbb{I}_{\mathbf{R}_+}(y).$$

Fix a triplet (k, l, m) of positive integers such that $k \geq l \geq m$ and $k < l + m$. Put $\mathcal{J}_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Define

$$Z_\infty(s) := Z\left(\rho(\mathcal{J}_\infty)W_k, \rho(\mathcal{J}_\infty)W_l^{\left[\frac{k-l-\lambda_2}{2}\right]}, \rho(\mathcal{J}_\infty)W_m^{\left[\frac{k-m-\lambda_3}{2}\right]}, f_{s,\infty}^{[k,\lambda]}\right),$$

where λ is the parity type of (k, l, m) . Recall that

$$\begin{aligned} L(s, \sigma_k \times \sigma_l \times \sigma_m) &= \Gamma_{\mathbf{C}}\left(s + \frac{k+l+m-3}{2}\right) \Gamma_{\mathbf{C}}\left(s + \frac{k-l+m-1}{2}\right) \\ &\quad \times \Gamma_{\mathbf{C}}\left(s + \frac{k+l-m-1}{2}\right) \Gamma_{\mathbf{C}}\left(s + \frac{m+l-k-1}{2}\right). \end{aligned}$$

Put

$$\gamma_{(k,m,l)}^*(s) = (\sqrt{-1})^{k+2\lambda_2+\lambda_1} \frac{\Gamma\left(s + \frac{k-m-l}{2} + 1\right)}{\Gamma\left(s - \frac{k-\lambda_1}{2} + \lambda_2 + 1\right)} \cdot \frac{\Gamma\left(s + \frac{k+\lambda_1}{2}\right)}{\Gamma\left(s + \frac{k+\lambda_1}{2} + 1\right)} \cdot \frac{\pi^{3s+1}(4\pi)^{l+m-\frac{k-\lambda_1}{2}+\lambda_2}}{4\Gamma\left(s + \frac{m+l-k}{2}\right)\Gamma(2s+k)}.$$

Lemma 4.8. *If λ is the parity type of (k, l, m) , then*

$$Z_\infty(s) = \chi_\infty(-1) \text{vol}(\text{SO}(2))^3 \cdot \frac{\gamma_{(k,m,l)}^*(s)}{2^{5+(k+m+l)}} L\left(s + \frac{1}{2}, \sigma_k \times \sigma_l \times \sigma_m\right).$$

PROOF. For $a = (a_1, a_2, a_3) \in \mathbf{R}_+^3$ and $x \in \mathbf{R}$, we set

$$\begin{aligned} z &= x + \sqrt{-1}(a_1^2 + a_2^2 + a_3^2), \\ t(a) &= \text{diag}\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_3^{-1} \end{pmatrix}\right), \\ u(x) &= \text{diag}\left(\begin{pmatrix} 1 & \frac{x}{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{x}{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{x}{3} \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

When $x \neq 0$, the Iwasawa decomposition of $\eta\iota(u(x)t(a))$ can be described as follows: Put

$$P = \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{pmatrix}.$$

We write $\eta\iota(u(x)t(a)) = \mathbf{n}(z)\mathbf{m}(A)\mathbf{u}$ with $z \in \text{Sym}_3(\mathbf{R})$, $A \in \text{GL}_3(\mathbf{R})$ and $\mathbf{u} = \begin{pmatrix} D & -C \\ C & D \end{pmatrix} \in \text{U}(3)$. Since $D^{-1}C = x^{-1}P$, we can choose $U \in \text{GL}_3(\mathbf{R})$ so that

$$U^t U = (x^2 \mathbf{1}_3 + P^2)^{-1}, \quad \mathbf{u} = \begin{pmatrix} Ux & -UP \\ UP & Ux \end{pmatrix} \in \text{U}(3).$$

Put $u = Ux - \sqrt{-1}UP$. Then $u^t u = (x\mathbf{1}_3 - P\sqrt{-1})(x\mathbf{1}_3 + P\sqrt{-1})^{-1}$. By direct computations we get

$$\det A = a_1 a_2 a_3 |z|^{-1}, \quad \det u = \frac{\bar{z}}{|z|}, \quad H_{23}(u) = -2\sqrt{-1} \frac{a_2 a_3}{z}, \quad H_{12}(u) = -2\sqrt{-1} \frac{a_1 a_2}{z}$$

(see [GK92, (6.7), (6.8)]). Put

$$b = \frac{1}{2}(k-l-\lambda_2), \quad c = \frac{1}{2}(k-m-\lambda_3), \quad (\mathbf{s}, \mathbf{k}, \mathbf{l}, \mathbf{m}) = \left(s + \frac{\lambda_3}{2}, k - \lambda_2, l, m - \lambda_1\right).$$

It follows that

$$\begin{aligned} f_{s,\infty}^{[k,\lambda]}(\eta\iota(u(x)t(a)\mathbf{d}(-1))) &= \chi_\infty(-1)(-1)^{k-\lambda_1} \left(2\sqrt{-1} \frac{a_1 a_2}{\bar{z}}\right)^{\lambda_1} \left(-2\sqrt{-1} \frac{a_2 a_3}{z}\right)^{\lambda_2} \left(\frac{a_1 a_2 a_3}{|z|}\right)^{2s+2} \left(\frac{z}{|z|}\right)^{k-\lambda_1} \\ &= \chi_\infty(-1) 2^{\lambda_1+\lambda_2} \sqrt{-1}^{\lambda_2-\lambda_1} (a_1 a_2 a_3)^{2s+2} a_1^{-\lambda_2} a_3^{-\lambda_1} |z|^{-2s-2-\mathbf{k}} (-z)^{\mathbf{k}}. \end{aligned}$$

From (4.2) and (4.5) $\frac{2Z_\infty(s)}{\text{vol}(\text{SO}(2))^3}$ equals

$$\begin{aligned} & \hat{\omega}(-1) \int_{\mathbf{R}} \int_{\mathbf{R}_+^3} W_k(\mathbf{n}(x/3)\mathbf{m}(a_1)) W_l^{[b]}(\mathbf{n}(x/3)\mathbf{m}(a_2)) W_m^{[c]}(\mathbf{n}(x/3)\mathbf{m}(a_3)) f_{s,\infty}^{[k,\lambda]}(\boldsymbol{\eta}(u(x)t(a)\mathbf{d}(-1))) dx \prod_{j=1}^3 \frac{d^\times a_j}{|a_j|^2} \\ & = \chi_\infty \hat{\omega}(-1) 2^{\lambda_1+\lambda_2} \sqrt{-1}^{\lambda_2-\lambda_1} (-4\pi)^{-b-c} \sum_{A=0}^{\infty} \sum_{B=0}^{\infty} (-4\pi)^{A+B} \binom{b}{A} \binom{c}{B} \frac{\Gamma(l+b)}{\Gamma(l+A)} \frac{\Gamma(m+c)}{\Gamma(m+B)} \\ & \quad \times \int_{\mathbf{R}} \int_{\mathbf{R}_+^3} a_1^{2s+\mathbf{k}} a_2^{2s+1+2A} a_3^{2s+\mathbf{m}+2B} |z|^{-2s-2-\mathbf{k}} (-z)^{\mathbf{k}} e^{2\pi\sqrt{-1}x} e^{-2\pi(a_1^2+a_2^2+a_3^2)} dx \prod_{j=1}^3 d^\times a_j. \end{aligned}$$

Put $\alpha = \mathbf{s} + 1 + \frac{\mathbf{k}}{2}$ and $\beta = \mathbf{s} + 1 - \frac{\mathbf{k}}{2}$. The last integral equals

$$\frac{(-2\pi\sqrt{-1})^\alpha (2\pi\sqrt{-1})^\beta}{\Gamma(\alpha)\Gamma(\beta)} \int_{\mathbf{R}_+^4} a_1^{\mathbf{k}+2s} a_2^{1+2A+2s} a_3^{\mathbf{m}+2B+2s} \frac{(1+t)^{\alpha-1} t^{\beta-1}}{e^{4\pi(a_1^2+a_2^2+a_3^2)}(1+t)} dt \prod_{j=1}^3 d^\times a_j.$$

We here use the identity

$$\int_{\mathbf{R}} \frac{e^{-2\pi\sqrt{-1}x} dx}{(x + \sqrt{-1}y)^\alpha (x - \sqrt{-1}y)^\beta} = \frac{(-2\pi\sqrt{-1})^\alpha (2\pi\sqrt{-1})^\beta}{\Gamma(\alpha)\Gamma(\beta)} \int_{\mathbf{R}_+} \frac{(t+1)^{\alpha-1} t^{\beta-1}}{e^{2\pi y(1+2t)}} dt$$

(see [GK92, (6.11)]). The quadruple integral above equals

$$\begin{aligned} & \frac{1}{(4\pi)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} \int_{\mathbf{R}_+^4} \frac{a_1^{\mathbf{k}+2s} a_2^{1+2A+2s} a_3^{\mathbf{m}+2B+2s} (1+t)^{\alpha-1} t^{\beta-1}}{e^{a_1^2+a_2^2+a_3^2} (1+t)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} dt \prod_{j=1}^3 d^\times a_j \\ & = \frac{\Gamma(\frac{\mathbf{k}}{2} + \mathbf{s}) \Gamma(\frac{1}{2} + \mathbf{s} + A) \Gamma(\frac{\mathbf{m}}{2} + \mathbf{s} + B)}{2^3 (4\pi)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} \int_{\mathbf{R}_+} \frac{(1+t)^{\alpha-1} t^{\beta-1}}{(1+t)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} dt. \end{aligned}$$

Recall that

$$\begin{aligned} \int_0^\infty \frac{(1+t)^{\alpha-1} t^{\beta-1}}{(1+t)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} dt & = B \left(\beta, 1 - \alpha - \beta + \frac{\mathbf{k}+1+\mathbf{m}}{2} + 3s + A + B \right) \\ & = \frac{\Gamma(\beta)\Gamma(1 - \alpha - \beta + \frac{\mathbf{k}+1+\mathbf{m}}{2} + 3s + A + B)}{\Gamma(1 - \alpha + \frac{\mathbf{k}+1+\mathbf{m}}{2} + 3s + A + B)}. \end{aligned}$$

We finally get

$$\begin{aligned} Z_\infty(s) & = \text{vol}(\text{SO}(2))^3 \chi_\infty(-1) 2^{\lambda_1+2\lambda_2-2-4s-3k} \pi^{2-s+\frac{\lambda_2+\lambda_3-3k}{2}} (\sqrt{-1})^{k+\lambda_1} (-1)^{\lambda_2+b+c} \\ & \quad \times \frac{\Gamma(\mathbf{s} + \frac{\mathbf{k}}{2})}{\Gamma(\mathbf{s} + \frac{\mathbf{k}}{2} + 1)} \Gamma(l+b)\Gamma(m+c) \sum_{A,B} (-1)^{A+B} \binom{b}{A} \binom{c}{B} \frac{\Gamma_\infty(s; A, B)}{\Gamma(l+A)\Gamma(m+B)}, \end{aligned}$$

where

$$\Gamma_\infty(\mathbf{s}; A, B) = \frac{\Gamma(\mathbf{s} + \frac{1}{2} + A) \Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + B) \Gamma(\mathbf{s} + \frac{\mathbf{k}+1+\mathbf{m}}{2} - 1 + A + B)}{\Gamma(2\mathbf{s} + \frac{1+\mathbf{m}}{2} + A + B)}.$$

Lemma 3 of [Orl87] with $\alpha = l = 1$, $t = \mathbf{s} + \frac{1}{2}$, $\beta = B + \frac{\mathbf{m}-l}{2}$ and $N = b$ gives

$$\begin{aligned} & \Gamma(l+b) \sum_{A=0}^b (-1)^A \binom{b}{A} \frac{\Gamma(\mathbf{s} + \frac{1}{2} + A) \Gamma(\mathbf{s} + \frac{\mathbf{k}+1+\mathbf{m}}{2} - 1 + A + B)}{\Gamma(l+A) \Gamma(2\mathbf{s} + \frac{l+\mathbf{m}}{2} + A + B)} \\ & = (-1)^b \frac{\Gamma(\mathbf{s} + \frac{1}{2}) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + l + b - 1) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + b) \Gamma(\mathbf{s} - \frac{1}{2} + 1)}{\Gamma(2\mathbf{s} + B + \frac{\mathbf{m}+l}{2} + b) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2}) \Gamma(\mathbf{s} - \frac{1}{2} - b + 1)}. \end{aligned}$$

It follows that

$$\begin{aligned} & \Gamma(l+b)\Gamma(m+c) \sum_{A,B} (-1)^{A+B} \binom{b}{A} \binom{c}{B} \frac{\Gamma_\infty(s; A, B)}{\Gamma(l+A)\Gamma(m+B)} \\ &= (-1)^b \frac{\Gamma(\mathbf{s} + \frac{l}{2})\Gamma(\mathbf{s} - \frac{l}{2} + 1)}{\Gamma(\mathbf{s} - \frac{l}{2} - b + 1)} \Gamma(m+c) \sum_B (-1)^B \binom{c}{B} \frac{\Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + l + b - 1)\Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + b)}{\Gamma(2\mathbf{s} + B + \frac{\mathbf{m}+l}{2} + b)\Gamma(m+B)}. \end{aligned}$$

Again we apply Lemma 3 of [Orl87] with $\alpha = m$, $t = \mathbf{s} + \frac{\mathbf{m}}{2} + b$, $\beta = \frac{l-\mathbf{m}}{2} - b$ and $N = c$ to obtain

$$\begin{aligned} & \Gamma(m+c) \sum_B (-1)^B \binom{c}{B} \frac{\Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + l + b - 1)\Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + b)}{\Gamma(2\mathbf{s} + B + \frac{\mathbf{m}+l}{2} + b)\Gamma(m+B)} \\ &= (-1)^c \frac{\Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + b)\Gamma(\mathbf{s} + \frac{l}{2} + m + c - 1)\Gamma(\mathbf{s} + \frac{l}{2} + c)\Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + b - m + 1)}{\Gamma(2\mathbf{s} + \frac{\mathbf{m}+l}{2} + b + c)\Gamma(\mathbf{s} + \frac{l}{2})\Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} - m + b - c + 1)}. \end{aligned}$$

Then we can see that the double summation equals

$$\begin{aligned} & (-1)^{b+c} \frac{\Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + b)\Gamma(\mathbf{s} + \frac{l}{2} + m + c - 1)\Gamma(\mathbf{s} + \frac{l}{2} + c)\Gamma(\mathbf{s} + c - \frac{l}{2} + 1)}{\Gamma(\mathbf{s} - \frac{l}{2} - b + 1)\Gamma(2\mathbf{s} + \frac{\mathbf{m}+l}{2} + b + c)} \\ &= (-1)^{b+c} \frac{\Gamma(\mathbf{s} + \frac{k-l+m}{2})\Gamma(\mathbf{s} + \frac{k+l+m}{2} - 1)\Gamma(\mathbf{s} + \frac{k-l-m}{2} + 1)\Gamma(\mathbf{s} + \frac{k-m+l}{2})}{\Gamma(2\mathbf{s} + k)} \cdot \frac{1}{\Gamma(\mathbf{s} - \frac{k}{2} + 1)}. \end{aligned}$$

The last equality uses $b = \frac{k-1}{2}$, $m+c = \frac{k+\mathbf{m}}{2}$, $\mathbf{s} + c = \mathbf{s} + \frac{k-m}{2}$ and $2\mathbf{s} + \mathbf{k} + \mathbf{m} = 2\mathbf{s} + k + m$. \square

5. CLASSICAL AND p -ADIC MODULAR FORMS

5.1. Conventions. Besides the standard symbols \mathbf{Z} , \mathbf{Q} , \mathbf{R} , \mathbf{C} , \mathbf{Z}_ℓ , \mathbf{Q}_ℓ we denote by \mathbf{R}_+ the group of strictly positive real numbers. Fix algebraic closures of \mathbf{Q} and \mathbf{Q}_p , denoting them by $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_p$. Let \mathbf{A} be the ring of adèles of \mathbf{Q} and μ_n the group of n -th roots of unity in $\overline{\mathbf{Q}}$. Put $\widehat{\mathbf{Z}} = \prod_\ell \mathbf{Z}_\ell$. Given a place v of \mathbf{Q} , we write \mathbf{Q}_v for the completion of \mathbf{Q} with respect to v . We shall regard \mathbf{Q}_v and \mathbf{Q}_v^\times as subgroups of \mathbf{A} and \mathbf{A}^\times in a natural way. We denote by the formal symbol ∞ the real place of \mathbf{Q} and do not use ℓ for the infinite place. Let $\psi_{\mathbf{Q}} : \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{C}^\times$ be the additive character with the archimedean component $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ and $\psi_\ell : \mathbf{Q}_\ell \rightarrow \mathbf{C}^\times$ the local component of $\psi_{\mathbf{Q}}$ at ℓ .

Denote by $\alpha_{\mathbf{Q}_v} = |\cdot|_v$ the absolute value on \mathbf{Q}_v normalized so that $\alpha_{\mathbf{R}}$ is the usual absolute value on \mathbf{R} , and $|\ell|_\ell = \ell^{-1}$ if $v = \ell$ is finite. For $a \in \mathbf{A}^\times$, let $a_v \in \mathbf{Q}_v^\times$ denote the v -component of a . Define the character $\alpha_{\mathbf{A}} = \mathbf{Q}^\times \backslash \mathbf{A}^\times \rightarrow \mathbf{R}_+$ by $\alpha_{\mathbf{A}}(a) = |a|_{\mathbf{A}} = \prod_v |a_v|_v$. Recall the local Riemann zeta functions

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2), \quad \zeta_\ell(s) = (1 - \ell^{-s})^{-1}.$$

Define the completed Riemann zeta function $\zeta_{\mathbf{Q}}(s)$ by $\zeta_{\mathbf{Q}}(s) = \prod_v \zeta_v(s)$. In particular, $\zeta_{\mathbf{Q}}(2) = \frac{\pi}{6}$. For each rational prime ℓ , let $v_\ell : \mathbf{Q}_\ell^\times \rightarrow \mathbf{Z}$ denote the valuation normalized so that $v_\ell(\ell) = 1$. To avoid possible confusion, denote by $\varpi_\ell = (\varpi_{\ell,v}) \in \mathbf{A}^\times$ the idèle defined by $\varpi_{\ell,\ell} = \ell$ and $\varpi_{\ell,v} = 1$ if $v \neq \ell$.

If $\omega : \mathbf{A}^\times \rightarrow \mathbf{C}^\times$ is a quasi-character, then we denote by $\omega_v : \mathbf{Q}_v^\times \rightarrow \mathbf{C}^\times$ the local component of ω at v . If $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \overline{\mathbf{Q}}^\times$ is a Dirichlet character modulo N , then we denote the ℓ -exponent of the conductor of χ by $c_\ell(\chi) \leq v_\ell(N)$. We can associate to a Dirichlet character χ of conductor N a Hecke character $\chi_{\mathbf{A}}$, called the *adèlic lift* of χ , which is the unique finite order Hecke character $\chi_{\mathbf{A}} : \mathbf{Q}^\times \mathbf{R}_+ \backslash \mathbf{A}^\times / (1 + N\widehat{\mathbf{Z}}) \cap \widehat{\mathbf{Z}}^\times \rightarrow \overline{\mathbf{Q}}^\times$ of conductor N such that $\chi_{\mathbf{A}}(\varpi_\ell) = \chi(\ell)^{-1}$ for any prime number $\ell \nmid N$.

Fix an odd prime number p and an isomorphism $\iota_p : \mathbf{Q}_p \simeq \mathbf{C}$ once and for all.

Definition 5.1 (Teichmüller and cyclotomic characters). The action of $G_{\mathbf{Q}}$ on $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$ gives rise to a continuous homomorphism $\varepsilon_{\text{cyc}} : G_{\mathbf{Q}} \rightarrow \mathbf{Z}_p^\times$, called the p -adic cyclotomic character, defined by $\sigma(\zeta) = \zeta^{\varepsilon_{\text{cyc}}(\sigma)}$ for every $\zeta \in \mu_{p^\infty}$. The character ε_{cyc} splits into the p -adic Teichmüller character $\omega : G_{\mathbf{Q}} \rightarrow \text{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \rightarrow \mathbf{Z}_p^\times$ and $\langle \cdot \rangle : G_{\mathbf{Q}} \rightarrow \text{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p$. The character ω sends σ to the unique solution in \mathbf{Z}_p^\times of $\omega(\sigma)^p = \omega(\sigma) \equiv \varepsilon_{\text{cyc}}(\sigma) \pmod{p}$. We often regard ω and $\langle \cdot \rangle^s$ with $s \in \mathbf{Z}_p$ as characters of \mathbf{Z}_p^\times . We sometimes identify ω with the Dirichlet character $\iota_p \circ \omega : (\mathbf{Z}/p\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$.

Remark 5.2. (1) Let χ be a Dirichlet character. If χ_v stands for the restriction of $\chi_{\mathbf{A}}$ to \mathbf{Q}_v^\times , then $\chi_\ell(\ell) = \chi(\ell)^{-1}$ for each prime number $\ell \nmid N$. Furthermore, if N is a power of p and b is not divisible by p , then $\chi_p(b) = \chi(b)$.

(2) Let χ be a character of \mathbf{Z}_p^\times of finite order, which can be regarded as either a complex character or a p -adic character via composition with ι_p . We view χ as a character of $G_{\mathbf{Q}}$ via composition with the cyclotomic character ε_{cyc} . Let $\mathbf{Q}^{\text{ab}} = \bigcup_{N=1}^{\infty} \mathbf{Q}(\mu_N)$ be the maximal abelian extension of \mathbf{Q} and

$$\text{rec}_{\mathbf{Q}} : \mathbf{Q}^\times \mathbf{R}_+ \backslash \mathbf{A}^\times \xrightarrow{\sim} \text{Gal}(\mathbf{Q}^{\text{ab}}/\mathbf{Q})$$

the geometrically normalized reciprocity law map, i.e., $\text{rec}_{\mathbf{Q}}(\varpi_\ell)|_{\mathbf{Q}^{\text{ab}}} = \text{Frob}_\ell$. Since χ factors through the quotient $\mathbf{Z}_p^\times \rightarrow (\mathbf{Z}/p^{c(\chi)}\mathbf{Z})^\times$, we can identify χ with a Dirichlet character of p -power conductor. Then since $\chi_{\mathbf{A}}(\varpi_\ell) = \chi(\ell)^{-1} = \chi(\varepsilon_{\text{cyc}}(\text{Frob}_\ell))$ for $\ell \neq p$,

$$\chi_{\mathbf{A}} = \chi \circ \varepsilon_{\text{cyc}} \circ \text{rec}_{\mathbf{Q}}, \quad \chi_p|_{\mathbf{Z}_p^\times} = \chi.$$

5.2. Differential operators and nearly holomorphic modular forms. Let $\text{GL}_2^+(\mathbf{R})$ be the subgroup of $\text{GL}_2(\mathbf{R})$ consisting of matrices with positive determinant and \mathfrak{H}_1 the upper half plane on which $\text{GL}_2^+(\mathbf{R})$ acts via fractional transformation. Define a subgroup of $\text{SL}_2(\mathbf{Z})$ of finite index

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbf{Z}) \mid N|c \right\}.$$

The Lie group $\text{GL}_2^+(\mathbf{R})$ acts on the complex vector space of complex valued functions f on \mathfrak{H}_1 as in (4.1).

The Maass-Shimura differential operators δ_k and λ_z on $C^\infty(\mathfrak{H}_1)$ are given by

$$\delta_k = \frac{1}{2\pi\sqrt{-1}} \left(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y} \right), \quad \lambda_z = -\frac{1}{2\pi\sqrt{-1}} y^2 \frac{\partial}{\partial \bar{z}}$$

with $y = \text{Im } z \in \mathbf{R}_+$. Let $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$ be a Dirichlet character, which we extend to a character $\chi^\downarrow : \Gamma_0(N) \rightarrow \mathbf{C}^\times$ by $\chi^\downarrow \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \chi(d)$. For a non-negative integer m the space $\mathcal{N}_k^{[m]}(N, \chi)$ of nearly holomorphic modular forms of weight k , level N and character χ consists of slowly increasing functions $f \in C^\infty(\mathfrak{H}_1)$ such that $\lambda_z^{m+1} f = 0$ and $f|_k \gamma = \chi^\downarrow(\gamma) f$ for $\gamma \in \Gamma_0(N)$ (cf. [Hid93, page 314]). Put $\mathcal{N}_k(N, \chi) = \bigcup_{m=0}^{\infty} \mathcal{N}_k^{[m]}(N, \chi)$ (cf. [Hid93, (1a), page 310]). By definition $\mathcal{N}_k^{[0]}(N, \chi) = \mathcal{M}_k(N, \chi)$ is the space of elliptic modular forms of weight k , level N and character χ . Denote the space of elliptic cusp forms in $\mathcal{M}_k(N, \chi)$ by $\mathcal{S}_k(N, \chi)$. Put $\delta_k^m = \delta_{k+2m-2} \cdots \delta_{k+2} \delta_k$. If $f \in \mathcal{N}_k(N, \chi)$, then $\delta_k^m f \in \mathcal{N}_{k+2m}(N, \chi)$ (see [Hid93, page 312]).

Define an open compact subgroup of $\text{GL}_2(\widehat{\mathbf{Z}})$ by

$$U_0(N) = \left\{ g \in \text{GL}_2(\widehat{\mathbf{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N\widehat{\mathbf{Z}}} \right\}.$$

We extend $\chi_{\mathbf{A}}$ to a character $\chi_{\mathbf{A}}^\downarrow$ of $U_0(N)$ by $\chi_{\mathbf{A}}^\downarrow(g) = \prod_{\ell|N} \chi_\ell^\downarrow(g_\ell)$ (see (2.4) for the definition of χ_ℓ^\downarrow). Let $\mathcal{A}_k(N, \chi_{\mathbf{A}}^{-1})$ be the space of functions $\Phi : \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$ such that $V_-^m \Phi = 0$ for some m and such that

$$\Phi(z\gamma g \kappa_\theta u) = \chi_{\mathbf{A}}(z)^{-1} \Phi(g) e^{\sqrt{-1}k\theta} \chi_{\mathbf{A}}^\downarrow(u)^{-1} \quad (z \in \mathbf{A}^\times, \gamma \in \text{GL}_2(\mathbf{Q}), \theta \in \mathbf{R}, u \in U_0(N)).$$

Definition 5.3 (The adèlic lift). With each nearly holomorphic modular form $f \in \mathcal{N}_k(N, \chi)$ we can associate a unique automorphic form $\Phi(f) \in \mathcal{A}_k(N, \chi_{\mathbf{A}}^{-1})$ defined by the equation

$$\Phi(f)(\gamma g_\infty u) := (f|_k g_\infty)(\sqrt{-1}) \cdot \chi_{\mathbf{A}}^\downarrow(u)^{-1}$$

for $\gamma \in \text{GL}_2(\mathbf{Q})$, $g_\infty \in \text{GL}_2^+(\mathbf{R})$ and $u \in U_0(N)$ (cf. [Cas73, §3]). We call $\Phi(f)$ the *adèlic lift* of f . Conversely, we can recover f from $\Phi(f)$ by

$$f(x + \sqrt{-1}y) = y^{-k/2} \Phi(f) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right).$$

Recall that V_\pm are the operators as defined in §4.4. By definition we have

$$\Phi(\delta_k f) = V_+ \Phi(f), \quad \Phi(\lambda_z f) = V_- \Phi(f).$$

We define the Whittaker coefficient and the constant term of $\Phi \in \mathcal{A}_k(N, \chi_{\mathbf{A}}^{-1})$ by

$$W(g, \Phi) = \int_{\mathbf{Q} \setminus \mathbf{A}} \Phi(\mathbf{n}(x)g) \overline{\psi_{\mathbf{Q}}(x)} dx, \quad \mathbf{a}_0(g, \Phi) = \int_{\mathbf{Q} \setminus \mathbf{A}} \Phi(\mathbf{n}(x)g) dx.$$

5.3. Ordinary \mathbf{I} -adic modular forms. For any subring $A \subset \mathbf{C}$ the space $\mathcal{S}_k(N, \chi; A)$ consists of elliptic cusp forms $f = \sum_{n=1}^{\infty} \mathbf{a}(n, f)q^n \in \mathcal{S}_k(N, \chi)$ such that $\mathbf{a}(n, f) \in A$ for all n . For every subring $A \subset \overline{\mathbf{Q}}_p$ containing $\mathbf{Z}[\chi]$ we define the space of cusp forms over A by

$$\mathcal{S}_k(N, \chi; A) = \mathcal{S}_k(N, \chi; \mathbf{Z}[\chi]) \otimes_{\mathbf{Z}[\chi]} A.$$

Here we view χ as a p -adic Dirichlet character via ι_p^{-1} .

Definition 5.4 (p -stabilized newforms). We say that a normalized Hecke eigenform $f \in \mathcal{S}_k(Np, \chi)$ is an (ordinary) p -stabilized newform (with respect to $\iota_p : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$) if f is new outside p and the eigenvalue of U_p , i.e. the p -th Fourier coefficient $\iota_p(\mathbf{a}(p, f))$, is a p -adic unit. The prime-to- p part N' of the conductor of f is called the tame conductor of f . There is a unique decomposition $\chi = \chi' \omega^a \epsilon$ with $a \in \mathbf{Z}/(p-1)\mathbf{Z}$, where χ' is a Dirichlet character modulo N' and ϵ is a character of $1 + p\mathbf{Z}_p$. We call $\chi' \omega^a$ the tame nebentypus of f .

Let $f^\circ = \sum_{n=1}^{\infty} \mathbf{a}(n, f^\circ)q^n \in \mathcal{S}_k(Np, \chi)$ be a primitive Hecke eigenform of conductor N_{f° . We call f° ordinary if $\iota_p^{-1}(\mathbf{a}(p, f^\circ))$ is a p -adic unit. If this is the case, then precisely one of the roots of the polynomial $X^2 - \mathbf{a}(p, f^\circ)X + \chi(p)p^{k-1}$ (call it $\alpha_p(f)$) satisfies $|\iota_p(\alpha_p(f))|_p = 1$. We associate to an ordinary primitive form f° the p -stabilized newform by

$$(5.1) \quad f(\tau) = f^\circ(\tau) - \frac{\chi(p)p^{k-1}}{\alpha_p(f)} f^\circ(p\tau) \in \mathcal{S}_k(N_{f^\circ}p, \chi),$$

if N_{f° and p are coprime, and $f = f^\circ$ if p divides N_{f° .

Let \mathcal{O} be the ring of integers of a finite extension of \mathbf{Q}_p and \mathbf{I} a normal domain finite flat over $\Lambda = \mathcal{O}[[1 + p\mathbf{Z}_p]]$. A point $Q \in \text{Spec } \mathbf{I}(\overline{\mathbf{Q}}_p)$, a ring homomorphism $Q : \mathbf{I} \rightarrow \overline{\mathbf{Q}}_p$, is said to be locally algebraic if the restriction of Q to $1 + p\mathbf{Z}_p$ is of the form $Q(z) = z^{k_Q} \epsilon_Q(z)$ with k_Q an integer and $\epsilon_Q(z) \in \mu_{p^\infty}$. We shall call k_Q the *weight* of Q and ϵ_Q the *finite part* of Q . Let $\mathfrak{X}_{\mathbf{I}}$ be the set of locally algebraic points $Q \in \text{Spec } \mathbf{I}(\overline{\mathbf{Q}}_p)$ of weight $k_Q \geq 1$. A point $Q \in \mathfrak{X}_{\mathbf{I}}$ is said to be *arithmetic* if $k_Q \geq 2$. Let $\mathfrak{X}_{\mathbf{I}}^+$ be the set of arithmetic points, $\wp_Q = \text{Ker } Q$ the prime ideal of \mathbf{I} corresponding to Q and $\mathcal{O}(Q)$ the image of \mathbf{I} under Q .

Let N be a positive integer prime to p and $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \rightarrow \mathcal{O}^\times$ a Dirichlet character modulo Np . An \mathbf{I} -adic cusp form is a formal power series $\mathbf{f}(q) = \sum_{n=1}^{\infty} \mathbf{a}(n, \mathbf{f})q^n \in \mathbf{I}[[q]]$ with the following property: there exists an integer $a_{\mathbf{f}}$ such that for arithmetic points $Q \in \mathfrak{X}_{\mathbf{I}}^+$ with $k_Q \geq a_{\mathbf{f}}$, the specialization $\mathbf{f}_Q(q) = \sum_{n=1}^{\infty} Q(\mathbf{a}(n, \mathbf{f}))q^n$ is the Fourier expansion of a cusp form $\mathbf{f}_Q \in \mathcal{S}_{k_Q}(Np^e, \chi \omega^{-k_Q} \epsilon_Q; \mathcal{O}(Q))$. Denote by $\mathbf{S}(N, \chi, \mathbf{I})$ the space of \mathbf{I} -adic cusp forms of tame level N and (even) branch character χ . This space $\mathbf{S}(N, \chi, \mathbf{I})$ is equipped with the action of the Hecke operators T_ℓ for $\ell \nmid Np$ as in [Wil88, page 537] and the operators \mathbf{U}_ℓ for $\ell \mid pN$ given by $\mathbf{U}_\ell(\sum_n \mathbf{a}(n, \mathbf{f})q^n) = \sum_n \mathbf{a}(n\ell, \mathbf{f})q^n$.

Hida's ordinary projector e_{ord} is defined by

$$e_{\text{ord}} := \lim_{n \rightarrow \infty} \mathbf{U}_p^n.$$

It has a well-defined action on the space of classical modular forms preserving the cuspidal part as well as on the space $\mathbf{S}(N, \chi, \mathbf{I})$ (cf. [Wil88, page 537 and Proposition 1.2.1]). The space $\mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I}) := e_{\text{ord}}\mathbf{S}(N, \chi, \mathbf{I})$ is called the space of ordinary \mathbf{I} -adic forms with respect to χ . Put

$$\mathcal{M}_k^{\text{ord}}(N, \chi; A) = e_{\text{ord}}\mathcal{M}_k(Np^e, \chi; A), \quad \mathbf{S}_k^{\text{ord}}(N, \chi; A) = e_{\text{ord}}\mathcal{S}_k(Np^e, \chi; A)$$

where e is any integer that is greater than the exponent of the p -primary part of the conductor of χ . A key result in Hida's theory for ordinary \mathbf{I} -adic cusp forms is that if $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$, then for every arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$, we have $\mathbf{f}_Q \in \mathcal{S}_{k_Q}^{\text{ord}}(N, \chi \omega^{-k_Q} \epsilon_Q; \mathcal{O}(Q))$. We call $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$ a *primitive Hida family* if \mathbf{f}_Q is a cuspidal p -stabilized newform of tame level N for every arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$.

6. A p -ADIC FAMILY OF PULL-BACKS OF SIEGEL EISENSTEIN SERIES

6.1. Siegel Eisenstein series. We work in adèlic form, which allows us to assemble Eisenstein series out of local data. Put $K_n = \mathrm{U}(n) \mathrm{GSp}_{2n}(\widehat{\mathbf{Z}})$. Fix characters $\chi, \hat{\omega}$ of \mathbf{Z}_p^\times of finite order and extend them to Hecke characters $\chi_{\mathbf{A}}, \hat{\omega}_{\mathbf{A}} : \mathbf{Q}^\times \backslash \mathbf{A}^\times \rightarrow \mathbf{C}^\times$ by composition with the quotient map $\mathbf{Q}^\times \backslash \mathbf{R}_+ \backslash \mathbf{A}^\times \simeq \widehat{\mathbf{Z}}^\times \rightarrow \mathbf{Z}_p^\times$. We regard χ as either a p -adic character or a complex character via composition with ι_p . For each place v we write χ_v for the restriction of $\chi_{\mathbf{A}}$ to \mathbf{Q}_v^\times . Our setting means that $\chi_p = \chi$ and $\chi_\ell(\ell) = \chi(\ell)^{-1}$ for $\ell \neq p$. Let

$$I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s) = \mathrm{Ind}_{\mathcal{P}_3(\mathbf{A})}^{\mathrm{GSp}_6(\mathbf{A})} (\chi_{\mathbf{A}}^2 \hat{\omega}_{\mathbf{A}} \boxtimes \chi_{\mathbf{A}}^{-3} \hat{\omega}_{\mathbf{A}}^{-1} \alpha_{\mathbf{A}}^s) \simeq \otimes'_v I_3(\hat{\omega}_v^{-1}, \chi_v \hat{\omega}_v \alpha_{\mathbf{Q}_v}^s)$$

be the global degenerate principal series representation of $\mathrm{GSp}_6(\mathbf{A})$ on the space of right K_3 -finite functions $f : \mathrm{GSp}_6(\mathbf{A}) \rightarrow \mathbf{C}$ satisfying the transformation laws

$$f(\mathbf{n}(z) \mathbf{m}(A, \nu) g) = \hat{\omega}_{\mathbf{A}}(\nu^{-1} \det A) \chi_{\mathbf{A}}(\nu^{-3} (\det A)^2) |\nu^{-3} (\det A)^2|^{1+s} f(g)$$

for $A \in \mathrm{GL}_3(\mathbf{A})$, $\nu \in \mathbf{A}^\times$, $z \in \mathrm{Sym}_3(\mathbf{A})$ and $g \in \mathrm{GSp}_6(\mathbf{A})$. We define global holomorphic sections of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$ similarly. The Eisenstein series associated to a holomorphic section f_s of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$ is defined by

$$E_{\mathbf{A}}(g, f_s) = \sum_{\gamma \in \mathcal{P}_3(\mathbf{Q}) \backslash \mathrm{GSp}_6(\mathbf{Q})} f_s(\gamma g).$$

Such series is absolutely convergent for $\mathrm{Re} s > 1$ and can be continued to a meromorphic function in s on the whole plane.

Let k be an integer and λ a parity type. Fix a square-free integer N which is not divisible by p . We write $\hat{\omega} = \omega_1 \omega_2 \omega_3$ as a product of three characters $\omega_1, \omega_2, \omega_3$ of \mathbf{Z}_p^\times . Set

$$\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3).$$

Assume that $\hat{\omega}_\infty = \mathrm{sgn}^{k-\lambda_1}$. Now we define a holomorphic section of $I_3(\hat{\omega}_v^{-1}, \chi_v \hat{\omega}_v \alpha_{\mathbf{Q}_v}^s)$ for $v \nmid N$:

- In the archimedean case we consider the section $f_{s,\infty}^{[k,\lambda]}$ defined in §4.1;
- In the p -adic case we consider $f_{\mathcal{D},s,p}$, where the section $f_{\mathcal{D},s,p}$ of $I_3(\hat{\omega}_p^{-1}, \chi_p \hat{\omega}_p \alpha_{\mathbf{Q}_p}^s)$ is attached to the quadruplet \mathcal{D} in Definition 2.5;
- If ℓ and Np are coprime, then $f_{s,\ell}^0$ is the section with $f_{s,\ell}^0(\mathrm{GSp}_6(\mathbf{Z}_\ell)) = 1$.

Let $f_{s,N}$ be an arbitrary holomorphic section of $\otimes_{\ell \mid N} I_3(\hat{\omega}_\ell^{-1}, \chi_\ell \hat{\omega}_\ell \alpha_{\mathbf{Q}_\ell}^s)$ for the moment. We define the normalized Siegel Eisenstein series

$$E_{\mathbf{A}}^*(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = L^{(\infty p N)}(2s+2, \chi_{\mathbf{A}}^2 \hat{\omega}_{\mathbf{A}}) L^{(\infty p N)}(4s+2, \chi_{\mathbf{A}}^4 \hat{\omega}_{\mathbf{A}}^2) \gamma_{(k,l,m)}^*(s)^{-1} \cdot E_{\mathbf{A}}(g, f_{\mathcal{D},s,N}^{[k,\lambda]}),$$

where $\gamma_{(k,l,m)}^*(s)$ is defined in §4.4 and $f_{\mathcal{D},s,N}^{[k,\lambda]}$ is a global holomorphic section of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$ defined by

$$f_{\mathcal{D},s,N}^{[k,\lambda]}(g) = f_{s,\infty}^{[k,\lambda]}(g_\infty) f_{s,N}((g_\ell)_{\ell \mid N}) f_{\mathcal{D},s,p}(g_p) \prod_{\ell \nmid Np} f_{s,\ell}^0(g_\ell).$$

Since $f_{\mathcal{D},s,p}$ is supported in the big cell $\mathcal{P}_3(\mathbf{Q}_p) J_3 \mathcal{P}_3(\mathbf{Q}_p)$, we have the Fourier expansion

$$(6.1) \quad E_{\mathbf{A}}(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = \sum_{B \in \mathrm{Sym}_3(\mathbf{Q})} W_B(g, f_{\mathcal{D},s,N}^{[k,\lambda]})$$

if $g_p \in \mathcal{P}_3(\mathbf{Q}_p)$. Recall that for a holomorphic section f_s of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$

$$W_B(g, f_s) = \int_{\mathrm{Sym}_3(\mathbf{A})} f_s(J_3 \mathbf{n}(z) g) \psi_{\mathbf{Q}}(-\mathrm{tr}(Bz)) dz.$$

6.2. The Fourier expansion of the pull-back of Eisenstein series. Recall that

$$\Xi_p = \{(b_{ij}) \in \mathrm{Sym}_3(\mathbf{Z}_p) \mid b_{11}, b_{22}, b_{33} \in p\mathbf{Z}_p \text{ and } b_{12}, b_{23}, b_{31} \in \mathbf{Z}_p^\times\}.$$

Now we evaluate its pull-back at $s_0 = \frac{k-\lambda_1}{2} - r - 1$. Let $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N}) : \mathfrak{H}_1^3 \rightarrow \mathbf{C}$ be the modular form of weight $(k, k-\lambda_2, k-\lambda_3)$ defined by

$$\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(x + y\sqrt{-1}, f_{s_0,N}) := \lim_{s \rightarrow s_0} \frac{E_{\mathbf{A}}^*(\iota(\mathbf{n}(x_1) \mathbf{m}(\sqrt{y_1}), \mathbf{n}(x_2) \mathbf{m}(\sqrt{y_2}), \mathbf{n}(x_3) \mathbf{m}(\sqrt{y_3})), f_{\mathcal{D},s,N}^{[k,\lambda]})}{\sqrt{y_1}^k \sqrt{y_2}^{k-\lambda_2} \sqrt{y_3}^{k-\lambda_3}}$$

for $y = (y_1, y_2, y_3) \in \mathbf{R}_+^3$ and $x = (x_1, x_2, x_3) \in \mathbf{R}^3$.

Since ω_i factors through the quotient $\mathbf{Z}_p^\times \rightarrow (\mathbf{Z}/p^{c(\omega_i)}\mathbf{Z})^\times$, we can view ω_i as a Dirichlet character. The polynomial $F_{B,\ell}$ is defined in §2.2. We here set $\mathbf{Q}_N = \prod_{\ell|N} \mathbf{Q}_\ell$. Let Sym_3^+ denote the set of positive definite rational symmetric matrices of rank 3.

Proposition 6.1. *Put $n = \max\{1, c(\chi), c(\omega_i)\}$. The pull-back $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N})$ is a nearly holomorphic cusp form on \mathfrak{H}_1^3 of level $\Gamma_0(Np^{2n})^3$ and nebentypus $(\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1})$ with Fourier expansion given by*

$$\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N}) = \frac{C_1^{[k,r,\lambda]}}{\gamma_{(k,l,m)}^*(\frac{k-\lambda_1}{2} - r - 1)} \sum_{B \in \text{Sym}_3^+ \cap \Xi_p} \mathbf{W}_B^{[k,r,\lambda]}(y) \cdot \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) b_{B,N}^{[k,r,\lambda]} q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where

$$a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) := \prod_{\ell|Np} F_{B,\ell}(\chi_\ell(\ell)^2 \hat{\omega}_\ell(\ell) \ell^{2r+\lambda_1-k}),$$

$$b_{B,N}^{[k,r,\lambda]} := \lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} \int_{\text{Sym}_3(\mathbf{Q}_N)} f_{s,N}(J_3 \mathbf{n}(z)) \psi_{\mathbf{Q}}(-\text{tr}(Bz)) dz.$$

PROOF. The level and nebentypus are determined by Proposition 2.6. Note that $\det B \in \mathbf{Z}_p^\times$ for $B \in \Xi_p$. In particular, $W_B(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = 0$ unless $\det B \neq 0$. Lemma 4.4 says that $W_B(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = 0$ unless $B \in T_3^+$. We can derive the Fourier expansion formula from (6.1), recalling that local Whittaker functions

$$\lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} W_B(\mathbf{m}(A), f_{s,\infty}^{[k,\lambda]}), \quad W_B(\mathbf{1}_6, f_{s,\ell}^0), \quad W_B(\mathbf{1}_6, f_{\mathcal{D},s,p})$$

are computed in (2.3), Proposition 2.6 and Lemma 4.4, respectively. \square

6.3. Holomorphic and ordinary projections of $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}$. Recall that λ_z is the weight-lowering operator defined in §5.2. We write Hol for the holomorphic projection. Let T_3^+ denote the set of positive definite symmetric half-integral matrices of rank 3.

Definition 6.2. Define a holomorphic section $f_{s,\ell}$ of $I_3(\hat{\omega}_\ell^{-1}, \chi_\ell \hat{\omega}_\ell \alpha_{\mathbf{Q}_\ell}^s)$ by letting $f_{s,\ell} = f_{\Phi_\ell,s}$ with $\Phi_\ell = \mathbb{I}_{\text{Sym}_3(\mathbf{Z}_\ell)}$. When $f_{s,N} = \otimes_{\ell|N} f_{s,\ell}$, we write $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]} = \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N})$. If $B \in T_3^+$, then $b_{B,N}^{[k,r,\lambda]} = 1$ by (2.12). Proposition 6.1 gives

$$\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]} = \frac{C_1^{[k,r,\lambda]}}{\gamma_{(k,l,m)}^*(\frac{k-\lambda_1}{2} - r - 1)} \sum_{B \in T_3^+ \cap \Xi_p} \mathbf{W}_B^{[k,r,\lambda]}(y) \cdot \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

Proposition 6.3. *Let λ be the parity type of (k, l, m) with $k \geq l \geq m$, and r an integer which satisfies*

$$k - \frac{l+m+\lambda_1}{2} \leq r \leq \frac{l+m}{2} - 2.$$

Put $n = k - r - 2 + \frac{l+m-\lambda_1}{2}$. Then $e_{\text{ord}} \text{Hol} \left(\lambda_{z_2}^{\frac{k-l-\lambda_2}{2}} \lambda_{z_3}^{\frac{k-m-\lambda_3}{2}} \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]} \right)$ has the q -expansion

$$(-1)^{k+\frac{m+l+\lambda_1}{2}+\lambda_2} \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B(\chi \varepsilon_{\text{cyc}}^n, \omega_1 \varepsilon_{\text{cyc}}^{-k}, \omega_2 \varepsilon_{\text{cyc}}^{-l}, \omega_3 \varepsilon_{\text{cyc}}^{-m}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

PROOF. Put $b = \frac{k-l-\lambda_2}{2}$ and $c = \frac{k-m-\lambda_3}{2}$. If f is a holomorphic function on \mathfrak{H}_1 , then

$$\lambda_z^n (y^{-a} f) = \begin{cases} (4\pi)^{-n} n! \binom{a}{n} \cdot y^{n-a} f & \text{if } n \leq a, \\ 0 & \text{if } n > a. \end{cases}$$

By (4.4) the difference

$$\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(q) - C_1^{[k,r,\lambda]} \frac{b!c!}{(4\pi)^{b+c}} \sum_B Q_{0,b,c}^{[k,\lambda]}(B, r) \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}$$

belongs to $(y_1^{-1}, y_2^{-1}, y_3^{-1})\mathbf{C}[y_1^{-1}, y_2^{-1}, y_3^{-1}][[q_1, q_2, q_3]]$. On the other hand, we can write

$$\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D}, N}^{[k, r, \lambda]}(q) = \text{Hol}(\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D}, N}^{[k, r, \lambda]})(q) + \sum_{i+j+t \geq 1} \delta_{k-i}^i f_i(q_1) \delta_{l-j}^j g_j(q_2) \delta_{m-t}^t h_t(q_3),$$

where f_i , g_j and h_t are holomorphic modular forms. Equating the constant terms of this identity as a polynomial in $y_1^{-1}, y_2^{-1}, y_3^{-1}$ and employing the relation

$$(6.2) \quad \delta_k^t = \sum_{a=0}^t \binom{t}{a} \frac{\Gamma(t+k)}{\Gamma(a+k)} (-4\pi y)^{a-t} \left(\frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial z} \right)^a$$

(see [Hid93, (3), page 311]), we see that the holomorphic projection $\text{Hol}(\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D}, N}^{[k, r, \lambda]})(q)$ equals

$$C_1^{[k, r, \lambda]} \frac{b! c!}{(4\pi)^{b+c}} \sum_B Q_{0, b, c}^{[k, \lambda]}(B, r) \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}} - \sum_{i+j+t \geq 1} \theta^i f_i(q_1) \theta^j g_j(q_2) \theta^t h_t(q_3).$$

Here θ stands for the Serre's operator $\theta(\sum_i a_i q^i) = \sum_i i a_i q^i$. Since $e_{\text{ord}} \theta = 0$, the q -expansion of the ordinary projection $e_{\text{ord}} \text{Hol}(\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D}, N}^{[k, r, \lambda]})(q)$ equals

$$C_1^{[k, r, \lambda]} \frac{b! c!}{(4\pi)^{b+c}} \sum_B \mathbf{c}_B \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where

$$\mathbf{c}_B = \lim_{j \rightarrow \infty} Q_{0, b, c}^{[k, \lambda]}(B_j, r) \mathcal{Q}_{B_j}(\mathcal{D}) a_{B_j}(\chi^2 \hat{\omega}, k - 2r - \lambda_1), \quad B_j := \begin{pmatrix} p^{j^1} b_{11} & b_{12} & b_{13} \\ b_{12} & p^{j^1} b_{22} & b_{23} \\ b_{13} & b_{23} & p^{j^1} b_{33} \end{pmatrix}.$$

Since $p^{j^1} \rightarrow 1$ in \mathbf{Z}_ℓ as $j \rightarrow \infty$ for any rational prime $\ell \neq p$, we get

$$\mathcal{Q}_{B_j}(\mathcal{D}) = \mathcal{Q}_B(\mathcal{D}), \quad \lim_{j \rightarrow \infty} a_{B_j}(\chi^2 \hat{\omega}, k - 2r - \lambda_1) = a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1).$$

Since $Q_{0, b, c}^{[k, \lambda]}(B, r)$ is a polynomial in B , we find that

$$\mathbf{c}_B = Q_{0, b, c}^{[k, \lambda]}(B_\infty, r) \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1), \quad B_\infty = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{12} & 0 & b_{23} \\ b_{13} & b_{23} & 0 \end{pmatrix}.$$

Applying Lemma 4.7 to $Q_{0, b, c}^{[k, \lambda]}(B_\infty, r)$, we obtain

$$\mathbf{c}_B = w_{0, b, c} \cdot 2^{-3n+k+l+m} \mathcal{Q}_B(\chi \varepsilon_{\text{cyc}}^n, \omega_1 \varepsilon_{\text{cyc}}^{-k}, \omega_2 \varepsilon_{\text{cyc}}^{-l}, \omega_3 \varepsilon_{\text{cyc}}^{-m}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1)$$

in view of Definition 2.5 of \mathcal{Q}_B . We thus obtain the lemma by noting the equality

$$(-1)^{k + \frac{m+l+\lambda_1}{2} + \lambda_2} \gamma_{(k, l, m)}^* \left(\frac{k - \lambda_1}{2} - r - 1 \right) = C_1^{[k, r, \lambda]} b! c! (4\pi)^{-b-c} 2^{-3n+k+l+m} \cdot w_{0, b, c}.$$

The constant $C_1^{[k, r, \lambda]}$ is defined in Lemma 4.4. The equality can be checked by the following items:

- The power of 2:

$$3(3 + 2r - k - \lambda_2) + \{2(k - r) - 3\} - 2b - 2c + (k + l + m - 3n) \\ + (7M - 3b - 3c - 3\lambda_1) = -2 - k + 2(l + m) + \lambda_1 + 2\lambda_2.$$

- The power of π : $(6 - 2) - b - c + (3M - b - c - 2\lambda_1 - \lambda_2) = -3r + k + l + m + \lambda_2 - \lambda_1 - 2$.

□

6.4. The modular forms $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$.

Definition 6.4. Let (k_1, k_2, k_3) be a triplet of positive integers. Put $k^* = \max\{k_1, k_2, k_3\}$. We say that (k_1, k_2, k_3) is *balanced* if $2k^* < k_1 + k_2 + k_3$. An integer n is said to be *critical* for (k_1, k_2, k_3) if

$$k^* \leq n \leq k_1 + k_2 + k_3 - k^* - 2.$$

Definition 6.5. Fix a balanced triplet (k_1, k_2, k_3) of positive integers. Take a permutation σ of $\{1, 2, 3\}$ so that $k^* = k_{\sigma(1)} \geq k_{\sigma(2)} \geq k_{\sigma(3)}$. Denote the parity type of $(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})$ by $\delta = (\delta_1, \delta_2, \delta_3)$. For each critical integer n for (k_1, k_2, k_3) and quadruplet $\mathcal{D} = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ of finite-order p -adic characters of \mathbf{Z}_p^\times we define the modular form $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$ by

$$G_{k_1, k_2, k_3}^{[n]}(\mathcal{D}) := (-1)^{k + \frac{m+l+\lambda_1}{2} + \lambda_2} e_{\text{ord}} \text{Hol} \left(\lambda_{z_{\sigma(2)}}^{\frac{k^* - k_{\sigma(2)} - \delta_2}{2}} \lambda_{z_{\sigma(3)}}^{\frac{k^* - k_{\sigma(3)} - \delta_3}{2}} \mathbf{E}_{\mathcal{D}}^{[k^*, r, \delta]} \right),$$

where $r = \left\lceil \frac{k^* + k_1 + k_2 + k_3}{2} \right\rceil - n - 2$ and $\mathcal{D} = (\iota_p \circ \epsilon_0, \iota_p \circ \epsilon_1, \iota_p \circ \epsilon_2, \iota_p \circ \epsilon_3)$.

Corollary 6.6. *With notation in Definition 6.5, $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$ is an ordinary cusp form of weight (k_1, k_2, k_3) , level $\Gamma_0(Np^\infty)^3$ and nebentypus $(\epsilon_1^{-1}, \epsilon_2^{-1}, \epsilon_3^{-1})$ whose q -expansion at the infinity cusp is given by*

$$\sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B(\epsilon_0 \epsilon_{\text{cyc}}^n, \epsilon_1 \epsilon_{\text{cyc}}^{-k_1}, \epsilon_2 \epsilon_{\text{cyc}}^{-k_2}, \epsilon_3 \epsilon_{\text{cyc}}^{-k_3}) a_B(\epsilon_0^2 \epsilon_1 \epsilon_2 \epsilon_3, 2n - (k_1 + k_2 + k_3) + 4) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

PROOF. The assertion for the Fourier expansion is a direct consequence of Proposition 6.3 by symmetry. Lemma 6.7 below implies the cuspidality of $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$. \square

Lemma 6.7. *Let $f \in \mathcal{M}_k(N, \chi; A)$. Assume that $\mathbf{a}_0(g, \Phi(f)) = 0$ whenever $g_p \in B_2(\mathbf{Q}_p)$. Then $e_{\text{ord}} f \in \mathcal{S}_k^{\text{ord}}(N, \chi; A)$.*

PROOF. Our task is to prove that $\mathbf{a}_0(g, \Phi(f)) = 0$ for all $g \in \text{GL}_2(\mathbf{A})$. Since

$$\mathbf{a}_0(\gamma \mathbf{n}(x) \text{diag}(a, d) g \kappa_\theta, \Phi(f)) = (ad^{-1})^{k/2} e^{\sqrt{-1}k\theta} \mathbf{a}_0(g_f, \Phi(f))$$

for $\gamma \in B_2(\mathbf{Q})$, $x \in \mathbf{A}$, $a, d \in \mathbf{R}_+$ and $\theta \in \mathbf{R}$, it suffices to show that $\mathbf{a}_0(g, \Phi(f)) = 0$ for all $g \in \text{GL}_2(\widehat{\mathbf{Z}})$. Since

$$\text{GL}_2(\mathbf{Z}_p) = \mathbf{n}^-(p\mathbf{Z}_p) B_2(\mathbf{Z}_p) \sqcup \mathbf{n}(\mathbf{Z}_p) J_1 B_2(\mathbf{Z}_p),$$

where $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have only to show that $\mathbf{a}_0(h\mathbf{n}^-(y), \Phi(f)) = \mathbf{a}_0(hJ_1, \Phi(f)) = 0$ for all $h \in \text{GL}_2(\widehat{\mathbf{Z}}^{(p)})$ and $y \in p\mathbf{Z}_p$. Recall that the operator \mathbf{U}_p is defined by

$$[\mathbf{U}_p \Phi](g, f) = p^{(k-2)/2} \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \Phi \left(g \begin{pmatrix} \varpi_p & x \\ 0 & 1 \end{pmatrix}, f \right).$$

Recall that $\varpi_p \in \widehat{\mathbf{Q}}^\times$ is defined by $\varpi_{p,p} = p$ and $\varpi_{p,\ell} = 1$ for $\ell \neq p$. Since

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} \varpi_p^m & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varpi_p^m & x \\ 0 & 1+xy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi_p^m y & 1+xy \end{pmatrix} \in B_2(\mathbf{Q}_p) U_0(N)$$

for $y \in p\mathbf{Z}_p$, $x \in \mathbf{Z}_p$ and sufficiently large m , we get

$$\mathbf{a}_0(h\mathbf{n}^-(y), \mathbf{U}_p^m f) = p^{(k-2)m/2} \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \Phi \left(h \begin{pmatrix} \varpi_p^m & x \\ 0 & 1+xy \end{pmatrix}, f \right) = 0$$

by assumption. It follows that $\mathbf{a}_0(h\mathbf{n}^-(y), f) = \lim_{n \rightarrow \infty} \mathbf{a}_0(h\mathbf{n}^-(y), \mathbf{U}_p^n f) = 0$. If $x \in p^n \mathbf{Z}_p^\times$ with $n < m$, then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi_p^m & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varpi_p^{m-n} & -\varpi_p^n x^{-1} \\ 0 & \varpi_p^n \end{pmatrix} \begin{pmatrix} \varpi_p^n x^{-1} & 0 \\ \varpi_p^{m-n} & \varpi_p^{-n} x \end{pmatrix} \in B_2(\mathbf{Q}_p) \mathbf{n}^-(p\mathbf{Z}_p).$$

One can therefore see that

$$\mathbf{a}_0(hJ_1, \mathbf{U}_p^m f) = p^{(k-2)m/2} \mathbf{a}_0(hJ_1 \text{diag}(\varpi_p^m, 1), f) = p^{(k-1)m/2} \mathbf{a}_0(\text{diag}(1, \varpi_p^{-m}) hJ_1, f),$$

from which we conclude that

$$\mathbf{a}_0(hJ_1, f) = \lim_{n \rightarrow \infty} p^{(k-1)n!/2} \mathbf{a}_0(\text{diag}(1, \varpi_{(p)}^{-n!})hJ_1, f) = 0.$$

Here $\varpi_{(p)} \in \widehat{\mathbf{Z}}^\times$ is defined by $\varpi_{(p),p} = 1$ and $\varpi_{(p),\ell} = p$ for $\ell \neq p$. \square

6.5. The p -adic interpolation of $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$. Let $\mathbf{u} = 1 + p \in 1 + p\mathbf{Z}_p$ be a topological generator. We identify $\mathcal{O}[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]$ with $\mathcal{O}[X]$ where $X = [\mathbf{u}] - 1$ with the group-like element $[\mathbf{u}]$ in Λ . Put

$$\Lambda = \mathcal{O}[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})], \quad \Lambda_3 = \mathcal{O}[X_1, X_2, X_3], \quad \Lambda_4 = \Lambda_3[[T]].$$

For each ℓ and $B \in T_3^+$, let $F_{B,\ell}(X) \in \mathbf{Z}[X]$ be as defined in (2.3). Let $\alpha_X : \mathbf{Z}_p^\times \rightarrow \mathbf{Z}_p[[X]]^\times$ be the character $\alpha_X(z) = \langle z \rangle_X = (1 + X)^{\log_p z / \log_p \mathbf{u}}$. Let $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$ be a triplet of \mathcal{O} -valued finite-order characters of \mathbf{Z}_p^\times . For each $a \in \mathbf{Z}/(p-1)\mathbf{Z}$ we define the formal power series $\mathcal{G}_{\underline{\chi}}^{(a)} \in \Lambda_4[[q_1, q_2, q_3]]$ by

$$\mathcal{G}_{\underline{\chi}}^{(a)}(X_1, X_2, X_3, T) = \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) \cdot \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where $\mathcal{Q}_B^{(a)}$ and $\mathcal{F}_B^{(a)} \in \Lambda_3[[T]]$ are power series given by

$$\begin{aligned} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) &= \omega(8b_{23}b_{31}b_{12})^a \langle 8b_{23}b_{31}b_{12} \rangle_T \chi_1(2b_{23})^{-1} \langle 2b_{23} \rangle_{X_1}^{-1} \chi_2(2b_{31})^{-1} \langle 2b_{31} \rangle_{X_2}^{-1} \chi_3(2b_{12})^{-1} \langle 2b_{12} \rangle_{X_3}^{-1}, \\ \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) &= \prod_{\ell \nmid pN} F_{B,\ell}(\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)} \ell^{-2}), \end{aligned}$$

where

$$\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)} := (\omega^{-2a} \chi_1 \chi_2 \chi_3)(\ell) \ell^{-2} \cdot \langle \ell \rangle_{X_1} \langle \ell \rangle_{X_2} \langle \ell \rangle_{X_3} \langle \ell \rangle_T^{-2} \in \Lambda_4^\times.$$

Here the set $\mathfrak{X}_{\Lambda_4}^{\text{bal}}$ consists of $(Q, P) = (Q_1, Q_2, Q_3, P) \in (\mathfrak{X}_\Lambda^+)^3 \times \mathfrak{X}_\Lambda \subset \text{Spec } \Lambda_4(\overline{\mathbf{Q}}_p)$ such that $(k_{Q_1}, k_{Q_2}, k_{Q_3})$ is balanced and k_P is critical for $(k_{Q_1}, k_{Q_2}, k_{Q_3})$.

Proposition 6.8. *For every $(Q, P) \in \mathfrak{X}_{\Lambda_4}^{\text{bal}}$, we have*

$$\mathcal{G}_{\underline{\chi}}^{(a)}(Q, P) = G_{k_{Q_1}, k_{Q_2}, k_{Q_3}}^{[k_P]}(\epsilon_P \omega^{a-k_P}, \chi_1^{-1} \epsilon_{Q_1}^{-1} \omega^{k_{Q_1}}, \chi_2^{-1} \epsilon_{Q_2}^{-1} \omega^{k_{Q_2}}, \chi_3^{-1} \epsilon_{Q_3}^{-1} \omega^{k_{Q_3}}).$$

In particular, this implies that

$$\mathcal{G}_{\underline{\chi}}^{(a)} \in \mathbf{S}^{\text{ord}}(N, \chi_1, \mathcal{O}[X_1]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_2, \mathcal{O}[X_2]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_3, \mathcal{O}[X_3]) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[T].$$

PROOF. Set $\chi := \epsilon_P \omega^{a-k_P}$, $\omega_i = \chi_i^{-1} \epsilon_{Q_i}^{-1} \omega^{k_{Q_i}}$ and $\hat{\omega} = \omega_1 \omega_2 \omega_3$. One can check that

$$\begin{aligned} \mathcal{Q}_B^{(a)}(Q, P) &= \frac{(\epsilon_P \omega^a)(8b_{12}b_{23}b_{13}) \langle 8b_{12}b_{23}b_{13} \rangle^{k_P}}{(\chi_1 \epsilon_{Q_1})(2b_{23})(\chi_2 \epsilon_{Q_2})(2b_{13})(\chi_3 \epsilon_{Q_3})(2b_{12}) (2b_{23})^{k_{Q_1}} (2b_{31})^{k_{Q_2}} (2b_{12})^{k_{Q_3}}} \\ &= \mathcal{Q}_B(\chi \epsilon_{\text{cyc}}^{k_P}, \omega_1 \epsilon_{\text{cyc}}^{-k_{Q_1}}, \omega_2 \epsilon_{\text{cyc}}^{-k_{Q_2}}, \omega_3 \epsilon_{\text{cyc}}^{-k_{Q_3}}), \\ \langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}(Q, P) &= (\omega^{-2a} \chi_1 \chi_2 \chi_3)(\ell) \ell^{-2} \cdot (\epsilon_{Q_1} \epsilon_{Q_2} \epsilon_{Q_3} \epsilon_P^{-2} \omega^{2k_P - k_{Q_1} - k_{Q_2} - k_{Q_3}})(\ell)^{-1} \ell^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 2k_P} \\ &= \chi_\ell^2(\ell) |\ell|^{2k_P + 2} \hat{\omega}_\ell(\ell) |\ell|^{-(k_{Q_1} + k_{Q_2} + k_{Q_3})}, \\ \mathcal{F}_B^{(a)}(Q, P) &= a_B(\chi^2 \hat{\omega}, 2k_P - (k_{Q_1} + k_{Q_2} + k_{Q_3}) + 4) \end{aligned}$$

(see Definition 2.5 of \mathcal{Q}_B). Recall the convention that $\chi_\ell(\ell) = \iota_p(\chi(\ell))^{-1}$ and $\hat{\omega}_\ell(\ell) = \iota_p(\hat{\omega}(\ell))^{-1}$ (see Remark 5.2). From Corollary 6.6, we deduce the interpolation formula and that

$$(6.3) \quad \mathcal{G}_{\underline{\chi}}^{(a)}(Q, P) \in \mathbf{S}_{k_{Q_1}}^{\text{ord}}(N, \omega_1^{-1}; \mathcal{O}(Q_1)) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}_{k_{Q_2}}^{\text{ord}}(N, \omega_2^{-1}; \mathcal{O}(Q_2)) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}_{k_{Q_3}}^{\text{ord}}(N, \omega_3^{-1}; \mathcal{O}(Q_3)) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(P).$$

By the control theorem for ordinary Λ -adic forms [Hid93, Theorem 3, p.215], for any arithmetic point Q , the specialization map $X \mapsto \mathbf{u}^{k_Q} \epsilon_Q(\mathbf{u}) - 1$ yields an isomorphism

$$\mathbf{S}^{\text{ord}}(N, \chi, \mathcal{O}[X]) / (1 + X - \mathbf{u}^{k_Q} \epsilon_Q(\mathbf{u})) \simeq \mathbf{S}_{k_Q}^{\text{ord}}(N, \chi \omega^{-k_Q} \epsilon_Q; \mathcal{O}(Q)).$$

Hence, from (6.3) we find that for all P with $k_P = 2$

$$\mathcal{G}_{\underline{\chi}}^{(a)}(X_1, X_2, X_3, P) \in \mathbf{S}^{\text{ord}}(N, \chi_1, \mathcal{O}[X_1]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_2, \mathcal{O}[X_2]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_3, \mathcal{O}[X_3]) \otimes_{\mathcal{O}} \mathcal{O}(P).$$

Now we can deduce the second statement from the above equation combined with the argument in [Hid93, Lemma 1, page 328]. \square

7. FOUR-VARIABLE p -ADIC TRIPLE PRODUCT L -FUNCTIONS

7.1. Measures. We shall normalize the Haar measures dx_v on \mathbf{Q}_v and $d^\times x_v$ on \mathbf{Q}_v^\times as follows: Let dx_∞ denote the usual Lebesgue measure on \mathbf{R} and $d^\times x_\infty = \frac{dx_\infty}{|x_\infty|}$. If $v = \ell$ is finite, then $\text{vol}(\mathbf{Z}_\ell, dx_\ell) = \text{vol}(\mathbf{Z}_\ell^\times, d^\times x_\ell) = 1$. Define the compact subgroups \mathbf{K}_v of $\text{GL}_2(\mathbf{Q}_v)$ and \mathbf{K}'_v of $\text{SL}_2(\mathbf{Q}_v)$ by

$$\mathbf{K}_\infty = \text{O}(2, \mathbf{R}), \quad \mathbf{K}_\ell = \text{GL}_2(\mathbf{Z}_\ell), \quad \mathbf{K}'_\infty = \text{SO}(2, \mathbf{R}), \quad \mathbf{K}'_\ell = \text{SL}_2(\mathbf{Z}_\ell).$$

Let dk_v and dk'_v be the Haar measures on \mathbf{K}_v and \mathbf{K}'_v which have total volume 1.

The Haar measure dg_v on $\text{PGL}_2(\mathbf{Q}_v)$ is given by $dg_v = |y_v|_v^{-1} dx_v d^\times y_v dk_v$ for $g_v = \begin{pmatrix} y_v & x_v \\ 0 & 1 \end{pmatrix}$ with $y_v \in \mathbf{Q}_v^\times$, $x_v \in \mathbf{Q}_v$ and $k_v \in \mathbf{K}_v$. Define the Haar measure dg'_v on $\text{SL}_2(\mathbf{Q}_v)$ by $dg'_v = |y_v|_v^{-2} dx_v d^\times y_v dk'_v$ for $g_v = \mathbf{n}(x_v)\mathbf{m}(y_v)k'_v$ with $y_v \in \mathbf{Q}_v^\times$, $x_v \in \mathbf{Q}_v$ and $k'_v \in \mathbf{K}'_v$. The Tamagawa measures dg on $\text{PGL}_2(\mathbf{A})$ and dg' on $\text{SL}_2(\mathbf{A})$ are given by $dg = \zeta_{\mathbf{Q}}(2)^{-1} \prod_v dg_v$ and $dg' = \zeta_{\mathbf{Q}}(2)^{-1} \prod_v dg'_v$. Since $Z \backslash H \simeq \text{PGL}_2 \times \text{SL}_2 \times \text{SL}_2$, we can define the Tamagawa measure on $Z \backslash H$ by $dg_1 dg'_2 dg'_3$, where dg_1 is the Tamagawa measure on $\text{PGL}_2(\mathbf{A})$ and $dg'_2 = dg'_3$ are that on $\text{SL}_2(\mathbf{A})$. The Tamagawa numbers of PGL_2 , SL_2 and $Z \backslash H$ are 2, 1 and 2, respectively.

7.2. Garrett's integral representation. Let π_i ($i = 1, 2, 3$) be an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbf{A})$ generated by an elliptic cusp form of weight k_i and nebentypus ω_i^{-1} . Put $\hat{\omega} = \omega_1 \omega_2 \omega_3$ and $\tilde{\pi}_i = \pi_i \otimes \omega_i^{-1}$ for $i = 1, 2, 3$. Fix a character $\chi_{\mathbf{A}}$ of $\mathbf{A}^\times / \mathbf{Q}^\times \mathbf{R}_+$. For each triplet of cusp forms $\varphi_i \in \tilde{\pi}_i$ and a holomorphic section f_s of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}})$ we consider the global zeta integral defined by

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \int_{Z(\mathbf{A})H(\mathbf{Q}) \backslash H(\mathbf{A})} \varphi_1(g_1) \varphi_2(g_2) \varphi_3(g_3) E_{\mathbf{A}}(\iota(g_1, g_2, g_3), f_s) dg_1 dg_2 dg_3.$$

The integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic in s . Unfolding the Eisenstein series as in [PSR87], we get

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \int_{Z(\mathbf{A})U^0(\mathbf{A}) \backslash H(\mathbf{A})} W(g_1, \varphi_1) W(g_2, \varphi_2) W(g_3, \varphi_3) f_s(\delta \iota(g_1, g_2, g_3)) dg_1 dg_2 dg_3.$$

If $W(g, \varphi_i) = \prod_v W_{i,v}(g_v)$ and $f_s(g) = \prod_v f_{s,v}(g_v)$ are factorizable, then the integral factors into a product of local integrals and so by §2.2

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \frac{\zeta_{\mathbf{Q}}(2)^{-3} L(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})}{L^S(2s + 2, \chi_{\mathbf{A}}^2 \hat{\omega}_{\mathbf{A}}) L^S(4s + 2, \chi_{\mathbf{A}}^4 \hat{\omega}_{\mathbf{A}}^2)} \prod_{v \in S} \frac{Z(W_{1,v}, W_{2,v}, W_{3,v}, f_{s,v})}{L(s + \frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \chi_v)},$$

where S is a large enough set of places such that $\pi_{i,\ell}$, $W_{i,\ell}$, χ_ℓ and $f_{s,\ell}$ are unramified for all $\ell \notin S$. The complete L -function $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$ admits meromorphic continuation and a functional equation

$$L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}) = \varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}) L(1 - s, \pi_1 \times \pi_2 \times \pi_3 \otimes \hat{\omega}_{\mathbf{A}}^{-1} \chi_{\mathbf{A}}^{-1}).$$

By Theorem 2.7 of [Ike92] the L -function $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$ has a pole if and only if there exists an imaginary quadratic field E and characters of χ_i of $\mathbf{A}_E^\times / E^\times$ such that $\chi_1 \chi_2 \chi_3 \chi^E = 1$ and such that π_i is induced automorphically from χ_i , where χ^E denotes the base change of χ to E . Recall that $k^* = \max\{k_1, k_2, k_3\}$. In particular, if $k_1 + k_2 + k_3 \geq 2k^* + 2$, then $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$ is holomorphic everywhere. Let us put

$$\mathcal{J}_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbf{R}), \quad t_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix} \in \text{GL}_2(\mathbf{Q}_p) \hookrightarrow \text{GL}_2(\mathbf{A}).$$

Let $E_{\mathbf{A}}^*(f_{\mathcal{D},s,N}^{[k_1,\lambda]})$ be the Eisenstein series defined in §6.1.

Lemma 7.1. *Let $f_i \in \mathcal{S}_{k_i}(N_i, \omega_i^{-1})$ be an ordinary p -stabilized newform. Put*

$$\varphi_i = \Phi(f_i), \quad \check{\varphi}_i = \varphi_i \otimes \omega_{i,\mathbf{A}}^{-1}, \quad W(\varphi_i) = \prod_v W_{i,v}, \quad \check{W}_{i,v} = W_{i,v} \otimes \omega_{i,v}^{-1},$$

Let χ be a character of \mathbf{Z}_p^\times of finite order. Put $n = \{1, c(\chi), c(\omega_i)\}$. Let N be the least common multiple of N_1, N_2, N_3 . If $k_1 \geq k_2 \geq k_3$, λ is the parity type of (k_1, k_2, k_3) and N is square-free, then

$$\begin{aligned} & Z\left(\rho(\mathcal{J}_\infty t_n) \check{\varphi}_1, \rho(\mathcal{J}_\infty t_n) V_+^{\frac{k_1-k_2-\lambda_2}{2}} \check{\varphi}_2, \rho(\mathcal{J}_\infty t_n) V_+^{\frac{k_1-k_3-\lambda_3}{2}} \check{\varphi}_3, E_{\mathbf{A}}^*(f_{\mathcal{D},s,N}^{[k_1,\lambda]})\right) \\ &= L^{(N)}\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) E_p\left(s + \frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right) \\ & \quad \times \prod_{i=1}^3 \frac{\zeta_p(2)}{\zeta_p(1)} \left(\frac{\alpha_p(f_i)^2}{p^{k_i \omega_{i,p}(p)}}\right)^n \frac{\prod_{\ell|N} Z(\check{W}_{1,\ell}, \check{W}_{2,\ell}, \check{W}_{3,\ell}, f_{s,\ell})}{\zeta_{\mathbf{Q}}(2)^{3 \cdot 2^5 + (k_1+k_2+k_3)}} \end{aligned}$$

PROOF. By Garrett's integral representation of triple L -functions the left hand side equals

$$\begin{aligned} & \zeta_{\mathbf{Q}}(2)^{-3} L^{(\infty p N)}\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) \\ & \times \gamma_{(k_1, k_2, k_3)}^*(s)^{-1} Z\left(\rho(\mathcal{J}_\infty) \check{W}_{1,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_2-\lambda_2}{2}} \check{W}_{2,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_3-\lambda_3}{2}} \check{W}_{3,\infty}, f_{s,\infty}^{[k_1,\lambda]}\right) \\ & \times Z(\rho(t_n) \check{W}_{1,p}, \rho(t_n) \check{W}_{2,p}, \rho(t_n) \check{W}_{3,p}, f_{\mathcal{D},s,p}) \prod_{\ell|N} Z(\check{W}_{1,\ell}, \check{W}_{2,\ell}, \check{W}_{3,\ell}, f_{s,\ell}) \end{aligned}$$

in view of Definition 6.2 of $E_{\mathbf{A}}^*(f_{\mathcal{D},s}^{[k_1,\lambda]})$. Since $\check{W}_{i,\infty} = W_{i,\infty}$, Lemma 4.8 yields

$$\begin{aligned} & Z\left(\rho(\mathcal{J}_\infty) \check{W}_{1,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_2-\lambda_2}{2}} \check{W}_{2,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_3-\lambda_3}{2}} \check{W}_{3,\infty}, f_{s,\infty}^{[k_1,\lambda]}\right) \\ &= Z_\infty(s) = \chi_\infty(-1) \frac{\text{vol}(\text{SO}(2))^{3\gamma_{(k_1, k_2, k_3)}^*(s)}}{2^{5+(k_1+k_2+k_3)}} L\left(s + \frac{1}{2}, \sigma_{k_1} \times \sigma_{k_2} \times \sigma_{k_3} \otimes \chi_\infty\right). \end{aligned}$$

Proposition 2.6 calculates the p -adic part:

$$\begin{aligned} & \frac{Z(\rho(t_n) \check{W}_{1,p}, \rho(t_n) \check{W}_{2,p}, \rho(t_n) \check{W}_{3,p}, f_{\mathcal{D},s,p})}{L\left(s + \frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right)} \prod_{i=1}^3 \frac{\zeta_p(1)}{\zeta_p(2)} \left(\frac{p^{k_i \omega_{i,p}(p)}}{\alpha_p(f_i)^2}\right)^n \\ &= Z_p^*(f_{\mathcal{D},s,p}) = \chi_p(-1) E_p\left(s + \frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right). \end{aligned}$$

Since $\chi_{\mathbf{A}}$ is unramified outside p , we have $\chi_\infty(-1) = \chi_p(-1)$. \square

7.3. The congruence number. Put $\Delta = (\mathbf{Z}/Np\mathbf{Z})^\times$. Let $\hat{\Delta}$ be the group of Dirichlet characters modulo Np . Enlarging \mathcal{O} if necessary, we assume that every $\chi \in \hat{\Delta}$ takes value in \mathcal{O}^\times . Let

$$\mathbf{S}^{\text{ord}}(N, \mathbf{I}) := \bigoplus_{\chi \in \hat{\Delta}} \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$$

be the space of ordinary \mathbf{I} -adic cusp forms of tame level $\Gamma_1(N)$. Let σ_d denote the usual diamond operator for $d \in \Delta$ acting on $\mathbf{S}^{\text{ord}}(N, \mathbf{I})$ by $\sigma_d(\mathbf{f})_{\chi \in \hat{\Delta}} = (\chi(d)\mathbf{f})_{\chi \in \hat{\Delta}}$. The ordinary \mathbf{I} -adic cuspidal Hecke algebra $\mathbf{T}(N, \mathbf{I})$ is defined as the \mathbf{I} -subalgebra of $\text{End}_{\mathbf{I}} \mathbf{S}^{\text{ord}}(N, \mathbf{I})$ generated over \mathbf{I} by the Hecke operators T_ℓ with $\ell \nmid Np$, the operators \mathbf{U}_ℓ with $\ell | Np$ and the diamond operators σ_d with $d \in \Delta$. Let $\mathbf{T}_k^{\text{ord}}(N, \chi)$ denote the \mathcal{O} -subalgebra of $\text{End}_{\mathcal{O}} e_{\text{ord}} \mathcal{S}_k(N, \chi)$ generated over \mathcal{O} by the operators T_ℓ with $\ell \nmid Np$ and \mathbf{U}_ℓ with $\ell | Np$.

Let $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$ be a primitive Hida family of tame conductor N and character χ . The corresponding homomorphism $\lambda_{\mathbf{f}} : \mathbf{T}(N, \mathbf{I}) \rightarrow \mathbf{I}$ is defined by $\lambda_{\mathbf{f}}(T_\ell) = \mathbf{a}(\ell, \mathbf{f})$ for $\ell \nmid Np$, $\lambda_{\mathbf{f}}(\mathbf{U}_\ell) = \mathbf{a}(\ell, \mathbf{f})$ for $\ell | Np$ and $\lambda_{\mathbf{f}}(\sigma_d) = \chi(d)$ for $d \in \Delta$. We denote by $\mathfrak{m}_{\mathbf{f}}$ the maximal ideal of $\mathbf{T}(N, \mathbf{I})$ containing $\text{Ker } \lambda_{\mathbf{f}}$ and by $\mathbf{T}_{\mathfrak{m}_{\mathbf{f}}}$ the localization of $\mathbf{T}(N, \mathbf{I})$ at $\mathfrak{m}_{\mathbf{f}}$. It is the local ring of $\mathbf{T}(N, \mathbf{I})$ through which $\lambda_{\mathbf{f}}$ factors. Recall that the congruence ideal $C(\mathbf{f})$ of the morphism $\lambda_{\mathbf{f}} : \mathbf{T}_{\mathfrak{m}_{\mathbf{f}}} \rightarrow \mathbf{I}$ is defined by

$$C(\mathbf{f}) := \lambda_{\mathbf{f}}(\text{Ann}_{\mathbf{T}_{\mathfrak{m}_{\mathbf{f}}}}(\text{Ker } \lambda_{\mathbf{f}})) \subset \mathbf{I}.$$

It is well-known that $\mathbf{T}_{\mathfrak{m}_{\mathbf{f}}}$ is a local finite flat Λ -algebra, and there is an algebra direct sum decomposition

$$(7.1) \quad \tilde{\lambda}_{\mathbf{f}} : \mathbf{T}_{\mathfrak{m}_{\mathbf{f}}} \otimes_{\mathbf{I}} \text{Frac } \mathbf{I} \simeq \text{Frac } \mathbf{I} \oplus \mathcal{B}, \quad t \mapsto \tilde{\lambda}_{\mathbf{f}}(t) = (\lambda_{\mathbf{f}}(t), \lambda_{\mathcal{B}}(t)),$$

where \mathcal{B} is some finite dimensional $(\text{Frac } \mathbf{I})$ -algebra ([Hid88b, Corollary 3.7]). Then we have

$$C(\mathbf{f}) = \lambda_{\mathbf{f}}(\mathbf{T}_{\mathfrak{m}_{\mathbf{f}}} \cap \tilde{\lambda}_{\mathbf{f}}^{-1}(\text{Frac } \mathbf{I} \oplus \{0\}))$$

by definition. Now we consider the following hypothesis:

Hypothesis (CR). The residual Galois representation $\bar{\rho}_{\mathbf{f}}$ of $\rho_{\mathbf{f}}$ is absolutely irreducible and p -distinguished.

Under the hypothesis (CR), $\mathbf{T}_{\mathbf{m}_{\mathbf{f}}}$ is Gorenstein by [Wil95, Corollary 2, page 482]. With this property of $\mathbf{T}_{\mathbf{m}_{\mathbf{f}}}$ Hida in [Hid88a] proved that the congruence ideal $C(\mathbf{f})$ is generated by a non-zero element $\eta_{\mathbf{f}} \in \mathbf{I}$, called the congruence number for \mathbf{f} . Let $1_{\mathbf{f}}^*$ be the unique element in $\mathbf{T}_{\mathbf{m}_{\mathbf{f}}} \cap \tilde{\lambda}_{\mathbf{f}}^{-1}(\text{Frac } \mathbf{I} \oplus \{0\})$ such that $\lambda_{\mathbf{f}}(1_{\mathbf{f}}^*) = \eta_{\mathbf{f}}$. Then $1_{\mathbf{f}} := \eta_{\mathbf{f}}^{-1} 1_{\mathbf{f}}^*$ is the idempotent in $\mathbf{T}_{\mathbf{m}_{\mathbf{f}}} \otimes_{\mathbf{I}} \text{Frac } \mathbf{I}$ corresponding to the direct summand $\text{Frac } \mathbf{I}$ of (7.1) and $1_{\mathbf{f}}$ does not depend on any choice of a generator of $C(\mathbf{f})$. For $d \in \widehat{\Delta}$ we write $\wp_{Q, \chi}$ for the ideal of $\mathbf{T}(N, \mathbf{I})$ generated by $\wp_Q = \text{Ker } Q$ and $\{\sigma_d - \chi(d)\}_{d \in \Delta}$. A classical result in Hida theory asserts that

$$\mathbf{T}(N, \mathbf{I})/\wp_{Q, \chi} \simeq \mathbf{T}_{k_Q}^{\text{ord}}(Np^e, \chi \omega^{-k_Q} \epsilon_Q) \otimes_{\mathcal{O}} \mathcal{O}(Q)$$

(see Theorem 3.4 of [Hid88b]). Moreover, for each arithmetic point Q , it is also shown by Hida that the specialization $\eta_{\mathbf{f}}(Q) \in \mathcal{O}(Q)$ is the congruence number for \mathbf{f}_Q and

$$1_{\mathbf{f}} := \eta_{\mathbf{f}}^{-1} 1_{\mathbf{f}}^* \pmod{\wp_{\chi, Q}} \in \mathbf{T}_{k_Q}^{\text{ord}}(Np^e, \chi \omega^{-k_Q} \epsilon_Q) \otimes_{\mathcal{O}} \text{Frac } \mathcal{O}(Q)$$

is the idempotent with $\lambda_{\mathbf{f}}(1_{\mathbf{f}}) = 1$.

Definition 7.2. Let \mathbf{f} be a primitive Hida family satisfying (CR). To each choice of the congruence number $\eta_{\mathbf{f}}$ we associate Hida's canonical period $\Omega_{\mathbf{f}}$ of a p -ordinary newform f of weight k obtained by the specialization of \mathbf{f} defined by

$$\Omega_{\mathbf{f}} := \eta_{\mathbf{f}}^{-1} \cdot (-2\sqrt{-1})^{k+1} \|f^\circ\|_{\Gamma_0(N_{f^\circ})}^2 \cdot \mathcal{E}_p(f, \text{Ad}),$$

where $\eta_{\mathbf{f}}$ is the specialization of $\eta_{\mathbf{f}}$, f° the primitive form associated with f , N_{f° its conductor and $\mathcal{E}_p(f, \text{Ad})$ the modified p -Euler factor attached to the adjoint motive of f (see §1.4).

7.4. Hida's functional. When $\varphi \in \mathcal{A}_k(N, \omega_{\mathbf{A}})$ and $\varphi' \in \mathcal{A}_k(N, \omega_{\mathbf{A}}^{-1})$ are cuspidal, we define the pairing by

$$\langle \rho(\mathcal{J}_\infty)\varphi, \varphi' \rangle = \int_{\mathbf{A} \times \text{GL}_2(\mathbf{Q}) \backslash \text{GL}_2(\mathbf{A})} \varphi(g\mathcal{J}_\infty)\varphi'(g) dg.$$

Let $f \in \mathcal{S}_k(N_f, \omega^{-1})$ be an ordinary p -stabilized newform of level N_f , i.e., $\mathbf{U}_p f = \alpha_p(f)f$ with p -unit $\iota_p^{-1}(\alpha_p(f))$. Write $N_f = N_t p^c$ with N_t prime to p . For $n \geq c$, we define Hida's functional L_f on $\mathcal{S}_k(N_t p^{2n}, \omega; \mathcal{O})$ by

$$L_f(\mathcal{F}) = \left(\frac{\omega_p(p)p^k}{\alpha_p(f)^2} \right)^{n-1} \frac{\langle \rho(\mathcal{J}_\infty t_n)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \Phi(\mathcal{F}) \rangle}{\langle \rho(\mathcal{J}_\infty t_1)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \varphi \rangle},$$

where $\varphi = \Phi(f)$ is the adèlic lift of f . Note that for $\mathcal{F} \in \mathcal{S}_k(Np^{2n}, \omega)$ with $N_t \mid N$,

$$L_f(\mathcal{F}) = [\Gamma_0(N) : \Gamma_0(N_t)]^{-1} L_f(\text{Tr}_{N/N_t} \mathcal{F}).$$

Lemma 7.3. (1) $L_f(f) = 1$.

(2) If $\mathcal{F}_0 \in \mathcal{N}_{k+2m}(Np^{2n}, \omega)$ with $N_t \mid N$, then

$$\frac{L_f(\mathbf{1}_{\mathbf{f}}^* \text{Tr}_{N/N_t} e_{\text{ord}} \text{Hol}(\lambda_z^m \mathcal{F}_0))}{\zeta_{\mathbf{Q}}(2)[\text{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} = (-1)^{m+1} (2\sqrt{-1})^{k+1} \frac{\langle \rho(\mathcal{J}_\infty t_n)(V_+^m \varphi \otimes \omega_{\mathbf{A}}^{-1}), \Phi(\mathcal{F}_0) \rangle}{\Omega_{\mathbf{f}} \left(\frac{\alpha_p(f)^2}{p^k \omega_p(p)} \right)^n \frac{\zeta_p(2)}{\zeta_p(1)}}.$$

PROOF. The first assertion follows from the following formula stated in [Hsi21, Lemma 3.6]:

$$\begin{aligned} \langle \rho(\mathcal{J}_\infty t_n)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \varphi \rangle &= \omega_\infty(-1) \langle \rho(\mathcal{J}_\infty t_n)\varphi \otimes \omega_{\mathbf{A}}^{-1}, \varphi \rangle \\ &= \frac{(-1)^k \zeta_{\mathbf{Q}}(2)^{-1}}{[\text{SL}_2(\mathbf{Z}) : \Gamma_0(N_t)]} \cdot \|f^\circ\|_{\Gamma_0(N_{f^\circ})}^2 \cdot \mathcal{E}_p(f, \text{Ad}) \cdot \frac{\alpha_p(f)^{2n} \zeta_p(2)}{p^{kn} \omega_p(p)^n \zeta_p(1)} \\ &= \frac{\zeta_{\mathbf{Q}}(2)^{-1}}{[\text{SL}_2(\mathbf{Z}) : \Gamma_0(N_t)]} \cdot \frac{\eta_{\mathbf{f}} \Omega_{\mathbf{f}}}{(2\sqrt{-1})^{k+1}} \cdot \left(\frac{\alpha_p(f)^2}{p^k \omega_p(p)} \right)^n \frac{\zeta_p(2)}{\zeta_p(1)}. \end{aligned}$$

We remark that $\check{\varphi} = \varphi$ and $\omega_{(p)} = \omega_{\mathbf{A}}$ in the notation of [Hsi21].

To see the second part, we note that as a consequence of strong multiplicity one theorem for elliptic modular forms, the idempotent $1_{\mathbf{f}} = \eta_{\mathbf{f}}^{-1} 1_{\mathbf{f}}^*$ is generated by the Hecke operators T_ℓ with $\ell \nmid Np$, which implies that

1_f is the adjoint operator of $1_{\varphi \otimes \omega_{\mathbf{A}}^{-1}}$ with respect to the pairing. We are thus led to $L_f(\mathbf{1}_f^* \mathcal{F}) = \eta_f L_f(\mathcal{F})$. Moreover, $L_f(\mathbf{U}_p \mathcal{F}) = \alpha_p(f) L_f(\mathcal{F})$ (cf. the proof of Proposition 2.10 of [Kob13]) and hence

$$L_f(e_{\text{ord}} \mathcal{F}) = \lim_{j \rightarrow \infty} L_f(\mathbf{U}_p^{j!} \mathcal{F}) = \lim_{j \rightarrow \infty} \alpha_p(f)^{j!} L_f(\mathcal{F}) = L_f(\mathcal{F}).$$

One can easily verify that for $\phi \in \Phi(\mathcal{S}_k(M, \chi^{-1}))$, $\mathcal{F}_1 \in \mathcal{N}_k(M, \chi)$ and $\mathcal{F}_2 \in \mathcal{N}_{k+2}(M, \chi)$

$$\langle \rho(\mathcal{J}_{\infty}) \phi, \Phi(\text{Hol } \mathcal{F}_1) \rangle = \langle \rho(\mathcal{J}_{\infty}) \phi, \Phi(\mathcal{F}_1) \rangle, \quad \langle \rho(\mathcal{J}_{\infty}) \phi, \Phi(\lambda_z \mathcal{F}_2) \rangle = -\langle \rho(\mathcal{J}_{\infty}) V_+ \phi, \Phi(\mathcal{F}_2) \rangle.$$

The second part is a consequence of these results. \square

7.5. The construction of p -adic triple product L -functions. Let

$$\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h}) \in \mathbf{S}^{\text{ord}}(N_1, \chi_1, \mathbf{I}) \times \mathbf{S}^{\text{ord}}(N_2, \chi_2, \mathbf{I}) \times \mathbf{S}^{\text{ord}}(N_3, \chi_3, \mathbf{I})$$

be a triplet of primitive \mathbf{I} -adic Hida families of tame square-free level (N_1, N_2, N_3) and tame characters (χ_1, χ_2, χ_3) , where \mathbf{I} is a finite flat domain over $\Lambda = \mathcal{O}[\Gamma]$. Assuming that all \mathbf{f} , \mathbf{g} and \mathbf{h} satisfy Hypothesis (CR), we fix a choice of the congruence numbers $(\eta_{\mathbf{f}}, \eta_{\mathbf{g}}, \eta_{\mathbf{h}})$. Let

$$\mathbf{1}_{\mathbf{f}}^* \in \mathbf{T}(N_1, \mathbf{I}), \quad \mathbf{1}_{\mathbf{g}}^* \in \mathbf{T}(N_2, \mathbf{I}), \quad \mathbf{1}_{\mathbf{h}}^* \in \mathbf{T}(N_3, \mathbf{I})$$

be the idempotents multiplied by a fixed choice of congruence numbers $(\eta_{\mathbf{f}}, \eta_{\mathbf{g}}, \eta_{\mathbf{h}})$ in the Hecke algebras attached to the newforms $(\mathbf{f}, \mathbf{g}, \mathbf{h})$. Put

$$N := \text{lcm}(N_1, N_2, N_3), \quad N^- := \text{gcd}(N_1, N_2, N_3), \quad \mathbf{I}_3 := \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I}.$$

Definition 7.4. Define the p -adic triple product L -function $L_{\mathbf{F},(a)}$ in $\mathbf{I}_3[[T]]$ by

$$L_{\mathbf{F},(a)} := \text{the first Fourier coefficient of } \mathbf{1}_{\mathbf{f}}^* \otimes \mathbf{1}_{\mathbf{g}}^* \otimes \mathbf{1}_{\mathbf{h}}^* (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\chi}^{(a)})) \in \mathbf{I}_3[[T]].$$

We denote by $V_{\mathbf{f}}$ the associated p -adic Galois representation, and by $\text{WD}_{\ell}(V_{\mathbf{f}_Q})$ the representation of the Weil-Deligne group $W_{\mathbf{Q}_{\ell}}$ attached to $V_{\mathbf{f}_Q}$ for each prime ℓ . The epsilon factor of $\mathbf{V}_{(\underline{Q}, P)}$ at ℓ is defined by

$$\varepsilon_{\ell}(\mathbf{V}_{(\underline{Q}, P)}, s) = \varepsilon(s + k_P - w_{\underline{Q}}/2, \text{WD}_{\ell}(V_{\mathbf{f}_{Q_1}}) \otimes \text{WD}_{\ell}(V_{\mathbf{g}_{Q_2}}) \otimes \text{WD}_{\ell}(V_{\mathbf{h}_{Q_3}}) \otimes \omega^{a-k_P} \epsilon_P, \psi_{\ell}).$$

By the assumption (sf) and the rigidity of automorphic types of Hida families $\text{WD}_{\ell}(V_{\mathbf{f}_{Q_1}})$, $\text{WD}_{\ell}(V_{\mathbf{g}_{Q_2}})$, $\text{WD}_{\ell}(V_{\mathbf{h}_{Q_3}})$ are either unramified or the Steinberg representation twisted by an unramified character. Moreover, for $\ell | N_1$, there is an unramified finite order character $\xi_{\mathbf{f}, \ell} : G_{\mathbf{Q}_{\ell}} \rightarrow \overline{\mathbf{Q}}^{\times}$ such that $\xi_{\mathbf{f}, \ell}^2 = \chi_{1, \ell}^{-1}$ and

$$V_{\mathbf{f}}|_{G_{\mathbf{Q}_{\ell}}} \simeq \begin{pmatrix} \xi_{\mathbf{f}, \ell} \varepsilon_{\text{cyc}} \langle \varepsilon_{\text{cyc}} \rangle_{\mathbf{I}}^{-1/2} & * \\ 0 & \xi_{\mathbf{f}, \ell} \langle \varepsilon_{\text{cyc}} \rangle_{\mathbf{I}}^{-1/2} \end{pmatrix}.$$

Let $N_{\mathbf{V}} = N^- N^4$ be the tame conductor of \mathbf{V} . When $\ell | N^-$, we define $\sqrt{\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}} \in \mathbf{I}_4^{\times}$ by

$$\sqrt{\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}} := (\omega^{-a} \xi_{\mathbf{f}, \ell} \xi_{\mathbf{g}, \ell} \xi_{\mathbf{h}, \ell}) (\ell) \ell^{-1} \cdot \langle \ell \rangle_{X_1}^{1/2} \langle \ell \rangle_{X_2}^{1/2} \langle \ell \rangle_{X_3}^{1/2} \langle \ell \rangle_T^{-1}.$$

Let $t_{\mathbf{V}}$ denote the number of prime factors of N^- . We define the \mathbf{I}_4 -adic root number $\varepsilon^{(p\infty)}(\mathbf{V}) \in \mathbf{I}_4^{\times}$ by

$$(7.2) \quad \varepsilon^{(p\infty)}(\mathbf{V}) = (-1)^{t_{\mathbf{V}}} \prod_{\ell | N} \langle \ell^2 \rangle_{X_1, X_2, X_3, T}^{(a)} \prod_{\ell | N^-} \sqrt{\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}}.$$

Lemma 7.5. *Notation being as above, we get*

$$\varepsilon^{(p\infty)}(\mathbf{V}_{(\underline{Q}, P)}) = \prod_{\ell \neq p} \varepsilon_{\ell}(\mathbf{V}_{(\underline{Q}, P)}, 0).$$

PROOF. We retain the notation of the proof of Proposition 6.8. Remark 3.5 gives

$$\varepsilon_{\ell}(\mathbf{V}_{(\underline{Q}, P)}, 0) = \chi_{\ell}(\ell)^4 \hat{\omega}(\ell)^2 \ell^{-4k_P + 2(k_{Q_1} + k_{Q_2} + k_{Q_3}) - 4} = \langle \ell^2 \rangle_{X_1, X_2, X_3, T}^{(a)}(\underline{Q}, P)$$

if ℓ divides N/N^- . Put $\xi_{\ell} = \xi_{\mathbf{f}, \ell} \xi_{\mathbf{g}, \ell} \xi_{\mathbf{h}, \ell}$. If ℓ divides N^- , then

$$\begin{aligned} \varepsilon_{\ell}(\mathbf{V}_{(\underline{Q}, P)}, 0) &= -\chi_{\ell}(\ell)^5 \xi_{\ell}(\text{Frob}_{\ell})^5 \ell^{5(-2k_P + k_{Q_1} + k_{Q_2} + k_{Q_3} - 2)/2} \\ &= -\langle \ell^2 \rangle_{X_1, X_2, X_3, T}^{(a)}(\underline{Q}, P) \sqrt{\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}(\underline{Q}, P)}. \end{aligned}$$

We have thus completed our proof. \square

7.6. The interpolation formulae. Let $\mathbf{V} = V_{\mathbf{f}} \widehat{\otimes}_{\mathcal{O}} V_{\mathbf{g}} \widehat{\otimes}_{\mathcal{O}} V_{\mathbf{h}} \widehat{\otimes}_{\mathcal{O}} \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T$ be the triple tensor product of \mathbf{I} -adic Galois representations associated with primitive Hida families \mathbf{f}, \mathbf{g} and \mathbf{h} twisted by $\omega^a \langle \varepsilon_{\text{cyc}} \rangle_T$. Define the rank four $G_{\mathbf{Q}_p}$ -invariant subspace of \mathbf{V} by

$$\text{Fil}^+ \mathbf{V} = (\text{Fil}^0 V_{\mathbf{f}} \otimes \text{Fil}^0 V_{\mathbf{g}} \otimes V_{\mathbf{h}} + \text{Fil}^0 V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes \text{Fil}^0 V_{\mathbf{h}} + V_{\mathbf{f}} \otimes \text{Fil}^0 V_{\mathbf{g}} \otimes \text{Fil}^0 V_{\mathbf{h}}) \otimes \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T.$$

Recall that $w_{\underline{Q}} = k_{Q_1} + k_{Q_2} + k_{Q_3} - 3$ and $\Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(s) = L_{\infty} \left(s + k_P - \frac{w_{\underline{Q}}}{2}, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \right)$. The modified p -Euler factor $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)})$ is defined in the introduction.

Theorem 7.6. *Let $p > 3$. Assume that $N := \text{lcm}(N_1, N_2, N_3)$ is square-free and that the conductor of tame nebentypus χ_i divides p . Let t denote the number of prime factors of N . If \mathbf{f}, \mathbf{g} and \mathbf{h} satisfy Hypothesis (CR), then for each arithmetic point $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ we have*

$$L_{\mathbf{F},(a)}(\underline{Q}, P) = \Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(0) \cdot \frac{L(\mathbf{V}_{(\underline{Q}, P)}, 0)}{\Omega_{\mathbf{f}_{Q_1}} \Omega_{\mathbf{g}_{Q_2}} \Omega_{\mathbf{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}) \cdot \mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}(\underline{Q}, P),$$

where $\mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3} \in \mathbf{I}_4^{\times}$ is given by

$$\mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3} := \frac{(-1)^{t-t_{\mathbf{V}}}}{N \prod_{\ell|N} \sqrt{\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}}}.$$

PROOF. For brevity we write $(f_1, f_2, f_3) = (\mathbf{f}_{Q_1}, \mathbf{g}_{Q_2}, \mathbf{h}_{Q_3})$, $(k, l, m) = (k_{Q_1}, k_{Q_2}, k_{Q_3})$, $\pi_i = \pi_{f_i}$ and $N_i = N_{f_i}$. We may assume that $k \geq l \geq m$. Denote the parity type of (k, l, m) by λ . Put

$$\chi = \varepsilon_P \omega^{a-k_P}, \quad \omega_i = \omega^{k_{Q_i}} \chi_i^{-1} \varepsilon_{Q_i}^{-1}, \quad \mathcal{D} = (\chi, \omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}), \quad n = \max\{1, c(\omega_i), c(\chi)\}.$$

We define the functional L_{f_1, f_2, f_3} on

$$\mathcal{S}_k(N_1 p^{2n}, \omega_1^{-1}; \mathcal{O}(Q_1)) \otimes_{\mathcal{O}} \mathcal{S}_l(N_2 p^{2n}, \omega_2^{-1}; \mathcal{O}(Q_2)) \otimes_{\mathcal{O}} \mathcal{S}_m(N_3 p^{2n}, \omega_3^{-1}; \mathcal{O}(Q_3))$$

by

$$L_{f_1, f_2, f_3}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3) = L_{f_1}(\mathcal{F}_1) L_{f_2}(\mathcal{F}_2) L_{f_3}(\mathcal{F}_3).$$

Let $1_{f_i}^*$ be the specialization of $1_{\mathbf{f}}^*$ at Q_i . By definition and the theory of newforms

$$1_{f_1}^* \otimes 1_{f_2}^* \otimes 1_{f_3}^* (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\underline{\chi}}^{(a)}(\underline{Q}, P))) = L_{\mathbf{F},(a)}(\underline{Q}, P) \cdot f_1 \otimes f_2 \otimes f_3.$$

We apply the functional L_{f_1, f_2, f_3} to both the sides to get

$$L_{\mathbf{F},(a)}(\underline{Q}, P) = L_{f_1, f_2, f_3} (1_{f_1}^* \otimes 1_{f_2}^* \otimes 1_{f_3}^* (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\underline{\chi}}^{(a)}(\underline{Q}, P)))).$$

taking Lemma 7.3(1) into account. Let $\varphi_i = \Phi(f_i)$ and $G_{\mathbf{A}}(\mathcal{D}) = \Phi(\mathcal{G}_{\underline{\chi}}^{(a)}(\underline{Q}, P))$ be the adèlic lifts. Put $\check{\varphi}_i = \varphi_i \otimes \omega_{i, \mathbf{A}}^{-1}$. In the previous section we verified that

$$G_{\mathbf{A}}(\mathcal{D}) = \lim_{s \rightarrow -r + \frac{k-\lambda_1}{2} - 1} (-1)^{k + \frac{l+m+\lambda_1}{2} + \lambda_2} e_{\text{ord}} \text{Hol} \left(\left(1 \otimes V_-^{\frac{k-l-\lambda_2}{2}} \otimes V_-^{\frac{k-m-\lambda_3}{2}} \right) \iota^* E_{\mathbf{A}}^* (f_{\mathcal{D}, s, N}^{[k, \lambda]}) \right),$$

where $r = k - k_P + \frac{l+m-\lambda_1}{2} - 2$ (see Proposition 6.8 and Definitions 6.2, 6.5). Lemma 7.3 (2) therefore gives

$$\begin{aligned} \frac{L_{\mathbf{F},(a)}(\underline{Q}, P)}{\zeta_{\mathbf{Q}}(2)^3 [\text{SL}_2(\mathbf{Z}) : \Gamma_0(N)]^3} &= -(2\sqrt{-1})^{k+l+m+3} \frac{\zeta_p(1)^3}{\zeta_p(2)^3} \prod_{i=1}^3 \Omega_{f_i}^{-1} \left(\frac{p^{k_{Q_i}} \omega_{i,p}(p)}{\alpha_p(f_i)^2} \right)^n \\ &\quad \times 4 \lim_{s \rightarrow k_P - \frac{k+l+m}{2} + 1} Z \left(\rho(\mathcal{J}_{\infty} t_n) \check{\varphi}_1, \rho(\mathcal{J}_{\infty} t_n) V_+^{\frac{k-l-\lambda_2}{2}} \check{\varphi}_2, \rho(\mathcal{J}_{\infty} t_n) V_+^{\frac{k-m-\lambda_3}{2}} \check{\varphi}_3, E_{\mathbf{A}}^* (f_{\mathcal{D}, s, N}^{[k, \lambda]}) \right). \end{aligned}$$

Let $W(\varphi_i) = \prod_v W_{i,v}$ be the Whittaker function of φ_i . Put $\check{W}_{i,v} := W_{i,v} \otimes \omega_{i,v}^{-1}$. Let π_i be the automorphic representation generated by φ_i . We finally get

$$L_{\mathbf{F},(a)}(\underline{Q}, P) = \frac{L \left(k_P - \frac{k+l+m-3}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}} \right)}{(\sqrt{-1})^{3-(k+l+m)} \Omega_{f_1} \Omega_{f_2} \Omega_{f_3}} \mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}) \prod_{\ell|N} Z_{\ell}^*$$

by Lemma 7.1, where

$$Z_\ell^* = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(\ell)]^3 \lim_{s \rightarrow k_P - \frac{k+l+m}{2} + 1} \frac{Z(\check{W}_{1,\ell}, \check{W}_{2,\ell}, \check{W}_{3,\ell}, f_{\mathcal{D},s,\ell})}{L\left(s + \frac{1}{2}, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi_\ell\right)}.$$

Proposition 3.4 gives

$$Z_\ell^* = -\ell(\hat{\omega}_\ell^2 \chi_\ell^4)(\ell) |\ell|^{4k_P - 2(k+l+m) + 4} \varepsilon \left(k_P - \frac{k+l+m-3}{2}, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi_\ell, \psi_\ell \right)^{-1}.$$

By what we have seen in the proof of Proposition 6.8

$$\chi_\ell^2 \hat{\omega}_\ell(\ell) \ell^{-2k_P + (k+l+m) - 2} = \langle \ell \rangle_{X_1, X_2, X_3, T}(\underline{Q}, P).$$

The proof of Lemma 7.5 completes the proof. \square

Definition 7.7. We normalize p -adic triple product L -function by

$$L_{\mathbf{F},(a)}^* := L_{\mathbf{F},(a)} \cdot \mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}^{-1}.$$

Remark 7.8. Provided that $p > 3$, $\chi_1 \chi_2 \chi_3 = \omega^{2a}$ for some a , a three-variable p -adic L -function $\mathcal{L}_{\mathbf{F}}^{\mathrm{bal}} \in \mathbf{I}_3$ was constructed by a different approach in [Hsi21, Theorem B] such that for each balanced central point $\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathbf{I}_3}^{\mathrm{bal}}$

$$\left(\mathcal{L}_{\mathbf{F}}^{\mathrm{bal}}(\underline{Q})\right)^2 = \Gamma_{\mathbf{V}_{\underline{Q}}}(0) \cdot \frac{L(\mathbf{V}_{\underline{Q}}^\dagger, 0)}{\Omega_{\mathbf{f}_{Q_1}} \Omega_{\mathbf{g}_{Q_2}} \Omega_{\mathbf{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}}^\dagger),$$

where

$$\begin{aligned} \mathbf{V}^\dagger &:= \mathcal{V} \otimes \omega^a \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_1}^{1/2} \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_2}^{1/2} \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_3}^{1/2} \varepsilon_{\mathrm{cyc}}^{-1}, \\ \mathrm{Fil}^+ \mathbf{V}^\dagger &= \mathrm{Fil}^+ \mathcal{V} \otimes \omega^a \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_1}^{1/2} \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_2}^{1/2} \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_3}^{1/2} \varepsilon_{\mathrm{cyc}}^{-1}. \end{aligned}$$

We remark that $\det V_{\mathbf{f}} = (\chi_1 \circ \varepsilon_{\mathrm{cyc}})^{-1} \langle \varepsilon_{\mathrm{cyc}} \rangle_{\mathbf{I}}^{-1} \varepsilon_{\mathrm{cyc}}$. By the interpolation formulae, we find that

$$L_{\mathbf{F},(a-1)}^*(X_1, X_2, X_3, \mathbf{u}^{-1}\{(1+X_1)(1+X_2)(1+X_3)\}^{1/2} - 1) = \mathcal{L}_{\mathbf{F}}^{\mathrm{bal}}(X_1, X_2, X_3)^2.$$

This shows that the compatibility between p -adic L -functions constructed by different methods.

Without Hypothesis (CR) and the assumption $p > 3$, our method yields the construction of the p -adic L -function with denominators. For each p -stabilized newform f of weight k , define the p -modified period by

$$\Omega_f^\dagger := (-2\sqrt{-1})^{k+1} \cdot \|f^\circ\|_{\Gamma_0(N_{f^\circ})}^2 \cdot \mathcal{E}_p(f, \mathrm{Ad}).$$

By definition, $\Omega_f^\dagger \cdot \eta_f$ is equal to Hida's canonical period Ω_f up to p -adic units.

Corollary 7.9. *Let $p > 2$. There exists an element*

$$L_{\mathbf{F},(a)}^{**} \in \mathbf{I}_4 \otimes_{\mathbf{I}_3} (\mathrm{Frac} \mathbf{I} \otimes \mathrm{Frac} \mathbf{I} \otimes \mathrm{Frac} \mathbf{I})$$

such that

- for each balanced critical $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\mathrm{bal}}$,

$$L_{\mathbf{F},(a)}^{**}(\underline{Q}, P) = \frac{\Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(0) L(\mathbf{V}_{(\underline{Q}, P)}, 0)}{\Omega_{\mathbf{f}_{Q_1}}^\dagger \Omega_{\mathbf{g}_{Q_2}}^\dagger \Omega_{\mathbf{h}_{Q_3}}^\dagger} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{(\underline{Q}, P)});$$

- for any H_1, H_2 and H_3 in the congruence ideals of \mathbf{f}, \mathbf{g} and \mathbf{h} ,

$$H_1 H_2 H_3 \cdot L_{\mathbf{F},(a)}^{**} \in \mathbf{I}_4.$$

PROOF. For any H_1, H_2 and H_3 in the congruence ideals of \mathbf{f}, \mathbf{g} and \mathbf{h} , we let $L_H \in \mathbf{I}_3[[T]]$ be the first Fourier coefficient of

$$H_1 \mathbf{1}_{\mathbf{f}} \otimes H_2 \mathbf{1}_{\mathbf{g}} \otimes H_3 \mathbf{1}_{\mathbf{h}} \left(\mathrm{Tr}_{N/N_1} \otimes \mathrm{Tr}_{N/N_2} \otimes \mathrm{Tr}_{N/N_3}(\mathcal{G}_{\underline{\chi}}^{(a)}) \right) \in \mathbf{I}_3[[T]][[q_1, q_2, q_3]].$$

Then $L_{\mathbf{F},(a)}^{**} := L_H \cdot (H_1 H_2 H_3)^{-1} \cdot \mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}^{-1}$ enjoys the desired properties. \square

This p -adic L -function $L_{\mathbf{F},(a)}^{**}$ is more canonical in the sense that it does not depend on any particular choice of generators of the congruence ideal of \mathbf{f} , \mathbf{g} and \mathbf{h} .

7.7. The functional equation. Recall that we have fixed the topological generator $\mathbf{u} = 1 + p$ of $\Gamma = 1 + p\mathbf{Z}_p$ as in §6.5.

Proposition 7.10. *Assume that $\chi_1\chi_2\chi_3 = \omega^{a_0}$. Then*

$$L_{\mathbf{F},(a)}^*(X_1, X_2, X_3, T) = (-\varepsilon^{(p\infty)}(\mathbf{V})) \cdot L_{\mathbf{F},(a_0-a-2)}^* \left(X_1, X_2, X_3, \frac{(1+X_1)(1+X_2)(1+X_3)}{\mathbf{u}^2(1+T)} - 1 \right).$$

PROOF. Recall that $\chi = \epsilon_P \omega^{a-k_P}$ and $\omega_i = \chi_i^{-1} \epsilon_{Q_i}^{-1} \omega^{k_{Q_i}}$. Put

$$k_{\check{P}} = k_{Q_1} + k_{Q_2} + k_{Q_3} - k_P - 2, \quad \epsilon_{\check{P}} = \epsilon_P^{-1} \epsilon_{Q_1} \epsilon_{Q_2} \epsilon_{Q_3}, \quad \check{\chi} = \epsilon_{\check{P}} \omega^{a_0-a-2-k_{\check{P}}} = \chi^{-1} \omega_1^{-1} \omega_2^{-1} \omega_3^{-1}.$$

Thus the left hand side specialized at $(\underline{Q}, \check{P})$ equals

$$L_{\mathbf{F},(a_0-a-2)}^*(\underline{Q}, \check{P}) = \frac{L(1-s_0, \pi_1^{\vee} \times \pi_2^{\vee} \times \pi_3^{\vee} \otimes \chi_{\mathbf{A}}^{-1})}{(\sqrt{-1})^{3-(k_{Q_1}+k_{Q_2}+k_{Q_3})}} \Omega_{f_1} \Omega_{f_2} \Omega_{f_3} E_p(1-s_0, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p),$$

where $s_0 = k_P - \frac{k_{Q_1}+k_{Q_2}+k_{Q_3}-3}{2} = 1 - (k_{\check{P}} - \frac{k_{Q_1}+k_{Q_2}+k_{Q_3}-3}{2})$.

Since $(k_{Q_1}, k_{Q_2}, k_{Q_3})$ is balanced, we know that

$$\varepsilon(s, \pi_{1,\infty} \times \pi_{2,\infty} \times \pi_{3,\infty} \otimes \chi_{\infty}) = (-1)^{k_{Q_1}+k_{Q_2}+k_{Q_3}+1} = -\hat{\omega}_{\infty}(-1) = -\hat{\omega}_p(-1).$$

By the global functional equation we get

$$L_{\mathbf{F},(a_0-a-2)}^*(\underline{Q}, \check{P}) = \frac{L(s_0, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})}{(\sqrt{-1})^{3-(k_{Q_1}+k_{Q_2}+k_{Q_3})}} \Omega_{f_1} \Omega_{f_2} \Omega_{f_3} \cdot \frac{-E_p(s_0, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \check{\chi}_p)}{\prod_{\ell \neq p} \varepsilon(s_0, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi_{\ell}, \psi_{\ell})}$$

in view of Lemma 2.7. □

8. THE TRIVIAL ZERO FOR THE TRIPLE PRODUCT OF ELLIPTIC CURVES

8.1. The cyclotomic p -adic triple product L -functions for elliptic curves. Let $\mathbf{E} = E_1 \times E_2 \times E_3$ be the triple fiber product of rational elliptic curves E_i of square-free conductor M_i for $i = 1, 2, 3$. We denote the prime p -part of M_i by N_i . Recall the rank eight p -adic Galois representation $\mathbf{V}_{\mathbf{E}}$ defined in (1.1). We write $L(\mathbf{E} \otimes \chi, s)$ for the complex L -series attached to $\mathbf{V}_{\mathbf{E}}$ twisted by a Dirichlet character χ . Let M (resp. N) and M^- (resp. N^-) be the least common multiple and the greatest common divisor of M_1, M_2, M_3 (resp. N_1, N_2, N_3).

Remark 8.1. Let Σ^- be the set of prime factors ℓ of M^- such that $a_{\ell}(E_1)a_{\ell}(E_2)a_{\ell}(E_3) = 1$. From Remark 3.5, $\varepsilon(\mathbf{E}) = -(-1)^{\#\Sigma^-}$ is the sign in the functional equation for $L(s, \mathbf{E})$. From the formula (7.2) for the p -adic root number the p -adic sign $\varepsilon_p(\mathbf{E}) = -\varepsilon^{(p\infty)}(\mathbf{V}_{\mathbf{E}}(2))$ differs from $\varepsilon(\mathbf{E})$ if and only if $p \in \Sigma^-$.

Let $f_i^{\circ} = \sum_{n=1}^{\infty} a_n(E_i)q^n \in \mathcal{S}_2(M_i, 1; \mathbf{Z})$ be the primitive Hecke eigenform associated with the p -adic Galois representation $\mathbf{H}_{\text{ét}}^1(E_i/\bar{\mathbf{Q}}, \mathbf{Q}_p)$ by Wiles' modularity theorem. Hereafter, we assume that E_i has either good ordinary reduction or multiplicative reduction at p . Let $f_i \in \mathcal{S}_2(pM_i, 1; \mathbf{Z}_p)$ be the p -stabilization of f_i° (see (5.1)). If p and M_i are coprime, then $\alpha_i = \alpha_p(f_i) \in \mathbf{Z}_p^{\times}$ denotes the p -adic unit root of the Hecke polynomial $X^2 - a_p(E_i)X + p$ while if p divides M_i , then $\alpha_i = a_p(E_i)$. Define a period and a fudge factor by

$$\Omega(\mathbf{E}) = \prod_{i=1}^3 \Lambda(1, E_i, \text{Ad}), \quad c_p = \prod_{i=1}^3 \mathcal{E}_p(f_i, \text{Ad}),$$

where $\Lambda(s, E_i, \text{Ad})$ denotes the complete adjoint L -function for f_i

Let $\mathbf{T}_i = \mathbf{T}(N_i, \Lambda)$ be the big cuspidal ordinary Hecke algebra over $\Lambda = \mathbf{Z}_p[[X]]$ with $X = [\mathbf{u}] - 1$. Each f_i induces a surjective homomorphism $\lambda_{f_i} : \mathbf{T}_i \rightarrow \mathbf{Z}_p$. Let \mathfrak{m}_i be the maximal ideal of \mathbf{T}_i containing $\ker \lambda_{f_i}$ and $\mathbf{I}_i = (\mathbf{T}_i)_{\mathfrak{m}_i}$ be the localization at \mathfrak{m}_i . Let $\mathbf{f}_i = \sum_{n=1}^{\infty} \mathbf{a}(n, \mathbf{f}_i)q^n \in \mathbf{S}^{\text{ord}}(N_i, \omega^2, \mathbf{I}_i)$ be the primitive Hida family of tame level N_i such that f_i is the specialization $\mathbf{f}_{i, Q_i^{\circ}}$ at some arithmetic point Q_i° with $k_{Q_i^{\circ}} = 2$ and $\epsilon_{Q_i^{\circ}} = 1$. Now we consider the four-variable p -adic L -function $L_{\mathbf{F},(2)}^{**}$ in Corollary 7.9 with $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ and $a = 2$. Define the cyclotomic p -adic L -function by

$$L_p(\mathbf{E}, T) := c_p \cdot L_{\mathbf{F},(2)}^{**}(Q_1^{\circ}, Q_2^{\circ}, Q_3^{\circ}, \mathbf{u}^2(1+T) - 1) \in \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]] \otimes \mathbf{Q}_p.$$

Proposition 8.2. *The element $L_p(\mathbf{E}) \in \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \otimes \mathbf{Q}_p$ satisfies the following interpolation property*

$$\hat{\chi}(L_p(\mathbf{E})) = \frac{L(\mathbf{E} \otimes \hat{\chi}, 2)}{2^4 \pi^5 \Omega(\mathbf{E})} \mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{\mathbf{E}} \otimes \hat{\chi})$$

for all finite-order characters $\hat{\chi}$ of $\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Moreover, it satisfies the functional equation

$$L_p(\mathbf{E}, T) = \varepsilon_p(\mathbf{E}) \langle N^- N^4 \rangle_T^{-1} L_p(\mathbf{E}, (1+T)^{-1} - 1).$$

PROOF. Define $(Q^o, P) = (Q_1^o, Q_2^o, Q_3^o, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ with Q_i^o as above, $k_P = 2$ and $\epsilon_P = \hat{\chi}$. Then $\mathbf{V}_{(Q^o, P)} = \mathbf{V}_{\mathbf{E}}(2) \otimes \hat{\chi}$ and $\hat{\chi}(L_p(\mathbf{E})) = c_p \cdot L_{\mathbf{F}, (2)}^{**}(Q^o, P)$. The assertions follows from Corollary 7.9, Proposition 7.10 and the equation $2^2 \|f_i^o\| = M_i \Lambda(1, E_i, \text{Ad})$ by [CH18, Proposition 3.3]. \square

8.2. The trivial zero conjecture for the triple product of elliptic curves. We prove the trivial zero conjecture for the cyclotomic p -adic triple product L -function. We define a function on \mathbf{Z}_p by

$$L_p(\mathbf{E}, s) := L_p(\mathbf{E}, \mathbf{u}^{s-2} - 1).$$

We consider the case where $L_p(\mathbf{E}, s)$ has a trivial zero at the critical value $s = 2$. By Remark 8.3 below we essentially only need to consider the following two cases:

- (i) all E_1, E_2 and E_3 have multiplicative reduction at p such that $\alpha_1 \alpha_2 \alpha_3 = 1$.
- (ii) E_1 has multiplicative reduction at p ; E_2 and E_3 have good ordinary reduction at p such that $\alpha_2 = \alpha_1 \alpha_3$.

Remark 8.3. Let $\beta_i = p \alpha_i^{-1}$. Then $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{\mathbf{E}}(2)) = 0$ if and only if $L_p((\text{Fil}^+ \mathbf{V}_{\mathbf{E}}(2))^\vee, 1)^{-1} = 0$ if and only if one of the following equations holds:

$$\beta_1 \beta_2 \beta_3 = p^2, \quad \beta_1 \beta_2 \alpha_3 = p^2, \quad \beta_1 \alpha_2 \beta_3 = p^2, \quad \alpha_1 \beta_2 \beta_3 = p^2.$$

The ordinarity hypothesis rules out the first equation. The Ramanujan conjecture forces one or all of E_i to have multiplicative reduction at p . When E_1 is multiplicative at p , we will have $\alpha_1 \in \{\pm 1\}$ and $\alpha_2 = \alpha_1 \alpha_3$.

In the above cases (i) and (ii), the trivial zero conjecture predicts that the leading coefficient of the Taylor expansion of $L_p(\mathbf{E}, s)$ at $s = 2$ should be essentially the product of Greenberg's \mathcal{L} -invariant for \mathbf{E} and the central value $L(\mathbf{E}, 2)$. Note that the localization of \mathbf{I}_i at Q_i^o is that of Λ at P_2 , where P_2 is the principal ideal generated by $(1+X)\mathbf{u}^{-2} - 1$, so \mathbf{I}_i is contained in $\Lambda[\frac{1}{t_i}]$ with some $t_i(\mathbf{u}^2 - 1) \neq 0$. In what follows, we shall replace \mathbf{I}_i by $\Lambda[t_i^{-1}]$ with some $t_i(\mathbf{u}^2 - 1) \neq 0$. Let $\mathcal{U} \subset \mathbf{Z}_p$ be a neighborhood around 0 such that $(t_1 t_2 t_3)(\mathbf{u}^{s+2} - 1) \neq 0$ for any $s \in \mathcal{U}$. To introduce Greenberg's \mathcal{L} -invariants, we let

$$\mathbf{a}_i(s) := \mathbf{a}(p, \mathbf{f}_i)(\mathbf{u}^{s+2} - 1); \quad \ell_i := \alpha_i^{-1} \cdot \left. \frac{d\mathbf{a}_i(s)}{ds} \right|_{s=0} \quad (s \in \mathcal{U}).$$

Note that $\mathbf{a}_i(0) = \alpha_i$ by definition. If $\alpha_i = 1$, then $-2\ell_i = \frac{\log_p q_{E_i}}{\text{ord}_p q_{E_i}}$ by [GS93, Theorem 3.18]. According to the discussion in [Gre94b, §3], Greenberg's \mathcal{L} -invariant for the Galois representation (1.1) is given by

$$\mathcal{L}_p(\mathbf{E}) := \begin{cases} -8\ell_1 \ell_2 \ell_3 & \text{in Case (i);} \\ 4\ell_1^2 & \text{in Case (ii).} \end{cases}$$

The non-vanishing of these \mathcal{L} -invariants is known, thanks to the work [BSDGP96]. The aim of this section is to prove the following:

Theorem 8.4 (Trivial zero conjecture). *(1) In Case (i), $\text{ord}_{s=2} L_p(\mathbf{E}, s) \geq 3$, and*

$$\left. \frac{L_p(\mathbf{E}, s)}{(s-2)^3} \right|_{s=2} = -p \mathcal{L}_p(\mathbf{E}) \cdot \frac{L(\mathbf{E}, 2)}{2^4 \pi^5 \Omega(\mathbf{E})}.$$

(2) In Case (ii), $\text{ord}_{s=2} L_p(\mathbf{E}, s) \geq 2$ and

$$\left. \frac{L_p(\mathbf{E}, s)}{(s-2)^2} \right|_{s=2} = \mathcal{L}_p(\mathbf{E}) (-p \alpha_2^{-2}) (1 - \alpha_2^{-2})^2 \cdot \frac{L(\mathbf{E}, 2)}{2^4 \pi^5 \Omega(\mathbf{E})}.$$

8.3. Improved p -adic L -functions. We define an analytic function on $\mathcal{U}^3 \times \mathbf{Z}_p \subset \mathbf{Z}_p^4$ by

$$L_p(x, y, z, s) := c_p \cdot \langle N^- N^4 \rangle^{\frac{2s-(x+y+z)}{4}} L_{\mathbf{F},(2)}^{**}(\mathbf{u}^{x+2} - 1, \mathbf{u}^{y+2} - 1, \mathbf{u}^{z+2} - 1, \mathbf{u}^{s+2} - 1),$$

which satisfies

$$(8.1) \quad L_p(0, 0, 0, s) = \langle N^- N^4 \rangle^{s/2} L_p(\mathbf{E}, s + 2), \quad L_p(x, y, z, s) = \varepsilon_p(\mathbf{E}) \cdot L_p(x, y, z, x + y + z - s).$$

To follow the method used in [GS93] (cf. [BDJ]), we introduce p -adic L -functions which have only less variables but have better interpolation properties.

Lemma 8.5 (Improved p -adic L -functions). *Suppose that f_1° is special at p , i.e. $\alpha_1 = \mathbf{a}_1(0) = \pm 1$.*

(1) *There exist a two-variable improved p -adic L -function $L_p^\dagger(x, s)$ and a one-variable improved p -adic L -function $L_p^{\dagger\dagger}(s)$ such that*

$$L_p(x, s, s, s) = \left(1 - \frac{\mathbf{a}_2(s)}{\mathbf{a}_1(x)\mathbf{a}_3(s)}\right) \left(1 - \frac{\mathbf{a}_3(s)}{\mathbf{a}_1(x)\mathbf{a}_2(s)}\right) L_p^\dagger(x, s), \quad L_p^\dagger(s, s) = \left(1 - \frac{\mathbf{a}_1(s)}{\mathbf{a}_2(s)\mathbf{a}_3(s)}\right) L_p^{\dagger\dagger}(s).$$

(2) *For any positive integer k with $k \equiv 2 \pmod{p-1}$ and $k-2 \in \mathcal{U}$, we have the interpolation formula*

$$L_p^\dagger(0, k-2) = \mathcal{E}^\dagger(k-2) \cdot \frac{\Gamma(k-1)\Gamma(k)}{2^{2k-3}(\pi\sqrt{-1})^{2k+1}} \cdot \frac{L(\frac{1}{2}, \pi_{f_1} \times \pi_{f_{2,k}} \times \pi_{f_{3,k}})}{c_p^{-1} \Omega_{f_1}^\dagger \Omega_{f_{2,k}}^\dagger \Omega_{f_{3,k}}^\dagger},$$

where $\pi_{f_{i,k}}$ is the automorphic representation generated by $\mathbf{f}_{i,k} = \mathbf{f}_i(\mathbf{u}^k - 1) \in S_k(N_i p, 1; \overline{\mathbf{Q}})$, and

$$\mathcal{E}^\dagger(s) = (-\alpha_1)\mathbf{a}_2(s)^{-1}\mathbf{a}_3(s)^{-1}p^{s+1}(1 - \alpha_1 \cdot \mathbf{a}_2(s)^{-1}\mathbf{a}_3(s)^{-1}p^s)^2.$$

(3) *If $\varepsilon_p(\mathbf{E}) = -1$, then*

$$L_p^\dagger(0, s) = 0, \quad \frac{\partial L_p^\dagger}{\partial x}(0, 0) = (\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0), \quad \text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 3.$$

(4) *In Case (i), $L_p^{\dagger\dagger}(0) = -p^{\frac{L(\mathbf{E}, 2)}{2^4 \pi^5 \Omega(\mathbf{E})}}$.*

PROOF. The construction of these improved L -functions are similar to that of $L_{\mathbf{F},(a)}$ except that we need to replace the Λ_4 -adic modular form $\mathcal{G}_\chi^{(a)}$ in §6.5 with *improved* ones. To do so, we have to go back to §6.1 and modify the p -adic section $f_{\mathcal{D},s,p}$ used in the construction of the Siegel Eisenstein series $E_{\mathbf{A}}(g, f_{\mathcal{D},s,N}^{[k,\lambda]})$. In the notation of Definition 2.5, for a datum $\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3)$ of characters of \mathbf{Z}_p^\times and a Bruhat-Schwartz function $\varphi_1 \in \mathcal{S}(\mathbf{Q}_p)$, we modify the definition of Bruhat-Schwartz functions defined in (2.7) by

$$\Phi_{\mathcal{D}}(\varphi_1) \left(\begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix} \right) = \prod_{i=1}^3 \phi_i(u_i) \varphi_i(x_i),$$

where

$$\phi_1 = \phi_2 = \phi_3 = \widehat{\mathbb{I}}_{p\mathbf{Z}_p}, \quad \varphi_2 = \varphi_3 = \mathbb{I}_{\mathbf{Z}_p}.$$

Define the modified Bruhat-Schwartz functions by

$$\Phi_{\mathcal{D}}^\dagger = \Phi_{\mathcal{D}}(\widehat{\varphi}_{\chi\omega_1}), \quad \Phi_{\mathcal{D}}^{\dagger\dagger} = \Phi_{\mathcal{D}}(\mathbb{I}_{\mathbf{Z}_p}).$$

Following (2.5), we define the *improved* p -adic section $f_{\mathcal{D},s,p}^\bullet := f_{\Phi_{\mathcal{D}}^\dagger}(\chi\widehat{\omega}\alpha_{\mathbf{Q}_p}^s)$ for $\bullet \in \{\dagger, \dagger\dagger\}$. Then the local degenerate Whittaker functions for these improved p -adic sections are given by

$$\mathcal{W}_B(f_{\mathcal{D},s}^\dagger) = (\chi\omega_1)(2b_{23})\mathbb{I}_{\Xi_p^\dagger}(B), \quad \mathcal{W}_B(f_{\mathcal{D},s}^{\dagger\dagger}) = \mathbb{I}_{\Xi_p^{\dagger\dagger}}(B),$$

for $B = (b_{ij}) \in \text{Sym}_3(\mathbf{Q}_p)$, where

$$\Xi_p^{\dagger\dagger} := \{(b_{ij}) \in \text{Sym}_3(\mathbf{Z}_p) \mid b_{11}, b_{22}, b_{33} \in p\mathbf{Z}_p\}, \quad \Xi_p^\dagger := \{(b_{ij}) \in \Xi_p^{\dagger\dagger} \mid 2b_{23} \in \mathbf{Z}_p^\times\}.$$

Notation is as in §6.1. With the table of local Whittaker coefficients outside p in [LR20, page 210], one can apply the same argument in [Liu20, §5.2] to construct two power series $\mathcal{G}^\dagger(T, X) \in \mathbf{Z}_p[[T, X]][[q_1, q_2, q_3]]$ and $\mathcal{G}^{\dagger\dagger}(T) \in \mathbf{Z}_p[[T]][[q_1, q_2, q_3]]$ such that for arithmetic points (Q, P) with $k_Q = 2$ we have

$$\mathcal{G}^\dagger(Q, P) = e_{\text{ord}} \mathbf{E}_{\mathcal{D}^\dagger, N}^{[k_P, r, \lambda]}(\tau, f_{\mathcal{D}^\dagger, s, N}^\dagger)|_{s=0}, \quad \mathcal{G}^{\dagger\dagger}(P) = e_{\text{ord}} \mathbf{E}_{\mathcal{D}^{\dagger\dagger}, N}^{[k_P, r, \lambda]}(\tau, f_{\mathcal{D}^{\dagger\dagger}, s, N}^{\dagger\dagger})|_{s=0}$$

with $\lambda = (0, 0, 0)$ and $r = \frac{k_P}{2} - 1$, where we have written

$$\begin{aligned} \mathcal{D}^\dagger &:= (\epsilon_P \omega^{2-k_P}, \epsilon_Q^{-1} \omega^{k_Q-2}, \epsilon_P^{-1} \omega^{k_P-2}, \epsilon_P^{-1} \omega^{k_P-2}), \\ \mathcal{D}^{\dagger\dagger} &:= (\epsilon_P \omega^{2-k_P}, \epsilon_P^{-1} \omega^{k_P-2}, \epsilon_P^{-1} \omega^{k_P-2}, \epsilon_P^{-1} \omega^{k_P-2}), \\ f_{\mathcal{D}^\bullet, s, N}^\bullet &:= f_{s, \infty}^{[k_P, \lambda]} \otimes f_{\mathcal{D}^\bullet, s, p}^\bullet \otimes f_{s, N} \otimes_{\ell^1 N_p} f_{s, \ell}^0. \end{aligned}$$

Let $\Lambda_T := \mathbf{Z}_p[[T]]$. As in Proposition 6.8 we see that

$$\begin{aligned} \mathcal{G}^\dagger(T, X) &\in \mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_X) \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_T) \otimes_{\Lambda_T} \mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_T); \\ \mathcal{G}^{\dagger\dagger}(T) &\in \mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_T) \otimes_{\Lambda_T} \mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_T) \otimes_{\Lambda_T} \mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_T), \end{aligned}$$

where $\mathbf{M}^{\text{ord}}(N, \omega^2, \Lambda_T)$ is the space of ordinary Λ_T -adic modular forms of level N and character ω^2 . Let $\tilde{\mathbf{1}}_{f_i}$ be idempotent associated with f_i in the Hecke algebra acting on the ordinary Λ -adic module forms (not only cusp forms). Choose an element H_i with $H_i(\mathbf{u}^2 - 1) \neq 0$ in the congruence ideal of f_i among ordinary Λ -adic modular forms (or rather the ideal generated by the denominators of $\tilde{\mathbf{1}}_{f_i}$). We define the improved p -adic L -functions $L_{F, (2)}^\dagger(X, T)$ and $L_{F, (2)}^{\dagger\dagger}(T)$ as the first Fourier coefficients of

$$\begin{aligned} \tilde{\mathbf{1}}_{f_1} \otimes \tilde{\mathbf{1}}_{f_2} \otimes \tilde{\mathbf{1}}_{f_3} (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3}(\mathcal{G}^\dagger)) &\in \mathbf{Z}_p[[X, T]] \left[\frac{1}{H^\dagger} \right]; \\ \tilde{\mathbf{1}}_{f_1} \otimes \tilde{\mathbf{1}}_{f_2} \otimes \tilde{\mathbf{1}}_{f_3} (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3}(\mathcal{G}^{\dagger\dagger})) &\in \mathbf{Z}_p[[T]] \left[\frac{1}{H^{\dagger\dagger}} \right] \end{aligned}$$

respectively, where $H^\dagger = t_1 H_1(X) t_2 H_2 t_3 H_3(T)$ and $H^{\dagger\dagger} = t_1 H_1 t_2 H_2 t_3 H_3(T)$. Define

$$L_p^\dagger(x, s) := c_p \cdot \langle N^- N^4 \rangle^{\frac{-x}{4}} L_{F, (2)}^\dagger(\mathbf{u}^{x+2} - 1, \mathbf{u}^{s+2} - 1), \quad L_p^{\dagger\dagger}(s) := c_p \cdot \langle N^- N^4 \rangle^{\frac{-s}{4}} L_{F, (2)}^{\dagger\dagger}(\mathbf{u}^{s+2} - 1).$$

In view of the proof of Lemma 7.1, to prove the interpolation formulae for $L_p^\dagger(x, s)$ and $L_p^{\dagger\dagger}(s)$, we need to compute the quantity $Z_p^*(f_{\mathcal{D}, s}^\bullet)$ defined in (2.13) attached to our modified p -adic sections $f_{\mathcal{D}, s}^\bullet$ as well as a subrepresentation π_i of the induced representation $I(\mu_i, \nu_i)$ of $\text{GL}_2(\mathbf{Q}_p)$ with μ_i unramified for $i = 1, 2, 3$. Applying the computation in Proposition 2.3, we find that whenever $\chi\omega_2$ and $\chi\omega_3$ are unramified,

$$Z_p^*(f_{\mathcal{D}, s, p}^\dagger) = Z_p^*(f_{\mathcal{D}, s, pq}) \prod_{i=2,3} L\left(\frac{1}{2} - s, \chi^{-1} \mu_1^{-1} \mu_i^{-1} \nu_{5-i}^{-1}\right)$$

and that when $\chi\omega_i$ are unramified for $i = 1, 2, 3$,

$$Z_p^*(f_{\mathcal{D}, s, p}^{\dagger\dagger}) = Z_p^*(f_{\mathcal{D}, s, p}^\dagger) L\left(\frac{1}{2} - s, \chi^{-1} \nu_1^{-1} \mu_2^{-1} \mu_3^{-1}\right).$$

From the proof of Theorem 7.6 we can deduce the interpolation formulae for the improved L -functions. The formula for $\mathcal{E}^+(s)$ follows from that for $Z_p^*(f_{\mathcal{D}, s, p})$ proved in Proposition 2.6 and Remark 3.5. Observe that if $\pi_i \simeq \text{St} \otimes \chi_i$ for $i = 1, 2, 3$, i.e., $\mu_i = \chi_i \alpha_{\mathbf{Q}_p}^{1/2}$, then

$$E_p\left(\frac{1}{2} + s, \pi_1 \times \pi_2 \times \pi_3\right) = -(\chi_1 \chi_2 \chi_3)(p) p^{1+s} (1 - (\chi_1 \chi_2 \chi_3)(p) p^s)^3,$$

from which (4) follows.

Whenever $k > 2$, the central sign for $L(s, \pi_{f_1} \times \pi_{f_{2,k}} \times \pi_{f_{3,k}})$ is $\varepsilon_p(\mathbf{E})$. Therefore if $\varepsilon_p(\mathbf{E}) = -1$, then $L_p^\dagger(0, s) = 0$ by (2), which implies that $\frac{\partial L_p^\dagger}{\partial x}(0, 0) = \lim_{s \rightarrow 0} \frac{L_p^\dagger(s, s)}{s}$. The second equality of (1) gives the expression of $\lim_{s \rightarrow 0} \frac{L_p^\dagger(s, s)}{s}$. We write

$$L_p(x, y, z, s) = \sum_{j=0}^{\infty} A_j(x, y, z) \left(s - \frac{x+y+z}{2} \right)^j.$$

If $i \leq r := \text{ord}_{s=2} L_p(\mathbf{E}, s)$, then

$$(8.2) \quad r = \min\{j \mid A_j(0, 0, 0) \neq 0\}, \quad \lim_{s \rightarrow 2} \frac{L_p(\mathbf{E}, s)}{(s-2)^i} = A_i(0, 0, 0).$$

Letting $y = z = s = 0$, we see by (1) that the power series

$$\sum_{j=0}^{\infty} A_j(x, 0, 0) \left(-\frac{x}{2}\right)^j = (1 - \alpha_1 \mathbf{a}_1(x)^{-1})^2 L_p^\dagger(x, 0)$$

has at least a double zero at $x = 0$. If $\varepsilon_p(\mathbf{E}) = -1$, then since $A_{2n}(x, y, z) = 0$ for all non-negative integers n by the functional equation (8.1), we get $A_1(0, 0, 0) = 0$ and $r \geq 3$. \square

8.4. The proof of Theorem 8.4(1). We discuss Case (i). Then $\varepsilon_p(\mathbf{E}) = -\varepsilon(\mathbf{E})$ by Remark 8.1. First suppose that $\varepsilon(\mathbf{E}) = 1$. The functional equation (8.1) allows us to write

$$L_p(x, y, z, s) = A_1(x, y, z) \left(s - \frac{x+y+z}{2}\right) + A_3(x, y, z) \left(s - \frac{x+y+z}{2}\right)^3 + \dots$$

The proof of Lemma 8.5(3) gives $A_1(0, 0, 0) = 0$. From (8.2) and Lemma 8.5(4) the formula boils down to

$$A_3(0, 0, 0) = -8\ell_1\ell_2\ell_3L_p^{\dagger\dagger}(0).$$

If we denote the degree two term of $A_1(x, y, z)$ by $ax^2 + by^2 + cz^2 + dxy + eyz + fxz$, then the degree three term of $L_p(x, s, s, s)$ is given by

$$L^{(3)}(x, s) = \{ax^2 + (b+c+e)s^2 + (d+f)xs\}(-x/2) + A_3(0, 0, 0)(-x/2)^3.$$

On the other hand, from Lemma 8.5(1), (3) we find that

$$\begin{aligned} L^{(3)}(x, s) &= (\ell_1x + (\ell_3 - \ell_2)s) \cdot (\ell_1x + (\ell_2 - \ell_3)s)x \cdot \lim_{x \rightarrow 0} x^{-1} L_p^\dagger(x, 0) \\ &= (\ell_1^2x^2 - (\ell_2 - \ell_3)^2s^2)x \cdot (\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0). \end{aligned}$$

Comparing the coefficients of x^2s , xs^2 and x^3 , we obtain the relations

$$d + f = 0, \quad b + c + e = 2(\ell_2 - \ell_3)^2(\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0), \quad 4a + A_3(0, 0, 0) = -8\ell_1^2(\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0).$$

By symmetry we get

$$\begin{aligned} d + e &= 0, & e + f &= 0; \\ a + c + f &= 2(\ell_1 - \ell_3)^2(\ell_1 + \ell_3 - \ell_2)L_p^{\dagger\dagger}(0), & a + b + d &= 2(\ell_1 - \ell_2)^2(\ell_1 + \ell_2 - \ell_3)L_p^{\dagger\dagger}(0). \end{aligned}$$

From these equations we conclude that $d = e = f = 0$ and

$$\begin{aligned} a &= \{(\ell_1 - \ell_2)^2(\ell_1 + \ell_2 - \ell_3) + (\ell_1 - \ell_3)^2(\ell_1 + \ell_3 - \ell_2) - (\ell_2 - \ell_3)^2(\ell_2 + \ell_3 - \ell_1)\}L_p^{\dagger\dagger}(0), \\ A_3(0, 0, 0) &= -8\ell_1^2(\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0) - 4a = -8\ell_1\ell_2\ell_3L_p^{\dagger\dagger}(0). \end{aligned}$$

Next assume that $\varepsilon(\mathbf{E}) = -1$. Then $\varepsilon_p(\mathbf{E}) = 1$. By (8.1) and Lemma 8.5(1)

$$\sum_{n=0}^{\infty} A_{2n}(x, s, s) \left(\frac{s}{2}\right)^{2n} = \left(1 - \frac{\mathbf{a}_2(s)}{\mathbf{a}_1(x)\mathbf{a}_3(s)}\right) \left(1 - \frac{\mathbf{a}_3(s)}{\mathbf{a}_1(x)\mathbf{a}_2(s)}\right) L_p^\dagger(x, s).$$

Since $L_p^\dagger(0, 0) = 0$, every term in the right hand side has degree at least three. In particular, the constant term $A_0(0, 0, 0)$ of the left hand side is zero. If we denote the degree two term of $A_0(x, y, z)$ by $\alpha x^2 + \beta y^2 + \gamma z^2 + \xi xy + \eta yz + \zeta xz$, then the degree two term of the left hand side is

$$\alpha x^2 + (\beta + \gamma + \eta)s^2 + (\xi + \zeta)xs + A_2(0, 0, 0)(x/2)^2.$$

It is zero, and so by symmetry we get

$$\begin{aligned} A_2(0, 0, 0) &= -4\alpha, & \beta + \gamma + \eta &= 0, & \xi + \zeta &= 0; \\ A_2(0, 0, 0) &= -4\beta, & \alpha + \gamma + \zeta &= 0, & \xi + \eta &= 0; \\ A_2(0, 0, 0) &= -4\gamma, & \alpha + \beta + \xi &= 0, & \eta + \zeta &= 0. \end{aligned}$$

We arrive at $\xi = \eta = \zeta = \alpha = \beta = \gamma = A_2(0, 0, 0) = 0$. Hence $\text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 4$.

8.5. **The proof of Theorem 8.4(2).** We discuss Case (ii). Then $\varepsilon_p(\mathbf{E}) = \varepsilon(\mathbf{E})$ by Remark 8.1. If $\varepsilon(\mathbf{E}) = -1$, then $\text{ord}_{s=2} L_p(\mathbf{E}, s) \geq 3$ by Lemma 8.5(3), and both sides of the declared identity are zero. We will consider the case $\varepsilon(\mathbf{E}) = 1$, i.e. Σ^- has odd cardinality. Unlike Case (i) we cannot apply Lemma 8.5(3). Our proof relies on the three-variable p -adic triple product L -function in the balanced case constructed in [Hsi21].

Let D be the definite quaternion algebra over \mathbf{Q} of discriminant N^- and $\mathbf{S}^D(N, \Lambda)$ the space of Λ -adic modular forms on D^\times defined in [Hsi21, Definition 4.1]. Let $\mathbf{f}_i^D \in \mathbf{S}^D(N, \Lambda)[t_i^{-1}]$ be a Jacquet-Langlands lift of \mathbf{f}_i in the sense of [Hsi21, §4.5]. Since we do not assume Hypothesis (CR, Σ^-) of [Hsi21, §1.4], we cannot choose \mathbf{f}_i^D to be a primitive Jacquet-Langlands lift as in [Hsi21, Theorem 4.5]. Nonetheless, \mathbf{f}_i^D can be chosen so that $\mathbf{f}_i^D(\mathbf{u}^2 - 1)$ is a non-zero Jacquet-Langlands lift of f_i . Replacing the triple $\mathbf{F}^D = (\mathbf{f}_1^D, \mathbf{f}_2^D, \mathbf{f}_3^D)$ with the well-chosen test vectors in [Hsi21, Definition 4.8], we can associate to \mathbf{F}^D the three-variable *theta element* $\Theta_{\mathbf{F}^D}(X_1, X_2, X_3)$ in *loc.cit.* Define an analytic function on $\mathcal{U}^3 \subset \mathbf{Z}_p^3$ by

$$\Theta(x, y, z) = \Theta_{\mathbf{F}^D}(\mathbf{u}^{x+2} - 1, \mathbf{u}^{y+2} - 1, \mathbf{u}^{z+2} - 1).$$

By the interpolation formula for $\Theta_{\mathbf{F}^D}$ in [Hsi21, Theorem 7.1] (see Remark 7.8), we can find an analytic function $H(x, y, z)$ with $H(0, 0, 0) \neq 0$ such that

$$H(x, y, z) \cdot \Theta(x, y, z)^2 = L_p\left(x, y, z, \frac{x+y+z}{2}\right).$$

To proceed, we introduce two-variable *improved* theta elements.

Lemma 8.6 (Improved theta elements). *There exist analytic functions $\Theta_2^\dagger(x, z)$, $\Theta_3^\dagger(x, y)$ such that*

$$\begin{aligned} \Theta_2^\dagger(0, 0) &= -\Theta_3^\dagger(0, 0), \\ \Theta(x, x+z, z) &= \left(1 - \frac{\mathbf{a}_2(x+z)}{\mathbf{a}_1(x)\mathbf{a}_3(z)}\right) \Theta_2^\dagger(x, z), & \Theta(x, y, x+y) &= \left(1 - \frac{\mathbf{a}_3(x+y)}{\mathbf{a}_1(x)\mathbf{a}_2(y)}\right) \Theta_3^\dagger(x, y). \end{aligned}$$

PROOF. The idea of the proof is similar to [Hsi21, Proposition 8.3]. We give a sketch of the proof here. For every integer n , let R_n be the Eichler order of level $p^n N/N^-$ in D and let $X_0(p^n N) = D^\times \backslash \widehat{D}^\times / \widehat{R}_n^\times$, where $\widehat{D} = D \otimes \widehat{\mathbf{Q}}$ and $\widehat{R}_n = R_n \otimes \widehat{\mathbf{Z}}$. Through an isomorphism $R_0 \otimes \mathbf{Z}_p \simeq \mathbf{M}_2(\mathbf{Z}_p)$ we define

$$U_1(p^n) := \left\{ g \in \widehat{R}_n \mid g_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{p^n} \right\}.$$

Recall that $\mathbf{a}_i(Q) = \mathbf{a}(p, \mathbf{f}_{i,Q})$ and that $\varpi_p \in \widehat{\mathbf{Q}}^\times$ is the element with $\varpi_{p,p} = p$ and $\varpi_{p,\ell} = 1$ for $\ell \neq p$. For all but finitely many arithmetic points Q with $k_Q = 2$, the specialization $\mathbf{f}_{i,Q}^D : D^\times \backslash \widehat{D}^\times / U_1(p^n) \rightarrow \mathbf{C}_p$ is a p -stabilized form on \widehat{D}^\times with the same Hecke eigenvalues with $\mathbf{f}_{i,Q}$ and the central character $\epsilon_Q^{-1} : \mathbf{Q}^\times \backslash \widehat{\mathbf{Q}}^\times / (1 + p^n \widehat{\mathbf{Z}})^\times \rightarrow \mu_{p^\infty}$ for any sufficiently large n . In particular, $\mathbf{f}_{i,Q}^D$ is a \mathbf{U}_p -eigenform with eigenvalue $\mathbf{a}_i(Q)$. Namely,

$$(8.3) \quad \mathbf{U}_p \mathbf{f}_{i,Q}^D(g) := \sum_{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p} \mathbf{f}_{i,Q}^D\left(g \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix}\right) = \mathbf{a}_i(Q)^n \mathbf{f}_{i,Q}^D(g), \quad g \in \widehat{D}^\times.$$

In what follows, we shall write $(\mathbf{f}^D, \mathbf{g}^D, \mathbf{h}^D) = (\mathbf{f}_1^D, \mathbf{f}_2^D, \mathbf{f}_3^D)$. Let $N_D : D \rightarrow \mathbf{Q}$ be the reduced norm. Put $\tau_{p^n} = \begin{pmatrix} 0 & 1 \\ -\varpi_p^n & 0 \end{pmatrix} \in \text{GL}_2(\mathbf{Q}_p) \subset \widehat{D}^\times$. By definition,

$$(8.4) \quad \begin{aligned} \Theta(Q_1, Q_2, Q_3) &= \mathbf{a}_1(Q_1)^{-n} \mathbf{a}_2(Q_2)^{-n} \mathbf{a}_3(Q_3)^{-n} \\ &\sum_{[a] \in X_0(p^n N)} \sum_{\substack{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p \\ c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times}} \mathbf{f}_{Q_1}^D\left(a \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix}\right) \mathbf{g}_{Q_2}^D\left(a \begin{pmatrix} \varpi_p^n & b+c \\ 0 & 1 \end{pmatrix}\right) \mathbf{h}_{Q_3}^D(a \tau_{p^n}) \epsilon_{Q_1 Q_2 Q_3}^{\frac{1}{2}}(c) \epsilon_{Q_1 Q_2 Q_3}^{\frac{1}{2}}(N_D(a)). \end{aligned}$$

We replace the twisted diagonal cycle Δ_n in [Hsi21, Definition 4.6] by the *improved* diagonal cycle

$$\Delta_n^\dagger := \sum_{[a] \in X_0(Np^n)} \sum_{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p} \left[\left(a \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix}, a \tau_{p^n}, a \right) \right].$$

We can define the regularized improved diagonal cycle by

$$\Delta_\infty^\ddagger := \varprojlim_{n \rightarrow \infty} (\mathbf{U}_p^{-n} \otimes \mathbf{U}_p^{-n} \otimes 1) e_E(\Delta_n^\ddagger),$$

and the improved theta element

$$\Theta_2^\ddagger(X_1, X_3) := (\mathbf{F}^D)^*(\Delta_\infty^\ddagger)(X_1, (1+X_1)(1+X_3)-1, X_3) \in \mathbf{Z}_p[[X_1, X_3]][t^{-1}].$$

for $t = t_1 \cdot t_2((1+X_1)(1+X_3)-1) \cdot t_3$. Put $\Theta_2^\ddagger(x, z) := \Theta_2^\ddagger(\mathbf{u}^{x+2}-1, \mathbf{u}^{z+2}-1)$ for $(x, z) \in \mathcal{U}^2$. By definition and (8.3), for all but finitely many arithmetic points (Q_1, Q_3) with $k_{Q_1} = k_{Q_3} = 2$

$$\Theta_2^\ddagger(Q_1, Q_3) = \mathbf{a}_2(Q_1 Q_3)^{-n} \sum_{[a] \in X_0(Np^n)} \mathbf{f}_{Q_1}^D(a) \mathbf{g}_{Q_1 Q_3}^D(a \tau_{p^n}) \mathbf{h}_{Q_3}^D(a) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)).$$

The above expression holds for any n such that p^n is bigger than the conductors of ϵ_{Q_1} and ϵ_{Q_2} . Likewise we can define $\Theta_3^\ddagger \in \mathbf{Z}_p[[X_1, X_2]]$ and $\Theta_3^\ddagger(x, y)$ with the interpolation property:

$$\Theta_3^\ddagger(Q_1, Q_2) = \mathbf{a}_3(Q_1 Q_2)^{-n} \sum_{[a] \in X_0(Np^n)} \mathbf{f}_{Q_1}^D(a \tau_{p^n}) \mathbf{g}_{Q_2}^D(a) \mathbf{h}_{Q_1 Q_2}^D(a) \epsilon_{Q_1 Q_2}(\mathbf{N}_D(a)).$$

To see the first relation, we note that

$$\Theta_2^\ddagger(0, 0) = \alpha_2^{-1} \sum_{[a] \in X_0(Np)} \mathbf{f}_0^D(a) \mathbf{g}_0^D(a \tau_p) \mathbf{h}_0^D(a), \quad \Theta_3^\ddagger(0, 0) = \alpha_3^{-1} \sum_{[a] \in X_0(Np)} \mathbf{f}_0^D(a) \mathbf{g}_0^D(a) \mathbf{h}_0^D(a \tau_p).$$

Since \mathbf{f}_0^D is a newform that is special at p , $\mathbf{f}_0^D(x \tau_p) = (-\alpha_1) \mathbf{f}_0^D(x)$, and hence $\Theta_2^\ddagger(0, 0) = -\Theta_3^\ddagger(0, 0)$.

To prove the last relation, it suffices to verify the following equation

$$(8.5) \quad \Theta(Q_1, Q_1 Q_3, Q_3) = \left(1 - \frac{\mathbf{a}_2(Q_1 Q_3)}{\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3)}\right) \Theta_2^\ddagger(Q_1, Q_3)$$

for all but finitely many arithmetic points (Q_1, Q_3) with $k_{Q_1} = k_{Q_3} = 2$. The formula for Θ_3^\ddagger can be done by a similar computation, so we leave it to the reader. Let n be a sufficiently large integer. From (8.4), we get

$$\begin{aligned} & \mathbf{a}_1(Q_1)^n \mathbf{a}_2(Q_1 Q_3)^n \mathbf{a}_3(Q_3)^n p^{-n} \text{vol}(\widehat{R}_n^\times) \Theta(Q_1, Q_1 Q_3, Q_3) \\ &= \int_{D^\times \setminus \widehat{D}^\times} d^\times a \sum_{c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \mathbf{f}_{Q_1}^D\left(a \begin{pmatrix} 1 & -c \varpi_p^{-n} \\ 0 & 1 \end{pmatrix}\right) \mathbf{g}_{Q_1 Q_3}^D(a) \mathbf{h}_{Q_3}^D\left(a \tau_{p^n} \begin{pmatrix} 1 & 0 \\ 0 & \varpi_p^{-n} \end{pmatrix}\right) \epsilon_{Q_1}(c) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \int_{D^\times \setminus \widehat{D}^\times} d^\times a \sum_{c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \mathbf{f}_{Q_1}^D\left(a \tau_{p^n} \begin{pmatrix} 1 & -\varpi_p^{-n} \\ 0 & c^{-1} \end{pmatrix}\right) \mathbf{g}_{Q_1 Q_3}^D(a \tau_{p^n}) \mathbf{h}_{Q_3}^D\left(a \begin{pmatrix} 1 & 0 \\ 0 & \varpi_p^{-n} \end{pmatrix}\right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \end{aligned}$$

by change of variables. From the equations $\tau_{p^n} \begin{pmatrix} 1 & -\varpi_p^{-n} \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} \varpi_p^n & c^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ -\varpi_p^n & 1 \end{pmatrix}$, $\epsilon_{Q_1}(\varpi_p) = \epsilon_{Q_3}(\varpi_p) = 1$ and (8.3), the last integral equals

$$\begin{aligned} & \int_{D^\times \setminus \widehat{D}^\times} d^\times a \sum_{c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \mathbf{f}_{Q_1}^D\left(a \begin{pmatrix} \varpi_p^n & c \\ 0 & 1 \end{pmatrix}\right) \mathbf{g}_{Q_1 Q_3}^D(a \tau_{p^n}) \mathbf{h}_{Q_3}^D\left(a \begin{pmatrix} \varpi_p^n & 0 \\ 0 & 1 \end{pmatrix}\right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \int_{D^\times \setminus \widehat{D}^\times} d^\times a \mathbf{a}_1(Q_1)^n \cdot \left\{ \mathbf{f}_{Q_1}^D(a) - \mathbf{a}_1(Q_1)^{-1} \mathbf{f}_{Q_1}^D\left(a \begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix}\right) \right\} \mathbf{g}_{Q_1 Q_3}^D(a \tau_{p^n}) \mathbf{h}_{Q_3}^D\left(a \begin{pmatrix} \varpi_p^n & 0 \\ 0 & 1 \end{pmatrix}\right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \mathbf{a}_1(Q_1)^n \int_{D^\times \setminus \widehat{D}^\times} d^\times a \mathbf{f}_{Q_1}^D(a) \mathbf{g}_{Q_1 Q_3}^D(a \tau_{p^n}) p^{-n} \sum_{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p} \mathbf{h}_{Q_3}^D\left(a \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix}\right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ & \quad - \mathbf{a}_1(Q_1)^{n-1} \int_{D^\times \setminus \widehat{D}^\times} d^\times a \mathbf{f}_{Q_1}^D(a) \mathbf{g}_{Q_1 Q_3}^D(a \tau_{p^{n+1}}) p^{-(n-1)} \sum_{b \in \mathbf{Z}_p/p^{n-1} \mathbf{Z}_p} \mathbf{h}_{Q_3}^D\left(a \begin{pmatrix} \varpi_p^{n-1} & b \\ 0 & 1 \end{pmatrix}\right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \{(\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3) \mathbf{a}_2(Q_1 Q_3)/p)^n \text{vol}(\widehat{R}_n^\times) - (\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3)/p)^{n-1} \mathbf{a}_2(Q_1 Q_3)^{n+1} \text{vol}(\widehat{R}_{n+1}^\times)\} \Theta_2^\ddagger(Q_1, Q_3) \\ &= \mathbf{a}_1(Q_1)^n \mathbf{a}_3(Q_3)^n \mathbf{a}_2(Q_1 Q_3)^n \left(1 - \frac{\mathbf{a}_2(Q_1 Q_3)}{\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3)}\right) p^{-n} \text{vol}(\widehat{R}_n^\times) \Theta_2^\ddagger(Q_1, Q_3). \end{aligned}$$

This verifies (8.5). \square

Now we return to the proof of Theorem 8.4(2). Write Θ_x for the partial derivative $\frac{\partial \Theta}{\partial x}$. Put

$$a = \Theta_x(0, 0, 0), \quad b = \Theta_y(0, 0, 0), \quad c = \Theta_z(0, 0, 0).$$

Taking derivatives $\Theta(x, y, x + y)$ with respect to x and y at $(0, 0)$ in Lemma 8.6, we have

$$a + c = (\ell_1 - \ell_3)\Theta_3^\dagger(0, 0), \quad b + c = (\ell_2 - \ell_3)\Theta_3^\dagger(0, 0).$$

Similarly, we have

$$a + b = (\ell_1 - \ell_2)\Theta_2^\dagger(0, 0) = (\ell_2 - \ell_1)\Theta_3^\dagger(0, 0).$$

These imply that

$$a = 0, \quad b = (\ell_2 - \ell_1)\Theta_3^\dagger(0, 0), \quad c = (\ell_1 - \ell_3)\Theta_3^\dagger(0, 0).$$

On the other hand, by the functional equation (8.1) we obtain the Taylor expansion

$$L_p(x, y, z, s) = H(x, y, z)\Theta(x, y, z)^2 + A_2(x, y, z) \cdot \left(s - \frac{x + y + z}{2}\right)^2 + \cdots.$$

By Lemma 8.5(1), we find

$$(1 - \alpha_1 \mathbf{a}_1(x)^{-1})^2 L_p^\dagger(x, 0) = H(x, 0, 0)\Theta(x, 0, 0)^2 + A_2(x, 0, 0) \cdot x^2/4 + \cdots.$$

From the vanishing of $\Theta_x(0, 0, 0)$ we deduce that

$$A_2(0, 0, 0) = 4\ell_1^2 L_p^\dagger(0, 0).$$

Lemma 8.5(2) and (8.2) complete the proof of Theorem 8.4(2).

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INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI 10617, TAIWAN
E-mail address: mlhsieh@math.sinica.edu.tw

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA CITY UNIVERSITY, 3-3-138 SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN
E-mail address: yamana@sci.osaka-cu.ac.jp