RESEARCH INTEREST

Derchyi Wu
Institute of Mathematics, Academia Sinica, Taipei, Taiwan

The modern research of integrable systems was unleashed by the discovery of the solution procedure for the KdV equation via the inverse scattering method (IST). The $1+1$-dimensional IST has been developed or influenced by works of Gelfand-Levitan, Faddeev, Goldberg, Krein, etc., and completed by Beals, Coifman, Zhou. Their theory shows that the direct problem is the construction of bounded eigenfunction normalized at infinity and meromorphic outside contours on $\mathbb{C}$ and the inverse problem can be formulated as a Riemann-Hilbert problem where the scattering data is the $\bar{\partial}$-data of eigenfunctions.

1 Integrable dPDEs

Integrable dPDEs, including the dispersionless Kadomtsev-Petviashvili equation, the first and second heavenly equations of Plebanski, the dispersionless 2D Toda (or Boyer-Finley) equation, and the Pavlov equation, are defined by a commutation $[L, M] = 0$ of pairs of one-parameter families of vector fields. They arise in various problems of mathematical physics and are intensively studied recently.

Due to the lack of dispersion, integrable dPDEs may or may not exhibit a gradient catastrophe at finite time. Since the Lax operators are vector fields, the methods used in soliton theory for proving the existence of eigenfunctions fail and the inverse problem is intrinsically nonlinear for integrable dPDEs, unlike the $\bar{\partial}$-problem formulated for general soliton equations. At last, no explicit regular localized solutions, like solitons or lumps, exist for integrable dPDEs. Therefore, it is important to solve the inverse scattering theory for integrable dPDEs.

In an illustrative example, i. e., the Pavlov equation,

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \quad v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R},$$

we solve the forward problem via a Beltrami-type equation, a first order PDE, and a shifted Riemann-Hilbert problem and the inverse problem by a nonlinear integral equation and obtain a global bounded solution for the Cauchy problem under a small data constraint. Furthermore, transforming the nonlinear integral equation into a nonlinear Riemann-Hilbert problem and solving it via a Newtonian iteration scheme, we complete the inverse scattering theory and prove a short time unique solvability of the Cauchy problem of the Pavlov equation with large initial data.

2 Twisted Hierarchies Associated with the Generalized sinh-Gordon Equation

Symmetry is a characteristic phenomenon in the theory of integrable systems. Studying symmetries has been one of the central problems and yields rewarding results even beyond the field itself. One successful attempt in classification theory is the study of reduction groups. Recent results have characterized Lax pairs with finite reduction groups of fractional-linear transformations, i.e., $\mathbb{Z}_N$, $\mathbb{D}_N$, $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$ and aroused interest in the classification theory of automorphic Lie algebras. Despite progress made in the classification theory of algebraic structures, the analytic properties, such as the construction of solutions and the investigation of the inverse scattering theory, of integrable systems with reduction groups remain mostly open.

Twisted U/K-hierarchies were introduced via a loop group approach by Terng to study symmetries of the generalized sine-Gordon equations, famous for being connected to submanifold geometry in Euclidean spaces. Twisted U/K-hierarchies are integrable hierarchies with $\mathbb{D}_2$ reduction. The loop group theory of twisted U/K-hierarchies offers a systematic and transparent approach to investigate the symmetries and the associated inverse scattering theory. Via this approach, we have succeeded in analyzing algebraic, geometric and analytic structures of several prototypical integrable hierarchies with $\mathbb{D}_2$ reduction, in particular, the 1-dimensional twisted $\frac{O(J,J)}{O(J)\times O(J)}$-system, or the generalized sinh-Gordon equation,

$$\begin{aligned}
A = \left(a_i^j\right) &\in O(q, n-q), && \\
\partial_{x_j} a_i^k &= a_j^k f_{ij},\ \ f_{ii} = 0, && i \neq j, \\
\epsilon_j \partial_{x_j} f_{ij} + \epsilon_i \partial_{x_i} f_{ji} + \textstyle\sum_{k\neq i,\, j} \epsilon_k f_{ik} f_{jk} &= -a_i^1 a_j^1, && i \neq j, \\
\partial_{x_k} f_{ij} &= f_{ik} f_{kj}, && 1 \leq i,\, j,\, k \leq n, \quad \textit{distinct}.
\end{aligned}$$

3 The Cauchy Problem of the Ward Equation

Taking a dimension reduction and a gauge fixing of the self-dual Yang-Mills equation in the space-time with signature $(2,2)$, one derives a $2+1$

dimensional $SU(N)$ chiral field equation with an additional torsion term,

$$
\begin{aligned}
&-\left(J^{-1}J_t\right)_t+\left(J^{-1}J_x\right)_x+\left(J^{-1}J_y\right)_y+\nu_0\left\{\left(J^{-1}J_y\right)_x-\left(J^{-1}J_x\right)_y\right\}\\
&+\nu_1\left\{\left(J^{-1}J_t\right)_y-\left(J^{-1}J_y\right)_t\right\}+\nu_2\left\{\left(J^{-1}J_x\right)_t-\left(J^{-1}J_t\right)_x\right\}=0.
\end{aligned}
$$

Here J lies in $SU(N)$ and $\nu=(\nu_0,\nu_1,\nu_2)$ is a constant unit vector. Letting $\nu=(1,0,0)$ (time-like) and $\nu=(0,1,0)$ (space-like), we obtain two integrable systems, the 3-dimensional relativistic-invariant system and the Ward equation.

One important class of solutions for integrable systems are solitons of which the associated eigenfunctions $\psi(x,y,t,\lambda)$ are λ-rational. The construction of simple solitons, and the study of their scattering properties was done by Manakov and Zakharov for the 3-dimensional relativistic-invariant system and by many mathematicians, for instance, Terng et. al., for the Ward equation. Besides, mathematicians, for example, Fokas, et. el., study the inverse scattering problem and solve the Cauchy problem of the 3-dimensional relativistic-invariant system if the initial potential is sufficiently small.

Our main contribution is solving the inverse scattering problem and the Cauchy problem of the Ward equation without small data constraints. Namely, for $\psi(x,y,0,\lambda)$ possessing no poles, three important algebraic properties of the Lax pair of the Ward equation, (1) derivation property; (2) translating invariant property; (3) the principal part being equivalent to a $\bar{\partial}$-operator, are used in resolving the large data difficulties into solving various types of large data Riemann-Hilbert problem. Furthermore, if the set of poles of $\psi(x,y,0,\lambda)$ is of finite number and contained in $\mathbb{C}\backslash\mathbb{R}$, then we use previous solvability of the Cauchy problem (with purely continuous scattering data), loop group factorizations, and Backlund transformation to construct the Cauchy problem solution with mixed scattering data.