AN INDEX THEOREM FOR CR MANIFOLDS WITH $S^1$ ACTION

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Abstract. Let $(X, T^{1,0} X)$ be a compact CR manifold of dimension $2n + 1$ with a transversal CR locally free $S^1$ action and let $E$ be a rigid CR vector bundle over $X$. For every $m \in \mathbb{Z}$, let $H_{b,m}^j(X, E)$ be the $m$-th $S^1$ Fourier coefficient of the $j$-th $\bar{\partial}_b$ Kohn-Rossi cohomology group with values in $E$. In this paper, we prove that the Euler characteristic $\sum_{j=0}^{n} (-1)^j \dim H_{b,m}^j(X, E)$ can be computed in terms of the tangential Chern character of $E$, the tangential Todd class of $T^{1,0} X$, and the Chern polynomial of the Levi curvature of $X$. As applications, we can produce many CR functions on a weakly pseudoconvex CR manifold with such an $S^1$ action and many CR sections on some class of CR manifolds. In some cases, we can reinterpret Kawasaki’s Hirzebruch-Riemann-Roch formula for a complex orbifold with an orbifold holomorphic line bundle by an integral over a smooth CR manifold.

Contents

1. Introduction and statement of the results 2
   1.1. Introduction and Motivation 2
   1.2. Main theorem 3
   1.3. Applications 5
   1.4. Kawasaki’s Hirzebruch-Riemann-Roch and Grauert-Riemenschneider criterion for orbifold line bundles 8
   1.5. Examples 10
   1.6. Proof of Theorem 1.2 12
   1.7. The idea of the proof of Theorem 1.3 14

2. Preliminaries 20
   2.1. Some standard notations 21
   2.2. Set up and terminology 21
   2.3. Tangential De-Rham cohomology group, Tangential Chern character and Tangential Todd class 23
   2.4. BRT trivializations and rigid Hermitian metrics 24
3. Hodge theory for $\square^{(q)}_{b,m}$ 26
4. Modified Kohn Laplacians 30
5. Asymptotic expansions for the heat kernels of the modified Kohn Laplacians 38
   5.1. Heat kernels of the modified Kodaira Laplacians on BRT trivializations 38
   5.2. Heat kernels of the modified Kohn Laplacians 43
6. Local density for the Euler characteristic $\sum_{j=0}^{n} (-1)^j \dim H_{b,m}^j(X, E)$ 48

References 50

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1. Introduction and Statement of the Results

1.1. Introduction and Motivation. Let \((X, T^{1,0}X)\) be a compact CR manifold of dimension \(2n+1\) and let \(\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)\) be the tangential Cauchy-Riemann operator. For a smooth function \(u\), we say \(u\) is CR if \(\overline{\partial}_b u = 0\). In CR geometry, it is crucial to be able to produce many CR functions. When \(X\) is strongly pseudoconvex and dimension of \(X\) is greater than or equal to five, it is well-known that the space of global smooth CR functions \(H^0_b(X)\) is infinite dimensional. When \(X\) is weakly pseudoconvex or the Levi form of \(X\) has negative eigenvalues, the space of global CR functions could be trivial.

In general, it is very difficult to determine when the space \(H^0_b(X)\) is large. Since \(\overline{\partial}_b^2 = 0\), we have \(\overline{\partial}_b\)-complex: \(\cdots \to \Omega^{0,q-1}(X) \to \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \to \cdots\) and we can define cohomology group:

\[ H^q_b(X) := \frac{\text{Ker} \overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)}{\text{Im} \overline{\partial}_b : \Omega^{0,q-1}(X) \to \Omega^{0,q}(X)}. \]

As in complex geometry, to understand the space \(H^q_b(X)\), we could try to establish CR Hirzebruch-Riemann-Roch theorem for the Euler characteristic \(\sum_{j=0}^{n} (-1)^j \dim H^j_b(X)\) and vanishing theorems for \(H^j_b(X), j \geq 1\). But the difficulty comes from the fact that \(\dim H^j_b(X)\) could be infinite for some \(j\). Let's see some examples. If \(X\) is strongly pseudoconvex of dimension \(\geq 5\), it is well-known that \(\overline{\partial}_b : \Omega^{0,0}(X) \to \Omega^{0,1}(X)\) is not hypoelliptic but \(\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)\) is hypoelliptic, for \(q \geq 1\). We have \(\dim H^0_b(X) = \infty\) and \(\dim H^q_b(X) < \infty\), for \(q \geq 1\). If the Levi form of \(X\) has exactly one negative, \(n - 1\) positive eigenvalues on \(X\) and \(n > 3\), it is well-known that \(\dim H^0_b(X) = \infty\), \(\dim H^{n-1}_b(X) = \infty\) and \(\dim H^q_b(X) < \infty\), for \(q \notin \{1, n - 1\}\). In general, we don't know how to define the Euler characteristic \(\sum_{j=0}^{n} (-1)^j \dim H^j_b(X)\). On the other hand, even if \(\overline{\partial}_b\) is hypoelliptic, it is still difficult to calculate local index formula for \(\overline{\partial}_b\) operator. We are lead to ask the following question:

**Question 1.1.** Can we establish some kind of CR Hirzebruch-Riemann-Roch theorem (not the usual one) on some class of CR manifolds?

To answer this question, let's come back to complex geometry case. Consider a compact complex manifold \(M\) of dimension \(n\) and let \((L, h_L) \to M\) be a holomorphic line bundle over \(M\), where \(h_L\) denotes a Hermitian fiber metric of \(L\). Let \((L^*, h^{L*}) \to M\) be the dual bundle of \((L, h_L)\) and put \(X = \{v \in L^* : |v|_{h^{L*}}^2 = 1\}\). We call \(X\) the circle bundle of \((L^*, h^{L*})\). It is clear that \(X\) is a compact CR manifold of dimension \(2n + 1\). Given a local holomorphic frame \(s\) of \(L\) on an open subset \(U \subset M\) we define the associated local weight of \(h_L\) by \(|s(x)|_{h_L}^2 = e^{2\phi(x)}, \phi \in C^\infty(U, \mathbb{R})\). The CR manifold \(X\) is equipped with a natural \(S^1\) action. Locally \(X\) can be represented in local holomorphic coordinates \((z, \lambda) \in \mathbb{C}^{n+1}\), where \(\lambda\) is the fiber coordinate, as the set of all \((z, \lambda)\) such that \(|\lambda|^2 e^{2\phi(z)} = 1\), where \(\phi\) is a local weight of \(h_L\). The \(S^1\) action on \(X\) is given by \(e^{-i\theta} \circ (z, \lambda) = (z, e^{-i\theta} \lambda), e^{-i\theta} \in S^1, (z, \lambda) \in X\). Let \(T \in C^\infty(X, TX)\) be the real vector field induced by the \(S^1\) action, that is, \(Tu = \frac{\partial}{\partial u} (u(e^{-i\theta} \circ x))|_{\theta = 0}, u \in C^\infty(X)\). We can check that \([T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)\) and \(CT(x) \oplus T_{x}^{1,0}X \oplus T_{x}^{1,1}X = CT_x(X)\) (we say that the \(S^1\) action is CR and transversal). For every \(m \in \mathbb{Z}\) and \(q = 0, 1, 2, \ldots, n\), put

\[\Omega^q_m(X) := \{u \in \Omega^q(X) : Tu = -imu\}\]

\[= \{u \in \Omega^q(X) : u(e^{-i\theta} \circ x) = e^{-i\theta u}(x), \forall \theta \in [0, 2\pi]\}\].

Since \(\overline{\partial}_b T = T \overline{\partial}_b\), we have \(\overline{\partial}_b : \Omega^q_m(X) \to \Omega^{q+1}_m(X)\) and we could consider the cohomology group:

\[H^q_b(X) := \frac{\text{Ker} \overline{\partial}_b : \Omega^q_m(X) \to \Omega^{q+1}_m(X)}{\text{Im} \overline{\partial}_b : \Omega^{q-1}_m(X) \to \Omega^q_m(X)}\].

The following theorem can be seen as the starting point of this work.

**Theorem 1.2.** Let \(\Omega^{0,q}(M, L^m)\) be the space of smooth sections of \((0, q)\) forms of \(M\) with values in \(L^m\) and let \(H^q(M, L^m)\) be the \(q\)-th \(\overline{\partial}\) Dolbeault cohomology group with value in \(L^m\), where \(L^m\) is the \(m\)-th power of \(L\). For every \(q = 0, 1, 2, \ldots, n\), and every \(m \in \mathbb{Z}\), there is a bijection map \(A^q_m : \Omega^q_m(X) \to\)
\[ \Omega^{0,q}(M, L^m) \text{ such that } A_{b,m}^{(q+1)j} = \overline{\partial} A_{b,m}^{(q)} \text{ on } \Omega^{0,q}(X). \]  
Hence, \( \Omega^{0,q}(X) \cong \Omega^{0,q}(M, L^m) \) and \( H^{q}_{b,m}(X) \cong H^q(M, L^m). \)  
In particular \( \dim H^{q}_{b,m}(X) < \infty \), for every \( m \) and every \( q \) and
\[
\sum_{j=1}^{n} (-1)^j \dim H^{j}_{b,m}(X) = \sum_{j=1}^{n} (-1)^j \dim H^{j}(M, L^m).
\]

Theorem 1.2 is probably well-known. Since it is difficult to indicate a precise reference, we will give a proof of Theorem 1.2 in Section 1.6 for the convenience of the reader.

Hirzebruch-Riemann-Roch Theorem or Atiyah-Singer index Theorem tells us that
\[
\left(1.1\right) \sum_{j=0}^{n} (-1)^j \dim H^{j}_{b,m}(X) = \int_M Td(T^{1,0}M) \mathrm{ch}(L^m),
\]
where \( Td(T^{1,0}M) \) denotes the Todd class of \( T^{1,0}M \) the holomorphic tangent bundle of \( M \) and \( \mathrm{ch}(L^m) \) denotes the Chern character of \( L^m \). Let’s reformulate (1.1) in terms of characteristic classes on \( X \):
\[
\left(1.2\right) \sum_{j=0}^{n} (-1)^j \dim H^{j}_{b,m}(X) = \frac{1}{2\pi} \int_X Td_b(T^{1,0}X) \wedge e^{-m \frac{dim H^0}{2\pi}} \wedge \omega_0,
\]
where \( Td_b(T^{1,0}X) \) denotes the tangential Todd class of \( T^{1,0}X \) and \( e^{-m \frac{dim H^0}{2\pi}} \) denotes the Chern polynomial of the Levi curvature and \( \omega_0 \) is the uniquely determined global real 1-form (see Section 2.2 and Section 2.3 for the precise definitions). The goal of this work is to generalize (1.2) to any CR manifold with \( S^1 \) action. It turns out that to establish (1.1) on any CR manifold with \( S^1 \) action, we need to calculate local density function for the Euler characteristic and Section 2.3 for the precise definitions.

There is another index theory of geometric significance, developed by Charles Epstein. He studied so-called relative index of a pair of CR structures through their Szegö projectors in a series of papers (see [11], [12], [13], [14] and [15]). On the other hand, Erik van Erp derived an index formula for subelliptic operators on a contact manifold (see [16], [17]).

1.2. Main theorem. We now formulate the main results. We refer to Section 2.2 and Section 2.3 for some notations and terminology used here.

Let \( (X, T^{1,0}X) \) be a compact CR manifold with a transversal CR locally free \( S^1 \) action \( e^{-i\theta} \), where \( T^{1,0}X \) is a CR structure of \( X \). Let \( T \in C^\infty(X, TX) \) be the real vector field induced by the \( S^1 \) action and let \( \omega_0 \in C^\infty(X, T^*X) \) be the global real one form determined by \( \langle \omega_0, T \rangle = 1, \langle \omega_0, u \rangle = 0, \) for every \( u \in T^{1,0}X \oplus T^{0,1}X \).

Let \( E \) be a rigid CR vector bundle over \( X \) (see Definition 2.4) and let \( \overline{\partial}_b : \Omega^{0,q}(X, E) \to \Omega^{0,q+1}(X, E) \) be the tangential Cauchy-Riemann operator with values in \( E \). For every \( u \in \Omega^{0,q}(X, E) \), we can define \( Tu \in \Omega^{0,q}(X, E) \) and we have \( T \overline{\partial}_b = \overline{\partial}_b T \). For \( m \in \mathbb{Z}, \) put
\[
\Omega^{0,q}_m(X, E) := \left\{ u \in \Omega^{0,q}(X, E); Tu = -imu \right\} = \left\{ u \in \Omega^{0,q}(X, E); (e^{-i\theta})^*u = e^{-im\theta}u, \ \forall \theta \in [0, 2\pi[ \right\},
\]
where \( (e^{-i\theta})^* \) denotes the pull-back map by \( e^{-i\theta} \) (see (2.3)). Since \( T \overline{\partial}_b = \overline{\partial}_b T \), we have
\[
\overline{\partial}_{b,m} := \overline{\partial}_b : \Omega^{0,q}_m(X, E) \to \Omega^{0,q+1}_m(X, E).
\]
Thus, for every \( m \in \mathbb{Z} \), we have \( \overline{\partial}_{b,m} \)-complex:
\[
\overline{\partial}_{b,m} : \cdots \to \Omega^{0,q-1}_m(X, E) \to \Omega^{0,q}_m(X, E) \to \Omega^{0,q+1}_m(X, E) \to \cdots
\]
and we can consider $q$-th $\partial_{b,m}$ Kohn-Rossi cohomology group:

$$H^q_{b,m}(X,E) := \frac{\text{Ker} \partial_{b,m} : \Omega^q_{m}(X,E) \to \Omega^{q+1}_{m}(X,E)}{\text{Im} \partial_{b,m} : \Omega^{q-1}_{m}(X,E) \to \Omega^{q}_{m}(X,E)}.$$ 

When $E$ is trivial, for every $m \in \mathbb{Z}$, we denote $\Omega^q_{m}(X) := \Omega^q_{m}(X,E)$ and $H^q_{b,m}(X) := H^q_{b,m}(X,E)$.

We will prove in Theorem 3.7 that without any Levi curvature assumption, $\dim H^q_{b,m}(X,E) < \infty$, for every $m \in \mathbb{Z}$ and every $q = 0, 1, 2, \ldots, n$.

For $x \in X$, we say that the period of $x$ is $\frac{2\pi}{\ell}$, $\ell \in \mathbb{N}$, if $e^{-i\theta} \circ x \neq x$, for every $-\frac{\pi}{\ell} \leq \theta < \frac{\pi}{\ell}$ and $e^{-\frac{2\pi}{\ell}} \circ x = x$. For each $\ell \in \mathbb{N}$, put

$$X_\ell = \{x \in X; \text{ the period of } x \text{ is } \frac{2\pi}{\ell}\}$$

and let $p = \min \{\ell \in \mathbb{N}; X_\ell \neq \emptyset\}$. The main result of this work is the following

**Theorem 1.3.** Suppose $(X,T^{1,0}X)$ is a compact, connected CR manifold with a transversal CR locally free $S^1$ action. Then, $X_p$ is dense in $X$ and

$$\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E)$$

$$= \sum_{s=1}^{p} e^{2\pi(x(s-1)-m)} \frac{1}{2\pi} \int_X Td_{b}(T^{1,0}X) \land ch_b(E) \land e^{-m \frac{\omega_0}{2\pi} \land \omega_0},$$

where $Td_{b}(T^{1,0}X)$ denotes the tangential Todd class of $T^{1,0}X$ and $ch_b(E)$ denotes the tangential Chern character of $E$.

We refer the reader to Section 2.3 for the precise meanings of $Td_{b}(T^{1,0}X)$ and $ch_b(E)$.

**Remark 1.4.** We note that there is topological obstruction for a CR manifold to admit a transversal $S^1$ action. For instance, a compact strictly pseudoconvex CR 3-manifold must have even first Betti number if admitting a transversal CR $S^1$ action. The reason is that such a manifold must have free pseudohermitian torsion (see [25]). On the other hand, Alan Weinstein showed that vanishing pseudohermitian torsion implies even first Betti number (see the Appendix in [6]). In this paper, we only consider that the $S^1$ action is transversal and locally free. It should be mentioned that the $S^1$ action on a circle bundle is globally free. We give some examples:

**Example I:** $X = \{(z_1, z_2, z_3) \in \mathbb{C}^3; |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1^2 + z_2|^4 + |z_2^3 + z_3|^6 = 1\}$. Then $X$ admits a transversal CR locally free $S^1$ action: $e^{-i\theta} \circ (z_1, z_2, z_3) = (e^{-i\theta} z_1, e^{-2i\theta} z_2, e^{-6i\theta} z_3)$. It is clear that this $S^1$ action is not globally free.

**Example II:** Let $X$ be a compact orientable Seifert 3-manifold. Kamishima and Tsuboi [23] proved that $X$ is a compact CR manifold with a transversal CR locally free $S^1$ action. Moreover, $X$ is a circle bundle but with singular base.

In Section 1.5, we collect more examples.

**Remark 1.5.** (I) The number $\sum_{s=1}^{p} e^{2\pi(x(s-1)-m)}$ is equal to 0 or $p$.

(II) $X_p$ is dense in $X$ is a famous result in topology (see Proposition 1.24 in Meinrenken [29] and Duistermaat-Heckman [9]). We prove this result by computing local density function of the Euler characteristic $\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E)$.

**Remark 1.6.** Let $p_1 = p < p_2 < p_3 < \cdots < p_N$ be the periods of $X$ with respect to the given $S^1$ action $S^1 \times X \to X$.

$$S^1 \times X \to X;$$

$$e^{-i\theta} \circ (x) \to e^{-i\theta} \circ x.$$
We claim that one has \( p_1 = p \) divides every others. The reason is that, the subgroup \( \mathbb{Z}_p \) acts trivially on the dense open subset which is this \( p \)-stratum or \( p_1 \)-stratum, hence it acts trivially on the whole \( X \) by continuity and density of this stratum. Hence, the fixed groups \( g, q = p_j, j = 2, \ldots, N \), of any other stratum must contain \( \mathbb{Z}_p \). Namely \( \mathbb{Z}_p \) is a subgroup of \( \mathbb{Z}_q \). Hence, \( \frac{q}{p} \in \mathbb{N} \). The claim follows.

By the above fact, one may re-normalize the given \( S^1 \)-action by lifting, so as to make the new \( S^1 \)-action have \( p_1 = 1 \). More precise, define

\[
S^1 \times X \to X,
\]

\[
(e^{-i\theta}, x) \to e^{-i\theta} \circ x := e^{-i\frac{\theta}{p}} \circ x.
\]

The new \( S^1 \)-action \( (e^{-i\theta}, \circ) \) has \( p_1 = 1 \). Let \( \tilde{\omega}_0 \) be the global real one form with respect to \( (e^{-i\theta}, \circ) \) and for every \( m \in \mathbb{Z} \), let \( \tilde{H}^q_{b,m}(X, E) \) be the \( m \)-th \( S^1 \) Fourier coefficient of the \( q \)-th Kohn-Rossi cohomology group with respect to \( (e^{-i\theta}, \circ) \). It is straightforward to see that

\[
\tilde{\omega}_0 = p\omega_0,
\]

\[
\tilde{H}^q_{b,m}(X, E) = H^q_{b,pm}(X, E), \quad \forall m \in \mathbb{Z}, \quad \forall q = 0, 1, 2, \ldots, n.
\]

From (1.5) and Theorem 1.3, we see that the index formulas can be transformed to each other: the one before the above reduction and the one after the reduction. The point is that, the old one shall have a sum of exponential terms, and the new one not.

1.3. Applications.

1.3.1. Applications in CR geometry. In CR geometry, it is crucial to be able to produce many CR functions or CR sections. Put

\[
H^0_b(X, E) = \{ u \in C^\infty(X, E); \overline{\partial}_b u = 0 \} .
\]

Kohn asked the following question

**Question 1.7.** Let \( X \) be a compact weakly pseudoconvex CR manifold. When is the space \( H^0_b(X, E) \) large?

See the discussion after Definition 2.2 for the meanings of weakly pseudoconvex and strongly pseudoconvex CR manifolds.

In [26], Lempert proved that a three dimensional compact strongly pseudoconvex CR manifold \( X \) with a locally free transversal CR \( S^1 \) action can be CR embedded into \( \mathbb{C}^N \). In [10], Epstein proved that a three dimensional compact strongly pseudoconvex CR manifold \( X \) with a global free transversal CR \( S^1 \) action can be embedded into \( \mathbb{C}^N \) by the positive Fourier coefficients. The embeddability of \( X \) by positive Fourier coefficients is related to the behavior of the \( S^1 \) action on \( X \). For example, suppose that we can find \( f_1 \in H^0_{b,m}(X), \ldots, f_{dm} \in H^0_{b,m}(X) \) and \( g_1 \in H^0_{b,m_1}(X), \ldots, g_{hm_1} \in H^0_{b,m_1}(X) \) such that the map

\[
\Phi_{m,m_1}: x \in X \to (f_1(x), \ldots, f_{dm}(x), g_1(x), \ldots, g_{hm_1}(x)) \in \mathbb{C}^{dm + hm_1}
\]

is a CR embedding. The \( S^1 \) action on \( X \) induces naturally a \( S^1 \) action on \( \Phi_{m,m_1}(X) \) and this \( S^1 \) action on \( \Phi_{m,m_1}(X) \) is simply given by the following:

\[
e^{-i\theta} \circ (z_1, \ldots, z_{dm}, z_{dm+1}, \ldots, z_{dm+hm_1}) = (e^{-im\theta} z_1, \ldots, e^{-im\theta} z_{dm}, e^{-im_1\theta} z_{dm+1}, \ldots, e^{-im_1\theta} z_{dm+hm_1}).
\]

Thus, if we could embed such CR manifold by positive Fourier coefficients, we can describe the \( S^1 \) action explicitly. To study the embedding theorem of such CR manifold by positive Fourier coefficients, it is crucial to be able to know

**Question 1.8.** When \( \dim H^0_{b,m}(X, E) \approx m^n \) for \( m \) large?

Combining our index theorems with some vanishing theorems, we could answer Question 1.7 and Question 1.8. From Theorem 1.3, we deduce
Corollary 1.9. Suppose we have the same assumption as in Theorem 1.3. Then,
\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E)
\]
(1.6)
\[
= r \sum_{s=1}^{p} e^{\frac{2n(s-1)}{p}} \frac{m^n}{n!(2\pi)^{n+1}} \int_X (-d\omega_0)^n \wedge \omega_0 + O(m^{n-1}),
\]
where \( r \) denotes the complex rank of the vector bundle \( E \).

We can repeat the proof of Theorem 2.1 in [22] with minor change and get

Theorem 1.10. Suppose we have the same assumption as in Theorem 1.3 and assume that \( X \) is weakly pseudoconvex. Then, \( \dim H^j_{b,m}(X, E) = o(m^n) \), for every \( j = 1, 2, \ldots, n \).

From Corollary 1.9 and Theorem 1.10, we obtain

Theorem 1.11. Suppose we have the same assumption as in Theorem 1.3 and assume that \( X \) is weakly pseudoconvex. Then for \( m \gg 1 \),
\[
\dim H^j_{b,m}(X, E)
\]
\[
= r \sum_{s=1}^{p} e^{\frac{2n(s-1)}{p}} \frac{m^n}{n!(2\pi)^{n+1}} \int_X (-d\omega_0)^n \wedge \omega_0 + o(m^n),
\]
where \( r \) denotes the complex rank of the vector bundle \( E \).

In particular, if the Levi form is strongly pseudoconvex at some point of \( X \), then \( \dim H^0_{b,m}(X) = O(m^n) \), for \( m \gg 1 \) and hence, \( \dim H^0_{b}(X, E) = \infty \).

From Theorem 1.11, we answer Question 1.7 and Question 1.8.

In some applications, we need to sum over \( m \). Given a CR manifold \( X \), when \( X \) can be CR embedded into complex space is an important problem in CR geometry. When \( X \) is strongly pseudoconvex and dimension of \( X \) is greater than or equal to five, a classical theorem of L. Boutet de Monvel [3] asserts that \( X \) can be globally CR embedded into \( \mathbb{C}^N \), for some \( N \in \mathbb{N} \). When \( X \) is not strongly pseudoconvex, the space of global CR functions could be even trivial. Since many interesting examples live in the projective space (e.g. the quadric \( \{z \in \mathbb{C}^{pN-1}; |z_1|^2 + \ldots + |z_q|^2 - |z_{q+1}|^2 - \ldots - |z_N|^2 = 0 \} \), it is thus natural to consider a setting analogous to the Kodaira embedding theorem and ask if \( X \) can be embedded into the projective space by means of CR sections of a CR line bundle. Inspired by Kodaira, we consider \( L^k \) the \( k \)-th power of a CR line bundle \( L \to X \). If the dimension of the space \( H^0_b(X, L^k) \) of CR sections of \( L^k \) is large, when \( k \to \infty \), one should find many CR sections to embed into the projective space. The following question is asked by Henkin and Marinescu [28, p.47-48]

Question 1.12. When is \( \dim H^0_b(X, L^k) = O(k^{n+1}) \) for \( k \) large?

We now assume that \( L \) is a rigid CR line bundle with a rigid Hermitian fiber metric \( h^L \) (see Definition 2.4 and Definition 2.9 for the precise meanings of rigid line bundle and rigid Hermitian metric) and let \( \mathcal{R}^L \in \Omega^2_b(X) \) be the curvature of \( L \) induced by \( h^L \). If \( s \) is a local trivializing section of \( L \) on an open subset \( D \subset X \), we have \( |s(x)|^2 = e^{-2\phi(x)} \), \( \forall x \in D \), where \( \phi \in C^\infty(D, \mathbb{R}) \) with \( T\phi = 0 \). Then, \( \mathcal{R}^L = 2\partial_b\bar{\partial}\phi \in \Omega^2_b(X) \). For \( k > 0 \), let \( (L^k, h^{L^k}) \) be the \( k \)-th power of \( (L, h^L) \).

Theorem 1.13. With the notations above, for \( k \) large, we have
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, L^k \otimes E)
\]
\[
= r(2\pi)^{-n-\frac{1}{2}} \frac{1}{n!} k^{n+1} \int_{X} \int_{[-\delta, \delta]} (i\mathcal{R}^L_s - s\omega_0(x))^n \wedge \omega_0(x)ds + o(k^{n+1}),
\]
where $\delta > 0$ and $r$ denotes the complex rank of the vector bundle $E$.

**Proof.** From Theorem 1.3, we can check that
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, L^k \otimes E)
\]
\[
= r(2\pi)^{-n-1} \sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{s=1}^{p} e^{2\pi(s-1)/p-mi} \frac{1}{n!} \int_X (ik^R_L - m\omega_0(x))^n \wedge \omega_0(x) + o(k^n)
\]
\[
= r(2\pi)^{-n-1} \sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{s=1}^{p} e^{2\pi(s-1)/p-mi} \frac{k^n}{n!} \int_X (iR^L_x - \frac{m}{k}\omega_0(x))^n \wedge \omega_0(x) + o(k^n).
\]

Note that $\sum_{s=1}^{p} e^{2\pi(s-1)/p-mi} = p$ if $m$ is divided by $p$ and $\sum_{s=1}^{p} e^{2\pi(s-1)/p-mi} = 0$ if $m$ is not divided by $p$. From this observation and (1.8), we get
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, L^k \otimes E)
\]
\[
= r(2\pi)^{-n-1} \sum_{\ell \in \mathbb{Z}, |\ell| \leq k\delta} \frac{p}{n!} \int_X (iR^L_x - \frac{p\ell}{k}\omega_0(x))^n \wedge \omega_0(x) + o(k^n).
\]

It is clear that the Riemann sum $\sum_{\ell \in \mathbb{Z}, |\ell| \leq k\delta} \frac{p}{n!} \int_X (iR^L_x - \frac{p\ell}{k}\omega_0(x))^n \wedge \omega_0(x)$ converges to
\[
\int_X \int_{[-\delta, \delta]} (iR^L_x - \sigma d\omega_0(x))^n \wedge \omega_0(x) ds
\]
as $k \to \infty$, and hence
\[
\sum_{\ell \in \mathbb{Z}, |\ell| \leq k\delta} \frac{p}{n!} \int_X (iR^L_x - \frac{p\ell}{k}\omega_0(x))^n \wedge \omega_0(x)
\]
\[
= \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta, \delta]} (iR^L_x - \sigma d\omega_0(x))^n \wedge \omega_0(x) ds + o(k^{n+1}).
\]
Combining (1.10) with (1.9), (1.7) follows. \qed

**Definition 1.14.** We say that $(L, h^L)$ is positive at $p \in X$ if $R^L_p$ is a positive Hermitian quadratic form over $T^1,0_p X$. We say that $(L, h^L)$ is semi-positive if for any $x \in X$ there exists a constant $\delta > 0$ such that $R^L_x - \sigma d\omega_0(x)$ is a semi-positive Hermitian quadratic form over $T^1,0_x X$ for any $|s| < \delta$.

We can repeat the proof of Theorem 1.24 in [21] with minor change and get

**Theorem 1.15.** Assume that $(L, h^L)$ is a semi-positive CR line bundle over $X$. Then, for $\delta > 0$, $\delta$ small, we have
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \dim H^j_{b,m}(X, L^k \otimes E) = o(k^{n+1}), \quad j = 1, 2, \ldots, n.
\]

From Theorem 1.13 and Theorem 1.15, we get

**Theorem 1.16.** Assume that $(L, h^L)$ is semi-positive. Then, for $\delta > 0$, $\delta$ small, we have
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \dim H^j_{b,m}(X, L^k \otimes E)
\]
\[
= r(2\pi)^{-n-1} \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta, \delta]} (iR^L_x - \sigma d\omega_0(x))^n \wedge \omega_0(x) ds + o(k^{n+1}),
\]
where $r$ denotes the complex rank of the vector bundle $E$. 

AN INDEX THEOREM FOR CR MANIFOLDS WITH S$^1$ ACTION
In particular, if \((L, h^L)\) is positive at some point of \(X\), then \(\dim H^0_k(X, L^k \otimes E) = O(k^{n+1})\).

From Theorem 1.16, we have answered Question 1.12.

1.4. Kawasaki’s Hirzebruch-Riemann-Roch and Grauert-Riemenschneider criterion for orbifold line bundles. Kawasaki’s Hirzebruch-Riemann-Roch formula, introduced by Kawasaki [24], is the Hirzebruch-Riemann-Roch formula for orbifolds. Comparing with Kawasaki’s formula, we get a simpler Hirzebruch-Riemann-Roch formula for some class of orbifold line bundles from our main result Theorem 1.3. Moreover, from Theorem 1.11, we establish Grauert-Riemenschneider criterion for orbifold line bundles.

Let \(M\) be a manifold and let \(G\) be a compact Lie group. Assume that \(M\) admits a \(G\)-action:

\[
G \times M \to M, \\
(g, x) \to g \circ x.
\]

We suppose that the action \(G\) on \(M\) is locally free, that is, for every point \(x \in M\), the stabilizer group \(G_x = \{g \in G; g \circ x = x\}\) of \(x\) is a finite subgroup of \(G\). If \(G\) is an action of this type the quotient space

\[\tag{1.12}X := M/G\]

is an orbifold. A theorem of Satake [30] says that the converse is true: every orbifold has a presentation of the form (1.12). We assume that \(M\) is a compact connected complex manifold with complex structure \(T^{1,0}M\). \(G\) induces an action on \(\mathbb{C}TM\):

\[
G \times \mathbb{C}TM \to \mathbb{C}TM, \\
(g, u) \to g^*u,
\]

where \(g^* = (g^{-1})^*, (g^{-1})_*\) denotes the push-forward by \(g^{-1}\) on \(\mathbb{C}TM\). We suppose that \(G\) is holomorphic, that is \(g^*(T^{1,0}M) \subset T^{1,0}M\), for every \(g \in G\). Let \(T^{0,1}M := \overline{T^{1,0}M}\). Put \(\mathbb{C}TM/G := \mathbb{C}TM/G, \mathbb{T}^{0,1}(M/G) := T^{0,1}M/G, T^{1,0}(M/G) := T^{1,0}M/G\). Assume that \(T^{0,1}(M/G) \cap T^{1,0}(M/G) = \{0\}\). Then, \(T^{1,0}(M/G)\) is a complex structure on \(M/G\) and \(M/G\) is a complex orbifold. Suppose

\[\dim_c T^{1,0}(M/G) = n.\]

Let \(L\) be a \(G\)-invariant holomorphic line bundle over \(M\), that is, for every transition function \(h\) of \(L\) on an open set \(U \subset M\), we have \(h(g \circ x) = h(x)\), for every \(g \in G\), \(x \in U\) with \(g \circ x \in U\). Suppose that \(L\) admits a locally free \(G\)-action:

\[
G \times L \to L, \\
(g, x) \to g \circ x
\]

with \(\pi(g \circ x) = g \circ (\pi(x))\), for every \(g \in G\), where \(\pi : L \to M\) denotes the natural projection, and where the action of \(G\) on \(L\) is linear on the fibers of \(L\), that is, for every \(g \in G\), every \(z \in M\), we have \(g \circ (s(z) \otimes \lambda) = s_1(g \circ z) \otimes \rho(g, z)\lambda\), for every \(\lambda \in \mathbb{C}\), where \(s\) and \(s_1\) are local sections of \(L\) defined near \(z\) and \(g \circ z\) respectively and \(\rho(g, z) \in \mathbb{C}\) depends smoothly on \(z\) and \(g\). Then, \(L/G\) is an orbifold holomorphic line bundle over \(M/G\). For every \(m \in \mathbb{N}\), let \(L^m\) be the \(m\)-th power of \(L\). Then, the \(G\)-action on \(L\) induces a locally free \(G\)-action on \(L^m\):

\[
G \times L^m \to L^m, \\
(g, x) \to g \circ x
\]

with \(\pi_m(g \circ x) = g \circ (\pi_m(x))\), for every \(g \in G\), where \(\pi_m : L^m \to M\) denotes the natural projection, and where the action of \(G\) on \(L^m\) is linear on the fibers of \(L^m\). Then, \(L^m/G\) is also an orbifold holomorphic line bundle over \(M/G\). Now, we fix \(m \in \mathbb{Z}\). Let \(T^{0,0}M\) denote the bundle of \((0, q)\) forms on \(M\). Since \(G\) is holomorphic, \(G\) induces a natural action on \((L^m \otimes T^{0,0}M)\):

\[
G \times (L^m \otimes T^{0,0}M) \to L^m \otimes T^{0,0}M, \\
(g, u) \to g^*u.
\]
For every $q = 0, 1, 2, \ldots, n$, put
\begin{equation}
\Omega^{0,q}(M/G, L^n/G) := \{u \in \Omega^{0,q}(M, L^n); g^*u = u, \ \forall g \in G\},
\end{equation}
where $\Omega^{0,q}(M, L^n)$ denotes the space of smooth sections with values in $T^{\ast 0,q}M \otimes L^n$. The Cauchy-Riemann operator $\overline{\partial} : \Omega^{0,q}(M, L^n) \rightarrow \Omega^{0,q+1}(M, L^n)$ is $G$-invariant and we have $\overline{\partial}$-complex:
\[\overline{\partial} : \cdots \rightarrow \Omega^{0,q-1}(M, L^n/G) \rightarrow \Omega^{0,q}(M, L^n/G) \rightarrow \Omega^{0,q+1}(M, L^n/G) \rightarrow \cdots\]
and we can consider $q$-th Dolbeault cohomology group:
\[H^q(M/G, L^n/G) := \text{Ker}\overline{\partial} : \Omega^{0,q}(M/G, L^n/G) \rightarrow \Omega^{0,q+1}(M/G, L^n/G) / \text{Im}\overline{\partial} : \Omega^{0,q-1}(M/G, L^n/G) \rightarrow \Omega^{0,q}(M/G, L^n/G).\]

Let $L^*$ be the dual bundle of $L$. Then, $L^*$ is also a $G$-invariant holomorphic line bundle and $L^*$ admits a locally free $G$-action:
\[G \times L^* \rightarrow L^*,\]
\[(g, x) \rightarrow g \circ x\]
with $\pi^*(g \circ x) = g \circ (\pi^*(x))$, for every $g \in G$, where $\pi^* : L^* \rightarrow M$ denotes the natural projection, and where the action of $G$ on $L^*$ is linear on the fibers of $L^*$. Then, $L^*/G$ is an orbifold holomorphic line bundle over $M/G$. Let $\text{Tot}(L^*)$ be the space of all non-zero vectors of $L^*$. Assume that $\text{Tot}(L^*)/G$ is a smooth manifold. Take any $G$-invariant Hermitian fiber metric $h^{\ast L}$ on $L^*$, set $\tilde{X} = \{v \in L^*; |v|_{h^{\ast L}} = 1\}$ and put $X = \tilde{X}/G$. Since $\text{Tot}(L^*)$ is a smooth manifold, $X = \tilde{X}/G$ is a smooth manifold. The natural $S^1$ action on $\tilde{X}$ induces a locally free $S^1$ action $e^{-i\theta}$ on $X$. Moreover, we can check that $X$ is a CR manifold and the $S^1$ action on $X$ is CR and transversal. We will use the same notations as before. We can repeat the proof of Theorem 1.2 with minor change and deduce that for every $q = 0, 1, 2, \ldots, n$, and every $m \in \mathbb{Z}$, we have
\begin{equation}
H^q(M/G, L^n/G) \cong H^q_{b,m}(X), \quad \dim H^q(M/G, L^n/G) = \dim H^q_{b,m}(X).
\end{equation}

We pause and introduce some notations. For every $x \in \text{Tot}(L^*)$ and $g \in G$, put $N(g, x) = 1$ if $g \notin G_x$ and $N(g, x) = \inf\{\ell \in \mathbb{N}; g^\ell = I\}$ if $g \in G_x$. Set
\[p = \inf\{N(g, x); x \in \text{Tot}(L^*), \ \ g \in G, \ \ g \neq I\},\]
where $I$ denotes the identity element of $G$. It is straightforward to see that $X_p$ is an open and dense subset of $X$, where $X_p$ is given by (1.3). From Theorem 1.3 and (1.18), we deduce

**Theorem 1.17.** With the notations used above and recall that we work with the assumptions that $M$ is connected and $\text{Tot}(L^*)/G$ is smooth. Then, for every $m \in \mathbb{Z}$, we have
\begin{equation}
\sum_{j=0}^{n} (-1)^j \dim H^j(M/G, L^n/G)
= \sum_{s=1}^{p} e^{2\pi i \frac{(s-1)m}{p}} \frac{1}{2\pi} \int_X \text{Td}_b(T^{1,0} X) \wedge e^{-m \frac{d\omega}{2\pi}} \wedge \omega_0.
\end{equation}

From Theorem 1.11 and (1.18), we establish Grauert-Riemenschneider criterion for orbifold line bundles

**Theorem 1.18.** With the notations and assumptions used above, assume that there is a $G$-invariant Hermitian fiber metric $h^{L}$ on $L$ such that the associated curvature $R^{L}$ is semi-positive and positive at some point of $M$. Then, $\dim H^0(M/G, L^p_m/G) = O(m^n)$, for $m \gg 1$.

In Section 1.5, we construct an orbifold holomorphic line bundle over a singular complex orbifold such that the assumptions of Theorem 1.17 and Theorem 1.18 are fulfilled (see the discussion after Corollary 1.20).
1.5. Examples. In this section, some examples of CR manifolds with locally free $S^1$ action are collected.

We first review the construction of generalized Hopf manifolds introduced by Brieskorn and Van de Ven [4].

Let $a = (a_1, \ldots, a_{n+2}) \in \mathbb{N}^{n+2}$, let $z = (z_1, \ldots, z_{n+2})$ be the standard coordinates of $\mathbb{C}^{n+2}$ and let $M(a)$ be the affine algebraic variety given by the equation

$$\sum_{j=1}^{n+2} z_j^{a_j} = 0.$$ 

If some $a_j = 1$, the variety $M(a)$ is non-singular. Otherwise $M(a)$ has exactly one singular point, namely $0 = (0, \ldots, 0)$. Put $\widetilde{M(a)} := M(a) - \{0\}$. Now we define a holomorphic $\mathbb{C}$-action on $\widetilde{M(a)}$ by

$$t \circ (z_1, \ldots, z_{n+2}) = (e^{i\pi t} z_1, \ldots, e^{i\pi t} z_{n+2}), \quad t \in \mathbb{C}, \quad (z_1, \ldots, z_{n+2}) \in \widetilde{M(a)}.$$ 

It is easy to see that the $\mathbb{Z}$-action on $\widetilde{M(a)}$ is globally free. The equivalence class of $(z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2}$ with respect to the $\mathbb{Z}$-action is denoted by $(z_1, \ldots, z_{n+2}) + \mathbb{Z}$ and hence

$$H(a) := \widetilde{M(a)}/\mathbb{Z} = \{ (z_1, \ldots, z_{n+2}) + \mathbb{Z}; (z_1, \ldots, z_{n+2}) \in \widetilde{M(a)} \}$$

is a compact complex manifold of complex dimension $n+1$. We call $H(a)$ a Hopf manifold. Let $\Gamma_a$ be the discrete subgroup of $\mathbb{C}$, generated by 1 and $2\pi \alpha i$, where $\alpha$ is the least common multiple of $a_1, a_2, \ldots, a_{n+2}$. Consider the complex 1-torus $T_a = \mathbb{C}/\Gamma_a$. $H(a)$ admits a natural $T_a$-action. Put $V(a) := H(a)/T_a$. By Holmann [18], $V(a)$ is a complex orbifold. Let $\pi_a : H(a) \to V(a)$ be the natural projection. For $m_1, m_2, \ldots, m_d \in \mathbb{N}$, let $[m_1, \ldots, m_d]$ denote the least common multiple of $m_1, \ldots, m_d$. The following is well-known (see the discussion before Proposition 4 in [4])

**Theorem 1.19.** Let $p = (z_1, \ldots, z_{n+2}) + \mathbb{Z} \in H(a)$. Assume that there are exactly $k$ coordinates $z_{j_1}, \ldots, z_{j_k}$ all different from zero, $k \geq 2$. Then, $V(a)$ is non-singular at $\pi_a(p)$ if and only if

$$\frac{[a_1, \ldots, a_{n+2}]}{[a_{j_1}, \ldots, a_{j_k}]} = \prod_{\ell \not\in \{j_1, \ldots, j_k\}} \frac{[a_1, \ldots, a_{n+2}]}{[a_1, \ldots, a_{j_1-1}, a_{j_1+1}, \ldots, a_{n+2}]}.$$ 

From Theorem 1.19, we deduce

**Corollary 1.20.** Assume $n \geq 2$ and $(a_1, a_2, \ldots, a_{n+2}) = (4b_1, 4b_2, 2b_3, 2b_4, \ldots, 2b_{n+2})$, where $b_j \in \mathbb{Z}$, $b_j$ is odd, $j = 1, \ldots, n+2$. Let $p = (0, 0, 1, i, 0, 0, \ldots, 0) + \mathbb{Z} \in H(a)$. Then, $V(a)$ is singular at $\pi_a(p)$.

Put

$$X := \{(z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2}; \quad z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+2}^{a_{n+2}} = 0, \quad |z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \cdots + |z_{n+2}|^{2a_{n+2}} = 1 \}.$$ 

It is easy to see that $X$ is a compact weakly pseudoconvex CR manifold of dimension $2n+1$ with CR structure $T^{1,0}X := T^{1,0} \mathbb{C}^{n+2} \cap CTX$, where $T^{1,0} \mathbb{C}^{n+2}$ denotes the standard complex structure on $\mathbb{C}^{n+2}$. Let $\alpha$ be the least common multiple of $a_1, \ldots, a_{n+2}$. Consider the following $S^1$ action on $X$:

$$S^1 \times X \to X,$$

$$(1.20) \quad e^{-i \theta} \circ (z_1, \ldots, z_{n+2}) \to (e^{-i \frac{\alpha \theta}{a_1}} z_1, \ldots, e^{-i \frac{\alpha \theta}{a_{n+2}}} z_{n+2}).$$

It is not difficult to check that the $S^1$ action is well-defined, locally free, CR and transversal. Moreover, it is straightforward to see that the quotient $X/S^1$ is equal to $V(a)$. Hence, $X/S^1$ is a complex orbifold. Moreover, from Corollary 1.20, we see that $X/S^1$ is singular if $n \geq 2$ and $(a_1, \ldots, a_{n+2}) = (4b_1, 4b_2, 2b_3, 2b_4, \ldots, 2b_{n+2})$, where $b_j \in \mathbb{Z}$, $b_j$ is odd, $j = 1, 2, \ldots, n+2$. We now show that $(X, T^{1,0}X)$
is CR-isomorphic to a circle bundle associated with an orbifold line bundle over \(X/S^1 = V(a)\). Let 
\[ L = (M(a) \times \mathbb{C})/\equiv, \]
where \((z_1, \ldots, z_{n+2}, \lambda) \equiv (\tilde{z}_1, \ldots, \tilde{z}_{n+2}, \tilde{\lambda})\) if
\[ \tilde{z}_j = e^{\frac{m}{n}} z_j, \quad j = 1, \ldots, n + 2, \]
\[ \tilde{\lambda} = e^{m \lambda}, \]
where \(m \in \mathbb{Z}\). We can check that \(\equiv\) is an equivalence relation and \(L\) is a holomorphic line bundle over \(H(a)\). The equivalence class of \((z_1, \ldots, z_{n+2}, \lambda) \in M(a) \times \mathbb{C}\) is denoted by \([(z_1, \ldots, z_{n+2}, \lambda)]\). The complex 1-torus \(T_a\) action on \(L\) is given by the following:
\[ T_a \times L \to L, \]
\[ (t + i\theta) \circ [(z_1, \ldots, z_{n+2}, \lambda)] \to [(e^{\frac{t+i\theta}{n+2}} z_1, \ldots, e^{\frac{t+i\theta}{n+2}} z_{n+2}, e^{t-i\pi \lambda})], \tag{1.22} \]
where \(\alpha\) is the least common multiple of \(a_1, \ldots, a_{n+2}\). It is easy to see that the torus action (1.22) is well-defined and \(L/T_a\) is an orbifold line bundle over \(H(a)/T_a = V(a)\). Let \(\tau : L \to L/T_a\) be the natural projection and for \([(z_1, \ldots, z_{n+2}, \lambda)] \in L\), we write \(\tau([(z_1, \ldots, z_{n+2}, \lambda)]) = [z_1, \ldots, z_{n+2}, \lambda] + T_a\). It is easy to check that the pointwise norm
\[ \|[(z_1, \ldots, z_{n+2}, \lambda)] + T_a\|_{h^{L/T_a}}^2 := |\lambda|^2 \left( |z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \cdots + |z_{n+2}|^{2a_{n+2}} \right)^{-1} \]
is well-defined as a Hermitian fiber metric on \(L/T_a\). The circle bundle \(C(L/T_a)\) with respect to \((L/T_a, h^{L/T_a})\) is given by
\[ C(L/T_a) := \left\{ v \in L/T_a; \ |v|^2_{h^{L/T_a}} = 1 \right\} \]
\[ = \left\{ [(z_1, \ldots, z_{n+2}, \lambda)] + T_a; \ |\lambda|^2 = |z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \cdots + |z_{n+2}|^{2a_{n+2}} \right\}. \tag{1.23} \]
It is easy to see that \(C(L/T_a)\) is a smooth CR manifold with CR structure
\[ T^{1,0}C(L/T_a) := T^{1,0}L/T_a \cap \mathbb{C}TC(L/T_a), \]
where \(T^{1,0}C(L/T_a)\) denotes the complex structure on \(L/T_a\). Moreover, the orbifold line bundle \(L/T_a \to V(a)\) satisfies the assumptions in Theorem 1.17 and Theorem 1.18.

The circle bundle \(C(L/T_a)\) admits a natural \(S^1\) action:
\[ e^{-i\theta} \circ ([z_1, \ldots, z_{n+2}, \lambda] + T_a) = [(z_1, \ldots, z_{n+2}, e^{-i\theta} \lambda)] + T_a, \quad [(z_1, \ldots, z_{n+2}, \lambda)] + T_a \in C(L/T_a). \]
Let \(\Phi : C(L/T_a) \to X\) be the smooth map defined as follows. For every \([(z_1, \ldots, z_{n+2}, \lambda)] + T_a \in C(L/T_a)\), there is a unique \((\hat{z}_1, \ldots, \hat{z}_{n+2}) \in X\) such that
\[ [(z_1, \ldots, z_{n+2}, \lambda)] + T_a = [(\hat{z}_1, \ldots, \hat{z}_{n+2}, 1)] + T_a. \]
Then, \(\Phi([(z_1, \ldots, z_{n+2}, \lambda)] + T_a) := (\hat{z}_1, \ldots, \hat{z}_{n+2}) \in X\). It is straightforward to check that \(\Phi\) is a CR embedding, globally one to one, onto, and the inverse \(\Phi^{-1} : X \to C(L/T_a)\) is also a CR embedding. Moreover, we have
\[ e^{-i\theta} \circ \Phi(x) = \Phi(e^{-i\theta} \circ x), \quad \forall x \in C(L/T_a). \]
We now consider another example. For simplicity, from now on, we assume that \(a_1 = 1\). Then,
\[ M(a) = \{(z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2}; \quad z_1 = -z_2^{a_2} - \cdots - z_{n+2}^{a_{n+2}} \}. \]
Fix \(q = 2, 3, \ldots, n + 2\). Put
\[ X_q := \{(z_1, \ldots, z_{n+2}) + Z \in H(a); \quad -|z_2|^{2a_2} - \cdots - |z_q|^{2a_q} - |z_{q+1}|^{2a_{q+1}} + \cdots + |z_{n+2}|^{2a_{n+2}} = 0 \}. \tag{1.24} \]
It is not difficult to check that $X_q$ is a compact CR manifold of dimension $2n + 1$ with CR structure $T^{1,0}X_q := T^{1,0}H(a) \cap CTX_q$, where $T^{1,0}H(a)$ denotes the natural complex structure on $H(a)$. Let $\tilde{a}$ be the least common multiple of $a_1, \ldots, a_q$. Consider the following $S^1$ action on $X_q$:

$$S^1 \times X_q \to X_q,$$

$$(1.25) \quad e^{-i\theta} \circ ((z_1, \ldots, z_{n+2} + \mathbb{Z}))$$

$$\to (e^{-i\tilde{a}\theta}(-z_2^{a_2} - \ldots - z_q^{a_q}) - z_{q+1}^{a_q+1} - \ldots - z_{n+2}^{a_{n+2}}, e^{-\frac{i\tilde{a}}{n} \theta} z_2, \ldots, e^{-\frac{i\tilde{a}}{n} \theta} z_q, z_{q+1}, \ldots, z_{n+2}) + \mathbb{Z}.$$ 

It is straightforward to check that the $S^1$ action is well-defined, locally free, CR and transversal. Moreover, the CR manifold $X_q$ is not pseudoconvex. 

Similarly, put

$$\tilde{X}_q := \{(z_1, \ldots, z_{n+2}) + \mathbb{Z} \in H(a) : -|z_2^{a_2} + z_3^{a_3}|^2 - |z_3^{a_3}|^{2a_3} - \ldots - |z_q|^{2a_q}$$

$$|z_{q+1}|^{2a_{q+1}} + \ldots + |z_{n+2}|^{2a_{n+2}} = 0\}.$$ 

It is not difficult to check that $\tilde{X}_q$ is a compact CR manifold of dimension $2n + 1$ with CR structure $T^{1,0}\tilde{X}_q := T^{1,0}H(a) \cap CT\tilde{X}_q$. Let $\tilde{a}$ be the least common multiple of $a_1, \ldots, a_q$. Consider the following $S^1$ action on $\tilde{X}_q$:

$$S^1 \times \tilde{X}_q \to \tilde{X}_q,$$

$$(1.27) \quad e^{-i\theta} \circ ((z_1, \ldots, z_{n+2} + \mathbb{Z}))$$

$$\to (e^{-i\tilde{a}\theta}(-z_2^{a_2} - \ldots - z_q^{a_q}) - z_{q+1}^{a_q+1} - \ldots - z_{n+2}^{a_{n+2}}, e^{-\frac{i\tilde{a}}{n} \theta} z_2, \ldots, e^{-\frac{i\tilde{a}}{n} \theta} z_q, z_{q+1}, \ldots, z_{n+2}) + \mathbb{Z}.$$ 

It is straightforward to check that the $S^1$ action is well-defined, locally free, CR and transversal. Let $f(x) \in C^\infty(\tilde{X}_q)$ with $Tf = 0$, where $T$ is the global real vector field induced by the $S^1$ action (1.27). Let $Z_1, \ldots, Z_n \in C^\infty(\tilde{X}_q, T^{1,0}\tilde{X}_q)$ be a basis for $T^{1,0}\tilde{X}_q$. Put

$$H^{1,0}\tilde{X}_q := \{Z_j + Z_j(f)T; j = 1, 2, \ldots, n\}.$$ 

We can check that $H^{1,0}\tilde{X}_q$ is a CR structure and the $S^1$ action (1.27) is locally free, CR and transversal with respect to this new CR structure $H^{1,0}\tilde{X}_q$.

1.6. **Proof of Theorem 1.2.** In this section, we will prove Theorem 1.2 and we will use the same notations as in Theorem 1.2. Let $s$ be a local trivializing section of $L$ defined on some open set $U$ of $M$, $|s|^2_{\tilde{H}_L} = e^{-2\phi}$. Let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates on $U$. We identify $U$ with an open set of $\mathbb{C}^n$. We have the local diffeomorphism:

$$\tau : U \times ]-\varepsilon_0, \varepsilon_0[ \to X, (z, \theta) \mapsto e^{-\phi(z)}s^\ast(z)e^{-i\theta}, 0 < \varepsilon_0 \leq \pi.$$ 

Put $D = U \times ]-\varepsilon_0, \varepsilon_0[$. We will identify $D$ with some open set in $X$ and we call $D$ a canonical coordinate patch with respect to local trivializing section $s$ and $(z, \theta)$ canonical coordinates of $D$. It is convenient to work with the local coordinates $(z, \theta)$. In terms of these coordinates, it is straightforward to see that $T = \frac{\partial}{\partial \theta}$. Recall that $T$ is the global real vector field induced by the $S^1$ action. Moreover, it is straightforward to check that on $(z, \theta)$ coordinates, we have

$$(1.28) \quad T^{1,0} = \left\{ \frac{\partial}{\partial z_j} - i \frac{\partial \phi}{\partial z_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \ldots, n \right\},$$

$$(1.29) \quad T^{0,1} = \left\{ \frac{\partial}{\partial \bar{z}_j} + i \frac{\partial \phi}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \ldots, n \right\},$$

and

$$(1.30) \quad T^{1,0} = \{dz_j; j = 1, 2, \ldots, n\}, \quad T^{0,1} = \{d\bar{z}_j; j = 1, 2, \ldots, n\}.$$
Let $f(z) \in \Omega^0 q(D)$. From (1.30), we may identify $f$ with an element in $\Omega^q_\theta(U)$. Let $A^{(q)}_m : \Omega^q_\theta(M, L^m) \rightarrow \Omega^q_\theta(M, L^m)$ be the map defined as follows. Let $u \in \Omega^q_\theta(X)$, $A^{(q)}_m u$ is an element in $\Omega^q_\theta(M, L^m)$ such that on a canonical coordinate patch $D = U \times \mathbb{C}^n \times \mathbb{R}$ with respect to local trivializing section $s$, if we write $u(z, \theta) = e^{-im\theta} u(z)$ on $D$, where $u(z) \in \Omega^q_\theta(U)$, then on $U$, $A^{(q)}_m u = s^m(z) e^{m\phi(z)} u(z) \in \Omega^q_\theta(U, L^m)$. We now check that $A^{(q)}_m$ is well-defined. Let $s$ and $s_1$ be local trivializing sections of $L$ on an open set $U$. Let $(z, \theta) \in \mathbb{C}^n \times \mathbb{R}$ and $(z, \eta) \in \mathbb{C}^n \times \mathbb{R}$ be canonical coordinates of $D$ with respect to $s$ and $s_1$ respectively, where $D = U \times \mathbb{C}^n \times \mathbb{R}$, $0 < \varepsilon_0, \varepsilon_1 < \pi$. Fix $u \in \Omega^q_\theta(U)$. Set $|s|^2_{h^L} = e^{-2\phi}$ and $|s_1|^2_{h^L} = e^{-2\phi_1}$. We write

$$u = e^{-im\theta} u(z) \text{ on } D,$$

$$u = e^{-im\eta} u_1(z) \text{ on } D.$$

From the construction of $A^{(q)}_m$, to check that $A^{(q)}_m$ is well-defined, we need to check that

$$s^m(z) e^{m\phi(z)} u(z) = s_1^m(z) e^{m\phi_1(z)} u_1(z), \quad \forall z \in U.$$

Let $s_1 = gs$, where $g \in C^\infty(U)$ is a non-zero holomorphic function. We have

$$|s_1|^2_{h^L} = e^{-2\phi_1} = |g|^2 |s|^2_{h^L} = e^{2\log |g| - 2\phi}.$$

Thus,

$$|s|^2_{h^L} = e^{-2\phi_1}.$$

We claim that

$$\phi_1 = \phi - \log |g|.$$

If $(z, \theta) = (z, \eta)$, then $e^{-i\theta} (\frac{g(z)}{g(z)}) \frac{1}{2} = e^{-i\eta}$.

**Proof of the claim.** From (1.32), we have

$$s^*(z) e^{-i\theta - \phi(z)} = s_1^*(z) g(z) e^{-i\eta - \phi(z)}$$

$$= s_1^*(z) g(z) e^{-i\phi_1(z) - \log |g(z)|}$$

$$= s_1^*(z) \left(\frac{g(z)}{g(z)}\right)^{\frac{1}{2}} e^{-i\phi_1(z)}.$$

Since $(z, \theta) = (z, \eta)$, we have

$$s^*(z) e^{-i\theta - \phi(z)} = s_1^*(z) e^{-i\eta - \phi_1(z)}.$$

From (1.34) and (1.35), we deduce that $e^{-i\theta} (\frac{g(z)}{g(z)}) \frac{1}{2} = e^{-i\eta}$. The claim follows.

From (1.33), we have

$$e^{-im\theta} u(z) = e^{-im\eta} \left(\frac{g(z)}{g(z)}\right)^{-m} u(z) = e^{-im\eta} u_1(z).$$

Thus,

$$\left(\frac{g(z)}{g(z)}\right)^{-m} u(z) = u_1(z).$$

From (1.32) and (1.36), we have

$$s^m(z) e^{m\phi(z)} u(z) = s_1^m(z) g^{-m}(z) e^{m\phi_1(z) + m \log |g(z)|} u(z)$$

$$= s_1^m(z) \left(\frac{g(z)}{g(z)}\right)^{m} e^{m\phi_1(z)} u(z)$$

$$= s_1^m(z) e^{m\phi_1(z)} u_1(z).$$

ACTION 13
Hence, (1.31) holds and $A_m^{(q)} : \Omega_m^{0,q}(X) \to \Omega^0(M, L^m)$ is well-defined. Moreover, it is easy to check that $A_m^{(q)} : \Omega_m^{0,q}(X) \to \Omega^0(M, L^m)$ is a bijection. We omit the detail.

We now prove that $\overline{\partial}A_m^{(q)} = A_m^{(q+1)}\overline{\partial}_b$. Let $D = U \times ] - \varepsilon_0, \varepsilon_0[ \setminus 0 < \varepsilon_0 < \pi$, be a canonical coordinate patch with respect to local trivializing section $s$ and let $(z, \theta)$ be canonical coordinates of $D$. Let $u \in \Omega_m^{0,q}(X)$. On $D$, we write $u(z, \theta) = e^{-im\theta} \hat{u}(z)$, where $\hat{u}(z) \in \Omega^0(U)$. In view of (1.29) and (1.30), we can check that on $D$,

$$
(1.38) \quad \overline{\partial}_b u = \overline{\partial}_b(e^{-im\theta} \hat{u}) = \sum_{j=1}^n e^{-im\theta} d\bar{z}_j(\frac{\partial \hat{u}}{\partial \bar{z}_j}(z) + m \frac{\partial \phi}{\partial \bar{z}_j}(z) \hat{u}(z)).
$$

From (1.38) and by the definition of $A_m^{(q+1)}$, it is easy to see that

$$
(1.39) \quad A_m^{(q+1)}(\overline{\partial}_b u) = s_m(z) e^{m\phi(z)} \sum_{j=1}^n d\bar{z}_j(\frac{\partial \hat{u}}{\partial \bar{z}_j}(z) + m \frac{\partial \phi}{\partial \bar{z}_j}(z) \hat{u}(z)) = s_m(z) \overline{\partial}(e^{m\phi(z)} \hat{u}(z)) \text{ on } U.
$$

Thus, $\overline{\partial}A_m^{(q)} = A_m^{(q+1)}\overline{\partial}_b$. Theorem 1.2 follows.

1.7. The idea of the proof of Theorem 1.3. In this section, we will give an outline of the idea of the proof of Theorem 1.3. We refer to Section 2.2 and Section 2.3 for some notations and terminology used here. To prove Theorem 1.3, we need to construct heat kernel for Kohn Laplacian associated to $m$-th $S^1$ Fourier coefficient.

1.7.1. Global difficulties. For simplicity, we assume that $E$ is trivial and $X$ is CR Kähler, that is, there is a closed form $\Theta \in C^\infty(X, T^{1,1}X)$ such that $\Theta(Z, \overline{Z}) > 0$, for all $Z \in C^\infty(X, T^{1,0}X)$. $\Theta$ induces a Hermitian metric $(\cdot | \cdot)$ on $\mathbb{T}X$. Let $\overline{\partial}_b$ be the adjoint of $\partial_b$ with respect to $(\cdot | \cdot)$. We fix $m \in \mathbb{Z}$. Then, $\overline{\partial}_b^* : \Omega_m^0(X) \to \Omega_m^0(X)$. Consider

$$
D_{b,m} := \overline{\partial}_b^* + \overline{\partial}_b^* : \Omega_m^{0,-}(X) \to \Omega_m^{0,+}(X),
$$

$$
D_{b,m} := \overline{\partial}_b^* + \overline{\partial}_b^* : \Omega_m^{0,+}(X) \to \Omega_m^{0,-}(X),
$$

and let

$$
\Box_{b,m}^+ := D_{b,m}^2 : \Omega_m^{0,+}(X) \to \Omega_m^{0,+}(X),
$$

$$
\Box_{b,m}^- := D_{b,m}^2 : \Omega_m^{0,-}(X) \to \Omega_m^{0,-}(X),
$$

where $\Omega_m^{0,+}(X)$ and $\Omega_m^{0,-}(X)$ are given by (3.21). Let $L_n^{2,+}(X)$, $L_n^{2,-}(X)$, $L_n^{2,+}(X)$ and $L_n^{2,-}(X)$ be as in the discussion after (3.22). We extend $\Box_{b,m}^+$ and $\Box_{b,m}^-$ to $L_n^{2,+}(X)$ and $L_n^{2,-}(X)$ in the standard way and we will show in Theorem 3.5 that $\text{Spec } \Box_{b,m}^+$ and $\text{Spec } \Box_{b,m}^-$ are discrete subsets of $[0, \infty]$ and for every $\nu \in \text{Spec } \Box_{b,m}^+$, $\mu \in \text{Spec } \Box_{b,m}^-$, $\nu$ and $\mu$ are eigenvalues of $\Box_{b,m}^+$ and $\Box_{b,m}^-$ respectively. For every $\nu \in \text{Spec } \Box_{b,m}^+$, let \{ $f_{1\nu}^+, \ldots, f_{d\nu}^+$ $\}$ be an orthonormal frame for the eigenspace of $\Box_{b,m}^+$ with eigenvalue $\nu$. The heat kernel $e^{-\Box_{b,m}^+}$ is given by

$$
(1.40) \quad e^{-\Box_{b,m}^+}(x, y) = \sum_{\nu \in \text{Spec } \Box_{b,m}^+} d\nu \sum_{j=1}^{d\nu} e^{-\nu t} f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger,
$$

where $f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger$ denotes the linear map:

$$
\begin{align*}
&f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger : \oplus_{q \in \{0,1,2,\ldots \}} T_y \to T_x, \quad e^{0,0,q} X \to T_x^{0,0,q} X, \\
&u(y) \oplus_{q \in \{0,1,2,\ldots \}} T_y \to f_j^\nu(x) (u(y) \mid f_j^\nu(y)) \in T_x^{0,0,q} X.
\end{align*}
$$
Similarly, we can define \( e^{-t\Box_{b,m}^+}(x, y) \) as (1.40). We will show in Theorem 4.10 (see also Remark 4.11) that we have CR Mckean-Singer type formula: for every \( t > 0 \),

\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X) = \int_X \left( \text{Tr} e^{-t\Box_{b,m}^+}(x, x) - \text{Tr} e^{-t\Box_{b,m}^+}(x, x) \right) d\nu_X.
\]

By using CR Mckean-Singer type formula, the proof of Theorem 1.3 is reduced to determine the small \( t \) behaviour of the function \( \left( \text{Tr} e^{-t\Box_{b,m}^+}(x, x) - \text{Tr} e^{-t\Box_{b,m}^+}(x, x) \right) \). Let \( e^{-t\Box_{b,m}^+} : \Omega^{0,+}(X) \rightarrow \Omega_0^{0,+}(X) \) be the operator with distribution kernel \( e^{-t\Box_{b,m}^+}(x, y) \). From (1.40), it is easy to see that \( e^{-t\Box_{b,m}^+} \) satisfies

\[
\frac{\partial e^{-t\Box_{b,m}^+}}{\partial t} + \Box_{b,m} e^{-t\Box_{b,m}^+} = 0
\]

and

\[
e^{-t\Box_{b,m}^+}|_{t=0} = Q_m^+,
\]

where \( Q_m^+ : L^{2,+}(X) \rightarrow L_0^{2,+}(X) \) is the orthogonal projection. The main difficulty is that the initial condition (1.43) is a projection rather than an identity operator. This is indeed a new point beyond the tradition, arising from the situation that we deal with part of the \( L^2 \) space (eigenspaces) rather than the whole \( L^2 \) space. This initial condition (1.43) causes that the heat kernels \( e^{-t\Box_{b,m}^+}(x, y) \) and \( e^{-t\Box_{b,m}^+}(x, y) \) do not have standard expansions with respect to \( t \). For the better understanding, let’s assume that \( X \) is an orbifold circle bundle of an orbifold line bundle \( L \) over a Kähler orbifold \( M \). Then, \( X \) is a CR manifold with a locally free \( S^1 \) action. We can repeat the proof of Theorem 1.2 (see Section 1.6) and conclude that there is a bijection map

\[
A_m^+ : \Omega_0^{0,+}(X) \rightarrow \Omega_0^{0,+}(M, L^m),
\]

\[
A_m^+ : \Omega_0^{0,-}(X) \rightarrow \Omega^{0,-}(M, L^m),
\]

such that \( A_m^+ \Box_b = \Box A_m^+ \). Let \( \Box_{b,m}^+ \) be the Kodaira Laplacian with values in \( T^{*0,+}M \otimes L^m \) and let \( e^{-t\Box_{b,m}^+} \) be the associated heat operator. Consider \( B_m(t) := (A_m^+)^{-1} \circ e^{-t\Box_{b,m}^+} \circ A_m^+ \). We can check that

\[
B_m(t) + \Box_{b,m} B_m(t) = 0 \quad \text{and} \quad B_m(0) = I \quad \text{on} \quad \Omega_0^{0,+}(X).
\]

But \( B_m(t) \) is not the heat operator \( e^{-t\Box_{b,m}^+} \). A trivial understanding is that, they are defined on different spaces, as \( B_m(t) \) is on \( \Omega_0^{0,+}(X) \) while the heat kernel \( e^{-t\Box_{b,m}^+} \) is on the whole \( \Omega^{0,+}(X) \). Moreover, it is straightforward to check that

\[
e^{-t\Box_{b,m}^+} = B_m(t) \circ Q_m^+ = Q_m^+ \circ B_m(t) \circ Q_m^+.
\]

Let \( B_m(t, x, y) \) and \( e^{-t\Box_{b,m}^+}(x, y) \) be the distribution kernels of \( B_m(t) \) and \( e^{-t\Box_{b,m}^+} \) respectively. From (1.44), we can check that (see (2.5) and (4.26))

\[
e^{-t\Box_{b,m}^+}(x, y) = \frac{1}{(2\pi)^2} \int B_m(t, e^{-i\theta} \circ x, e^{-iu} \circ y) e^{i(m\theta - u)} d\theta du.
\]

For \( x \notin X_p \), from (1.45), it is difficult to understand the asymptotic behavior of \( e^{-t\Box_{b,m}^+}(x, x) \) from the asymptotic expansion of \( B_m(t) \). For \( x \in X_p \), from (1.45), \( e^{-t\Box_{b,m}^+}(x, x) \) admits a standard expansion as \( t \to 0^+ \):

\[
e^{-t\Box_{b,m}^+}(x, x) \sim t^{-n} a_n^+(x) + t^{-(n-1)} a_{n-1}^+(x) + \cdots.
\]

Another problem is that the expansion (1.46) converges only locally uniformly on \( X_p \). It seems to us that even in orbifold circle bundle cases, to understand the asymptotic behavior of the heat operator \( e^{-t\Box_{b,m}^+} \), we still need to work directly on CR manifold \( X \). In this paper, we give a construction which is independent of the use of orbifold geometry and more adapted to CR geometry. Our choice of cut-off functions for the patching of local heat kernels is adapted to BRT trivializations for our CR manifolds. By this construction it becomes natural that our computation of local density need not
introduce orbifold contributions. It does not seem obvious how Kawasaki’s result can bypass the orbifold contributions in the case where the complex orbifold has a circle bundle whose total space is globally smooth, unless the complex orbifold is itself smooth.

In view of global difficulties above, it is not immediate that solely using the global argument can arrive at a detailed understanding of the heat kernel. Moreover, our CR manifold $X$ is not necessary to be an orbifold circle bundle of a complex orbifold. We thus turn our attention to work on it locally. The main ingredient here is BRT trivialization, which is first treated by Baouendi, Rothschild and Treves [1] in a more general context.

1.7.2. Transition to local situation. Let $B := (D, (z, \theta, \varphi))$ be a BRT trivialization (see Theorem 2.8). We may assume that $D = U \times [0, \varepsilon) \subset \mathbb{C}^n$, where $\varepsilon > 0$ and $U$ is an open set of $\mathbb{C}^n$. Consider $L \to U$ be a trivial line bundle with non-trivial Hermitian fiber metric $|1|_{h^L} = e^{-\varphi}$, where $\varphi \in C^\infty(D, \mathbb{R})$ is as in Theorem 2.8. Let $(L^m, h^L_m) \to U$ be the $m$-th power of $(L, h^L)$. $\Theta$ induces a Kähler form $\Theta_U$ on the complex manifold $U$ and let $(\cdot, \cdot)_m$ be the Hermitian metric on $\Theta_U$ induced by $\Theta_U$ and let $(\cdot, \cdot)_m$ be the $L^2$ inner product on $\Omega^{0,*}(U, L^m)$ induced by $(\cdot, \cdot)$ and $h^L_m$. Let

\[ D : \Omega^{0,q}(U, L^m) \to \Omega^{0,q+1}(U, L^m), \quad q = 0, 1, 2, \ldots, n-1, \]

be the Cauchy-Riemann operator and let

\[ \overline{\partial} : \Omega^{0,q}(U, L^m) \to \Omega^{0,q}(U, L^m), \quad q = 0, 1, 2, \ldots, n-1, \]

be the formal adjoint of $D$ with respect to $(\cdot, \cdot)_m$. Put

\[
D_{B,m} := \overline{\partial} + \overline{\partial}^* m : \Omega^{0,+}(U, L^m) \to \Omega^{0,-}(U, L^m),
\]

\[
D_{B,m} := \overline{\partial} + \overline{\partial}^* m : \Omega^{0,-}(U, L^m) \to \Omega^{0,+}(U, L^m).
\]

Let $u \in \Omega^{0,+}_m(X)$, $v \in \Omega^{0,-}_m(X)$. On $D$, we write $u(z, \theta) = e^{-im\theta} \tilde{u}(z)$, $v(z, \theta) = e^{-im\theta} \tilde{v}(z)$, $\tilde{u}(z) \in \Omega^{0,+}(U, L^m)$, $\tilde{v}(z) \in \Omega^{0,-}(U, L^m)$. Then, we will show in Lemma 5.1 that

\[
e^{-m\varphi} \overline{\partial}^*_m (e^{m\varphi} \tilde{u}) = e^{im\theta} \overline{\partial}^*_m(u),
\]

\[
e^{-m\varphi} \overline{\partial}_m (e^{m\varphi} \tilde{v}) = e^{im\theta} \overline{\partial}_m(v).
\]

From (1.49), we can guess that the heat kernel $e^{-\Box^+_m(x, y)}$ locally should be

\[
e^{-m\varphi(z) - im\theta} e^{-\Box^+_m(x, y)} e^{m\varphi(w) + im\eta}.
\]

By using the BRT trivializations we localize the construction and are inspired to obtain a localize heat kernel on these particular charts. We would like to patch up these local heat kernels. We will see that the standard construction for the heat kernel does not work in our case. Assume that $X = D_1 \cup D_2 \cup \cdots \cup D_N$, where $B_j := (D_j, (z, \theta, \varphi_j))$ is a BRT trivialization, for each $j$. We may assume that for each $j$, $D_j = U_j \times ]-\delta_j, \delta_j[ \subset \mathbb{C}^n \times \mathbb{R}$, $\delta_j > 0$, $\delta_j > 0$, $U_j$ is an open set in $\mathbb{C}^n$. Let $\chi_j, \bar{\chi}_j \in C^\infty_0(D_j)$, $j = 1, 2, \ldots, N$. Put

\[
A_m(t) = \sum_{j=1}^N \chi_j(x) (e^{-m\varphi_j(z) - im\theta} e^{-\Box^*_j(x, y)} e^{m\varphi_j(w) + im\eta}) \bar{\chi}_j(y),
\]

\[
\mathcal{P}_m(t) = Q^+_m \circ A_m(t) \circ Q^+_m.
\]

We hope that $\mathcal{P}_m(0) = Q^+_m$ and $\mathcal{P}'_m(t) + \Box^*_m \mathcal{P}_m(t)$ is small as $t \to 0^+$. 

1.7.3. Local difficulties. A necessary condition for \( P_m(0) = Q_m^+ \) is

\[
\sum_{j=1}^N \chi_j(x) \int \tilde{\chi}_j(z, \eta) d\eta = 1. 
\]

Unfortunately, it is unclear to see whether (1.52) can be enough to reach the desired heat kernel. A reasonable choice for the cut-off functions \( \chi_j \), \( \tilde{\chi}_j \) is the following: take \( \chi_j(z, \theta) \in C_0^\infty(D_j), j = 1, 2, \ldots, N, \) with \( \sum_{j=1}^N \chi_j = 1 \) on \( X \) and for each \( j = 1, 2, \ldots, N, \) take \( \tau_j(z) \in C_0^\infty(U_j) \) with \( \tau_j(z) = 1 \) if \( (z, \theta) \in \text{Supp} \chi_j \) and take \( \sigma_j \in C_0^\infty([-\delta_j, \tilde{\delta}_j]) \) with \( \int \sigma_j(\eta) d\eta = 1. \) Then, \( \chi_j(x), \tilde{\chi}_j(y) = \tau_j(w) \sigma_j(\eta), j = 1, 2, \ldots, N, \) satisfy (1.52). We can check that \( P_m(0) = Q_m^+ \) and

\[
P_m'(t) + \Box_{b,m}^+ P_m(t) = Q_m^+ \circ R_m(t) \circ Q_m^+, 
\]

where

\[
R_m(t) = \sum_{j=1}^N \sum_{\ell=1}^k L_{\ell,j} \left( \chi_j(x) e^{-m \varphi_j(z) - im\theta} \right) P_{\ell,j} \left( e^{-\Box_{b,m}^+} \right) e^{m \varphi_j(w) + im \eta} \tilde{\chi}_j(y),
\]

where \( L_{\ell,j} \) is a partial differential operator of order \( \leq 2 \) and \( \geq 1, \) for all \( \ell, j, \) \( P_{\ell,j} \) is a partial differential operator of order 1 acting on \( x, \) for all \( \ell, j. \) Since \( e^{-\Box_{b,m}^+} \) is self-adjoint, it could happen that

\[
P_{\ell,j} \left( e^{-\Box_{b,m}^+} \right) \sim e^{-\frac{|z-w|^2}{t}},
\]

From (1.55), we want \( P_m'(t) + \Box_{b,m}^+ P_m(t) \) to be small, we need

\[
L_{\ell,j} \left( \chi_j(x) e^{-m \varphi_j(z) - im\theta} \right) e^{m \varphi_j(w) + im \eta} \tilde{\chi}_j(y) = 0 \quad \text{if z is close to w} \quad |z - w| \lesssim \sqrt{t}.
\]

Since \( \chi_j \) is not constant on \( \text{Supp} \tilde{\chi}_j, \) it is difficult to see if (1.56) holds. To overcome the difficulty, instead of the original equation, we make use of an adjoint version of the original equation. It turns out that the aforementioned difficulty can be avoided, so that a global heat kernel can be obtained by the process of patching in this new context. Fix \( j = 1, 2, \ldots, N. \) There is \( A_{B_j,+}(t, z, w) \in C_0^\infty(\mathbb{R} \times U_j \times U_j, (T^0_\omega U \boxtimes T^0_\omega U)) \) such that

\[
\lim_{t \to 0^+} A_{B_j,+}(t) = I \quad \text{in} \quad \mathcal{D}'(U, T^{0,+} U),
\]

\[
A_{B_j,+}(t) u + A_{B_j,+}(t)(\Box_{B_j,m} u) = 0, \quad \forall u \in \Omega_0^{0,+}(U), \quad \forall t > 0,
\]

and \( A_{B_j,+}(t, z, w) \) admits an asymptotic expansion as \( t \to 0^+ \) (see (5.17)). Put

\[
H_j(t, x, y) = \chi_j(x) e^{-m \varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m \varphi_j(w) + im \eta} \tilde{\chi}_j(y).
\]

It should be noticed that there is no guarantee that \( H_j(t) \) still stay in the eigenspace \( \Omega_0^{0,+}(X). \) We thus set

\[
\Gamma(t) := \sum_{j=1}^N Q_m^+ \circ H_j(t) \circ Q_m^+: \Omega_0^{0,+}(X) \to \Omega_0^{0,+}(X).
\]

In Section 5.2, we will prove that \( \Gamma(t) \) is a successive approximation for the adjoint of \( e^{-\Box_{b,m}^+} \) (see Theorem 5.8) and since \( e^{-\Box_{b,m}^+} \) is self-adjoint, \( \Gamma(t) \) is a successive approximation for \( e^{-\Box_{b,m}^+}. \) More precisely, we have (see Theorem 5.10)

\[
\left\| e^{-\Box_{b,m}^+}(t, x, y) - \Gamma(t, x, y) \right\|_{C^0(X \times X)} \leq e^{-\frac{\gamma}{t}}, \quad \forall t \in (0, \epsilon),
\]

\[
\Gamma(t, x, y) \sim t^{-n} a_{n}^+(t, x, y) + t^{-n+1} a_{n-1}^+(t, x, y) + \cdots \quad \text{as} \ t \to 0^+,
\]
where \( \epsilon > 0, \epsilon_0 > 0 \) are constants and there is a constant \( C > 0 \) such that \( |a_+^j(t, x, y)| \leq C \), for every \( t > 0, (x, y) \in X \times X \) and every \( s = n, n - 1, \ldots \). Hence,

\[
(1.61) \quad e^{-\square} v_{b,m} \sim t^{-n} a_+^n(t, x, y) + t^{-n+1} a_{n-1}^+(t, x, y) + \cdots \text{ as } t \to 0^+.
\]

Similarly, we have

\[
(1.62) \quad e^{-\square} v_{b,m} \sim t^{-n} a_n^-(t, x, y) + t^{-n+1} a_{n-1}^-(t, x, y) + \cdots \text{ as } t \to 0^+.
\]

From (1.41), (1.61) and (1.62), we conclude that

\[
(1.63) \quad \sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, E) = \lim_{t \to 0^+} \int_X \sum_{\ell=0}^n t^{-\ell} \left( \text{Tr} a_\ell^+(t, x, x) - \text{Tr} a_\ell^-(t, x, x) \right) d\nu_X(x).
\]

It is indeed that we have already had a heat kernel which is put in the disguise of the spectral geometry (4.21). But we found no argument in the literature to claim that (4.21) shall have the asymptotic estimates as (1.61) and (1.62), which are essential for computation of the local density needed by our proof of the index theorem. The somewhat lengthy part of our reconstruction of the heat kernel beyond its spectral realization becomes indispensable as far as our approach is concerned.

1.7.4. Completion by evaluating of local density and by using Spin\(^c\) structure. From (1.63), to complete the proof of Theorem 1.3, we need to understand the small \( t \) behaviour of the local density \( \sum_{j=0}^n t^{-\ell} \left( \text{Tr} a_\ell^+(t, x, x) - \text{Tr} a_\ell^-(t, x, x) \right) \). Let's come back to local situation. Fix \( x_0 \in X_p \). Let \( B_j = (D_j(z, \theta, \varphi_j), j = 1, 2, \ldots, N \), be BRT trivializations as before. Fix \( j = 1, 2, \ldots, N \). Assume that \( e^{-i\theta_0} \circ x_0 \in D_j \) for some \( \theta_0 \in [-\pi, \pi] \) and suppose that \( e^{-i\theta_0} \circ x_0 = (z_j, 0) \in D_j \). By using some properties of BRT trivializations, our choice of cut-off functions and some tricks, we will show in Theorem 6.1 that

\[
(1.64) \quad (Q_m^{+} \circ H_j \circ Q_m^{+})(t, x_0, x_0) = \left( \sum_{s=1}^p e^{im\frac{2\pi}{p}(s-1)} \right) \left( \sum_{k=1}^p e^{-im\frac{2\pi}{p}(k-1)} \right) \int_{-\frac{\pi}{p}}^{\frac{\pi}{p}} \chi_j(e^{-i\theta} \circ x_0) A_{B_j,+}(t, z_j, z_j) d\theta,
\]

where \( A_{B_j,+}(t, z, w) \) is as in (1.57). Hence

\[
(1.65) \quad \sum_{\ell=0}^n t^{-\ell} \left( \text{Tr} a_\ell^+(t, x, x) - \text{Tr} a_\ell^-(t, x, x) \right) \]

\[
= \left( \sum_{s=1}^p e^{im\frac{2\pi}{p}(s-1)} \right) \left( \sum_{k=1}^p e^{-im\frac{2\pi}{p}(k-1)} \right) \]

\[
\times \sum_{j=1}^N \int_{-\frac{\pi}{p}}^{\frac{\pi}{p}} \chi_j(e^{-i\theta} \circ x_0) \left( \text{Tr} A_{B_j,+}(t, z_j, z_j) - \text{Tr} A_{B_j,-}(t, z_j, z_j) \right) d\theta + O(t).
\]

By using the rescaling technique in [2] and [8], we can show that for each \( j = 1, 2, \ldots, N \), we have

\[
(1.66) \quad \left( \text{Tr} A_{B_j,+}(t, z, z) - \text{Tr} A_{B_j,-}(t, z, z) \right) d\nu_{U_j}(z) = [\text{Td} (\nabla^{T^{1,0}U_j}, T^{1,0}U_j) \wedge \text{ch} (\nabla^{L^m}, L^m)]_{2n}(z) + O(t), \quad \forall z \in U_j,
\]

where \( d\nu_{U_j} \) is the induced volume form on \( U_j \), \( A_{B_j,+}(t, z, w) \) is as in (1.57), \( \text{Td} (\nabla^{T^{1,0}U_j}, T^{1,0}U_j) \) denotes the representative of the Todd class of \( T^{1,0}U_j \) induced by the given Hermitian metric on \( T^{1,0}U_j \) and \( \text{ch} (\nabla^{L^m}, L^m) \) denotes the representative of the Chern character induced by the Hermitian metric.
\( h^{1,0} \). In Section 2.3, we will introduce tangential characteristic classes, tangential Chern character and tangential Todd class on CR manifolds with \( S^1 \) action and we have

\[
(1.67) \quad \frac{\text{Td}(\nabla^{1,0}U_j, T^{1,0}U_j) \wedge \text{ch}(\nabla^{1,0}L^m)_{2n}(z_j)}{dv_{U_j}(z_j)} = \frac{[\text{Td}_b(\nabla^{1,0}X, T^{1,0}X) \wedge e^{-m}\omega_0]_{2n+1}(x_0)}{dv_X(x_0)},
\]

where \( \text{Td}_b(\nabla^{1,0}X, T^{1,0}X) \) denotes the representative of the tangential Todd class of \( T^{1,0}X \) induced by the given Hermitian metric (see (2.11)). From (1.65), (1.66) and (1.67), we deduce that

\[
\begin{aligned}
&\sum_{\ell=0}^n t^{-\ell} \left( \text{Tr} a_\ell^+(t, x, x) - \text{Tr} a_\ell^-(t, x, x) \right) dv_X(x) \\
&= \sum_{s=1}^p e^{2\pi i s/m} \left[ \text{Td}_b(\nabla^{1,0}X, T^{1,0}X) \wedge e^{-m}\omega_0 \right]_{2n+1}(x) + O(t), \quad \forall x \in X_p.
\end{aligned}
\]

By continuity argument, we see that (1.68) holds for all \( x \in X_p \). In Theorem 6.2, we will show that \( \overline{X}_p = X \) (recall that \( X \) is connected) and hence (1.68) holds for all \( x \in X \). From this observation, (1.68) and (1.63), we get Theorem 1.3 when \( X \) is CR Kähler.

When \( X \) is not CR Kähler, we still have (1.60), (1.61), (1.62) and (1.63). The problem is that the local operator \( \square_{B,m}^+ \) in (1.57) is not induced by Kähler form, it is difficult to understand small \( t \) behaviour of \( A_{B,+}(t, z, z) \) and (1.66) is not true in general. To overcome this difficulty, we introduce some kind of CR Spin\(^c\) Dirac operator:

\[
\tilde{D}_{b,m} = \overline{D}_b + b + \text{zero order term}
\]

on CR manifolds with \( S^1 \) action and modified Kohn Laplacians \( \tilde{\square}_{b,m}^+ = \tilde{D}_{b,m}^*\tilde{D}_{b,m}, \tilde{\square}_{b,m}^- = \tilde{D}_{b,m}\tilde{D}_{b,m}^* \).

We will show in Theorem 4.7 and Theorem 4.10 that we have homotopy invariance for the index of \( \overline{D}_b + b \) and Mckean-Singer formula for the modified Kohn Laplacians: for every \( t > 0 \),

\[
(1.69) \quad \sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, E) = \int_X \left( \text{Tr} e^{-\tilde{\square}_{b,m}^+(x, x)} - \text{Tr} e^{-\tilde{\square}_{b,m}^-(x, x)} \right) dv_X.
\]

Let \( B := (D, (z, \theta), \varphi) \) be a BRT trivialization. Let \( u \in \Omega^{0,+}_m(X), v \in \Omega^{0,-}_m(X) \). On \( D \), we write

\[
u(z, \theta) = e^{-im\theta}\tilde{u}(z), v(z, \theta) = e^{-im\theta}\tilde{v}(z), \quad \tilde{u}(z) \in \Omega^{0,+}(U, L^m), \quad \tilde{v}(z) \in \Omega^{0,-}(U, L^m).
\]

We will show in Lemma 5.1 that

\[
(1.70) \quad e^{-m\varphi}\tilde{\square}_{b,m}^+(e^{m\varphi}\tilde{u}, e^{m\varphi}\tilde{v}) = e^{im\theta}\tilde{\square}_{b,m}^+(\tilde{u}, \tilde{v}),
\]

\[
= e^{-m\varphi}\tilde{\square}_{b,m}^-(e^{m\varphi}\tilde{u}, e^{m\varphi}\tilde{v}) = e^{im\theta}\tilde{\square}_{b,m}^-(\tilde{u}, \tilde{v}),
\]

where \( \tilde{\square}_{b,m}^+ = D_{B,m}^*D_{B,m} : \Omega^{0,+}(U, L^m) \to \Omega^{0,+}(U, L^m), \tilde{\square}_{b,m}^- = D_{B,m}D_{B,m}^* : \Omega^{0,+}(U, L^m) \to \Omega^{0,+}(U, L^m) \), \( D_{B,m} : \Omega^{0,-}(U, L^m) \to \Omega^{0,-}(U, L^m) \) is the Spin\(^c\) Dirac operator with respect to the Chern connection on \( L^m \) induced by \( h^{1,0} \) and Clifford connection on \( \Lambda(T^{*,0,1}U) \) induced by the given Hermitian metric on \( A(T^{*,0,1}U) \) (see Definition 4.1).

Let \( B_j := (D_j, (z, \theta), \varphi_j) \), \( j = 1, 2, \ldots, N \) be BRT trivializations as before. Fix \( j = 1, 2, \ldots, N \). As (1.57), there is \( \tilde{A}_{B,j,+}(t, z, w) \in C^\infty(\mathbb{R} \times U_j \times U_j, (T_{w}^{*,0}U \boxtimes T_{z}^{*,0}U)) \) such that

\[
(1.71) \quad \lim_{t \to 0^+} \tilde{A}_{B,j,+}(t) = I \text{ in } \mathcal{D}'(U, T^{*,0,+}U),
\]

\[
\tilde{A}_{B,j,+}(t)u \to \tilde{A}_{B,j,+}(t)(\tilde{\square}_{B,j,m}u) = 0, \quad \forall u \in \Omega^{0,+}_0(U), \quad \forall t > 0.
\]
Hodge theory for Kohn Laplacian in the $L^2$-space of $m$-th $S^1$ Fourier coefficient. In Section 4, we introduce CR Spin$^c$ Dirac operator $\tilde{D}_{b,m}$, modified Kohn Laplacians $\tilde{\Delta}_{b,m}$, and we prove (1.69).

In Section 5, we construct approximate heat kernels for the operators $e^{-t\tilde{\Delta}_{b,m}^+}$, $e^{-t\tilde{\Delta}_{b,m}}$ and we prove that $e^{-t\tilde{\Delta}_{b,m}^+(x,y)}$, $e^{-t\tilde{\Delta}_{b,m}(x,y)}$ admit asymptotic expansions as (1.73). In Section 6, we prove (1.76) and we finish the proof of Theorem 1.3.

2. Preliminaries
2.1. Some standard notations. We use the following notations: \( \mathbb{N} = \{1, 2, \ldots\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mathbb{R} \) is the set of real numbers, \( \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\} \). For a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) we set \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). For \( x = (x_1, \ldots, x_n) \) we write
\[
x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial z_j = \frac{\partial}{\partial x_j}, \quad \partial z_\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_\alpha},
\]
\[
D_{x_j} = \frac{1}{i} \partial z_j, \quad D_\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D_x = \frac{1}{i} \partial z.
\]
Let \( z = (z_1, \ldots, z_n) \), \( z_j = x_{2j-1} + ix_{2j}, j = 1, \ldots, n \), be coordinates of \( \mathbb{C}^n \). We write
\[
z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \overline{z}^{\alpha} = \overline{z}_1^{\alpha_1} \cdots \overline{z}_n^{\alpha_n},
\]
\[
\partial z_j = \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right), \quad \partial z_j = \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}} \right),
\]
\[
\partial z^{\alpha} = \partial_{z_1}^{\alpha_1} \cdots \partial_z^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z}, \quad \partial \overline{z}^{\alpha} = \partial_{\overline{z}_1}^{\alpha_1} \cdots \partial_{\overline{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \overline{z}}.
\]
Let \( X \) be a \( C^\infty \) orientable paracompact manifold. We let \( TX \) and \( T^*X \) denote the tangent bundle of \( X \) and the cotangent bundle of \( X \) respectively. The complexified tangent bundle of \( X \) and the complexified cotangent bundle of \( X \) will be denoted by \( CTX \) and \( CT^*X \) respectively. We write \( \langle \cdot, \cdot \rangle \) to denote the pointwise duality between \( T^*X \) and \( TX \). We extend \( \langle \cdot, \cdot \rangle \) bilinearly to \( CT^*X \times CTX \). For \( u \in CT^*X, v \in CTX \), we also write \( u(v) := \langle u, v \rangle \).

Let \( E \) be a \( C^\infty \) vector bundle over \( X \). The fiber of \( E \) at \( x \in X \) will be denoted by \( E_x \). Let \( F \) be another vector bundle over \( X \). We write \( E \otimes F \) to denote the vector bundle over \( X \times X \) with fiber over \( (x, y) \in X \times X \) consisting of the linear maps from \( E_x \) to \( F_y \).

Let \( Y \subset X \) be an open set. The spaces of smooth sections of \( E \) over \( Y \) and distribution sections of \( E \) over \( Y \) will be denoted by \( C^\infty(Y, E) \) and \( \mathcal{D}(Y, E) \) respectively. Let \( \mathcal{D}'(Y, E) \) be the subspace of \( \mathcal{D}(Y, E) \) whose elements have compact support in \( Y \). For \( m \in \mathbb{R} \), we let \( H^m(Y, E) \) denote the Sobolev space of order \( m \) of sections of \( E \) over \( Y \). Put
\[
H^m_{\text{loc}}(Y, E) = \{ u \in \mathcal{D}'(Y, E); \varphi u \in H^m(Y, E), \varphi \in C^\infty_0(Y) \},
\]
\[
H^m_{\text{comp}}(Y, E) = H^m_{\text{loc}}(Y, E) \cap \mathcal{D}'(Y, E).
\]

2.2. Set up and terminology. Let \((X, T^{1,0}X)\) be a compact CR manifold of dimension \( 2n + 1, n \geq 1 \), where \( T^{1,0}X \) is a CR structure of \( X \). That is \( T^{1,0}X \) is a subbundle of rank \( n \) of the complexified tangent bundle \( CTX \), satisfying \( T^{1,0}X \cap T^{0,1}X = \{0\} \), where \( T^{0,1}X = \overline{T^{1,0}X} \), and \([Y, Y] \subset Y \), where \( Y = C^\infty(X, T^{1,0}X) \). We assume that \( X \) admits an \( S^1 \) action: \( S^1 \times X \to X \). We write \( e^{-i\theta} \) to denote the \( S^1 \) action. Let \( T \in C^\infty(X, TX) \) be the global real vector field induced by the \( S^1 \) action given by \((Tu)(x) = \frac{\partial}{\partial \theta} \langle u(e^{-i\theta} \cdot x) \rangle \big|_{\theta = 0}, u \in C^\infty(X) \).

**Definition 2.1.** We say that the \( S^1 \) action \( e^{-i\theta} \) is CR if \([T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X) \) and the \( S^1 \) action is transversal if for each \( x \in X \), \( C\langle T \rangle(x) \subset T_x^{1,0}X \subset T_x^{0,1}X = CT_xX \). Moreover, we say that the \( S^1 \) action is locally free if \( T \neq 0 \) everywhere.

We assume throughout that \((X, T^{1,0}X)\) is a CR manifold with a transversal CR locally free \( S^1 \) action \( e^{-i\theta} \) and we let \( T \) be the global vector field induced by the \( S^1 \) action. Let \( \omega_0 \in C^\infty(X, T^*X) \) be the global real one form determined by \( \langle \omega_0, u \rangle = 0 \), for every \( u \in T^{1,0}X \) and \( \langle \omega_0, T \rangle = 1 \).

**Definition 2.2.** For \( p \in X \), the Levi form \( L_p \) is the Hermitian quadratic form on \( T_p^{1,0}X \) given by
\[
L_p(U, \overline{V}) = -\frac{1}{2} \langle d\omega_0(p), U \wedge \overline{V} \rangle, U, V \in T_p^{1,0}X.
\]

If the Levi form \( L_p \) is semi-positive definite (positive definite), we say that \( X \) is weakly pseudoconvex (strongly pseudoconvex) at \( p \). If the Levi form is semi-positive definite (positive definite) at every point of \( X \), we say that \( X \) is weakly pseudoconvex (strongly pseudoconvex).
Denote by $T^{a,1.0}X$ and $T^{a,0.1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$ respectively. Define the vector bundle of $(p,q)$ forms by $T^{p,q}_{a}X = \Lambda^{p}(T^{a,1.0}X) \wedge \Lambda^{q}(T^{a,0.1}X)$. Put
\[ T^{a,0}X := \bigoplus_{j \in \{0,1,\ldots,n\}} J_{j} \] is even $T^{a,0,j}X$, $T^{a,0,-}X := \bigoplus_{j \in \{0,1,\ldots,n\}} J_{j}$ is odd $T^{a,0,j}X$.

Let $D \subset X$ be an open subset. Let $\Omega^{p,q}_{0}(D)$ denote the space of smooth sections of $T^{p,q}_{a}X$ over $D$ and let $\Omega^{p,q}_{0}(D)$ be the subspace of $\Omega^{p,q}_{0}(D)$ whose elements have compact support in $D$. Similarly, if $E$ is a vector bundle over $D$, then we let $\Omega^{p,q}(D,E)$ denote the space of smooth sections of $T^{p,q}_{a}X \otimes E$ over $D$ and let $\Omega^{p,q}_{0}(D,E)$ be the subspace of $\Omega^{p,q}_{0}(D,E)$ whose elements have compact support in $D$. Put
\[ \Omega^{0,+}(X,E) := \bigoplus_{j \in \{0,1,\ldots,n\}} J_{j} \] is even $\Omega^{0,j}(X,E)$,
\[ \Omega^{0,-}(X,E) := \bigoplus_{j \in \{0,1,\ldots,n\}} J_{j} \] is odd $\Omega^{0,j}(X,E)$.

Fix $\theta_{0} \in ]-\pi,\pi[, \theta_{0}$ small. Let
\[ de^{-i\theta_{0}} : \mathbb{C}T_{x}X \to \mathbb{C}T_{e^{-i\theta_{0}}X} \]
denote the differential map of $e^{-i\theta_{0}} : X \to X$. By the transversal property of the $S^{1}$ action, we can check that
\[ de^{-i\theta_{0}} : T^{1,0}_{x}X \to T^{1,0}_{e^{-i\theta_{0}}X}, \]
\[ de^{-i\theta_{0}} : T^{0,1}_{x}X \to T^{0,1}_{e^{-i\theta_{0}}X}, \]
\[ de^{-i\theta_{0}}(T(x)) = T(e^{-i\theta_{0}}x). \]

Let $(e^{-i\theta_{0}})^{*} : \Lambda^{r}(CT^{*}X) \to \Lambda^{r}(CT^{*}X)$ be the pull-back map by $e^{-i\theta_{0}}$, $r = 0, 1, \ldots, 2n + 1$. From (2.2), it is easy to see that for every $p, q = 0, 1, \ldots, n$,
\[ (e^{-i\theta_{0}})^{*} : T^{p,q}_{a}X \to T^{p,q}_{x}X. \]

Let $u \in \Omega^{p,q}(X)$. Define
\[ Tu := \frac{\partial}{\partial \theta} \left( (e^{-i\theta})^{*} u \right) \big|_{\theta = 0} \in \Omega^{p,q}(X). \]
(See also (2.14).) For every $\theta \in \mathbb{R}$ and every $u \in C^{\infty}(X, \Lambda^{r}(CT^{*}X))$, we write $u(e^{-i\theta} \circ x) := (e^{-i\theta})^{*} u(x)$. It is clear that for every $u \in C^{\infty}(X, \Lambda^{r}(CT^{*}X))$, we have
\[ u(x) = \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{-i\theta} \circ x) e^{im\theta} d\theta. \]

Let $\overline{\partial}_{b} : \Omega^{0,q}_{0}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. From the transversal property of the $S^{1}$ action, it is straightforward to see that (see also (2.15))
\[ T\overline{\partial}_{b} = \overline{\partial}_{b} T \] on $\Omega^{0,q}(X)$.

**Definition 2.3.** Let $D \subset U$ be an open set. We say that a function $u \in C^{\infty}(D)$ is rigid if $Tu = 0$. We say that a function $u \in C^{\infty}(X)$ is Cauchy-Riemann (CR for short) if $\overline{\partial}_{b} u = 0$. We call $u$ a rigid CR function if $\overline{\partial}_{b} u = 0$ and $Tu = 0$.

**Definition 2.4.** Let $F$ be a complex vector bundle over $X$. We say that $F$ is rigid (CR) if $X$ can be covered with open sets $U_{j}$ with trivializing frames $\{ f_{j}^{1}, f_{j}^{2}, \ldots, f_{j}^{r} \}$, $j = 1, 2, \ldots$, such that the corresponding transition matrices are rigid (CR). The frames $\{ f_{j}^{1}, f_{j}^{2}, \ldots, f_{j}^{r} \}$, $j = 1, 2, \ldots$, are called rigid frames.

From now on, let $E$ be a rigid CR vector bundle over $X$. We also write $\overline{\partial}_{b}$ to denote the tangential Cauchy-Riemann operator acting on forms with values in $E$:
\[ \overline{\partial}_{b} : \Omega^{0,q}(X, E) \to \Omega^{0,q+1}(X, E). \]
Since $E$ is rigid, we can also define $Tu$ for every $u \in \Omega^{0,q}(X, E)$ and we have
\[(2.6)\quad T\partial_b = \partial_b T \text{ on } \Omega^{0,q}(X, E).\]

For every $m \in \mathbb{Z}$, let
\[(2.7)\quad \Omega_m^{0,q}(X, E) := \{ u \in \Omega^{0,q}(X, E); Tu = -imu \}\]
and put $\partial_{b,m} := \partial_b : \Omega_m^{0,q}(X, E) \to \Omega_m^{0,q+1}(X, E)$. For every $m \in \mathbb{Z}$, we have $\partial_{b,m}$-complex: $\cdots \to \Omega_m^{0,q-1}(X, E) \to \Omega_m^{0,q}(X, E) \to \Omega_m^{0,q+1}(X, E) \to \cdots$. Define
\[H^q_{b,m}(X, E) := \frac{\ker \partial_{b,m} : \Omega_m^{0,q}(X, E) \to \Omega_m^{0,q+1}(X, E)}{\operatorname{Im} \partial_{b,m} : \Omega_m^{0,q-1}(X, E) \to \Omega_m^{0,q}(X, E)}.\]

We call $H^q_{b,m}(X, E)$ the $m$-th $S^1$ Fourier coefficient of the $q$-th $\partial_b$ Kohn-Rossi cohomology group. We will prove in Theorem 3.7 that $\dim H^q_{b,m}(X, E) < \infty$, for every $m \in \mathbb{Z}$ and every $q = 0, 1, 2, \ldots, n$. When $E$ is trivial, for every $m \in \mathbb{Z}$, we denote $\Omega_m^0(X) := \Omega_m^{0,0}(X, E)$ and $H^q_{b,m}(X) := H^q_{b,m}(X, E)$.

2.3. Tangential De-Rham cohomology group, Tangential Chern character and Tangential Todd class. For every $r = 0, 1, 2, \ldots, 2n$, put $\Omega^r_0(X) = \{ u \in \oplus_{p+q=r} \Omega^{p,q}(X); Tu = 0 \}$ and set $\Omega^r_0(X) = \oplus_{r=0}^n \Omega^r_0(X)$. Since $T \partial = dT$ (see (2.6)), we have $d$-complex:
\[d : \cdots \to \Omega^{r-1}_0(X) \to \Omega^r_0(X) \to \Omega^{r+1}_0(X) \to \cdots\]
and we define $r$-th tangential De-Rham cohomology group: $H^r_{b,0}(X) := \frac{\ker d : \Omega^r_0(X) \to \Omega^{r+1}_0(X)}{\text{Im} d : \Omega^{r-1}_0(X) \to \Omega^r_0(X)}$. Put $H^r_{b,0}(X) = \oplus_{r=0}^n H^r_{b,0}(X)$.

Let $F$ be a rigid complex vector bundle over $X$ of rank $r$. We will show in Theorem 2.11 that there is a connection $\nabla$ of $F$ such that for any rigid local frame $f = \{ f_1, f_2, \ldots, f_r \}$ of $F$ on an open set $D \subset X$, the connection matrix $\Theta(M, f) = \{ \Theta_{j,k} \}_{j,k=1}^r$ satisfies $\Theta_{j,k} \in \Omega^0(D)$, for every $j, k = 1, \ldots, r$. We call such $\nabla$ rigid connection. Let $\Theta(\nabla, F) \in C^\infty(X, \Lambda^2(CT^*X) \otimes \text{End} \; F)$ be the associated curvature. Let $h(z) = \sum_{j=0}^\infty a_j z^j, a_j \in \mathbb{R}$, for every $j$, be a real power series on $z \in \mathbb{C}$. Set
\[H(\Theta(\nabla, F)) = \text{Tr} \left( h\left( \frac{i}{2\pi} \Theta(\nabla, F) \right) \right).\]
It is clear that $H(\Theta(\nabla, F)) \in \Omega^r_0(X)$. The following is well-known (see Theorem B.5.1. in Ma-Marinescu [27]).

Theorem 2.5. $H(\Theta(\nabla, F))$ is a closed differential form.

We need

Theorem 2.6. Let $\nabla'$ be another rigid connection on $F$. Then, $H(\Theta(\nabla, F)) - H(\Theta(\nabla', F)) = da$, for some $A \in \Omega^r_0(X)$.

Proof. For each $t \in [0, 1]$, put $\nabla_t = (1-t)\nabla + t\nabla'$. Then, $\nabla_t$ is a rigid connection on $F$. Put
\[(2.8)\quad Q_t = \frac{i}{2\pi} \text{Tr} \left( \frac{\partial \nabla_t}{\partial t} h' \left( \frac{i}{2\pi} \Theta(\nabla_t, F) \right) \right).\]
Since $\nabla_t$ is rigid, it is easy to see that
\[(2.9)\quad Q_t \in \Omega^r_0(X).\]
It is well-known that (see Remark B.5.2. in Ma-Marinescu [27])
\[H(\Theta(\nabla, F)) - H(\Theta(\nabla', F)) = d \int_0^1 Q_t dt.\]
From (2.8) and (2.9), the theorem follows. \qed
From Theorem 2.5 and Theorem 2.6, we see that the tangential De-Rham cohomology class
\[ [H(\Theta(\nabla, F))] \in H^*_b(\Omega(X)) \]
does not depend on the choice of rigid connection \( \nabla \). Put
\begin{equation}
(2.10) \quad \text{ch}_b(\nabla, F) := H(\Theta(\nabla, F)) \in \Omega^*_0(X),
\end{equation}
where \( h(z) = e^z \) and set
\begin{equation}
(2.11) \quad \text{Td}_b(\nabla, F) = e^{H(\Theta(\nabla, F))} \in \Omega^*_0(X),
\end{equation}
where \( h(z) = \log(1 + ez) \). We can now introduce tangential Todd class and tangential Chern character

**Definition 2.7.** Tangential Chern character of \( F \) is given by
\[ \text{ch}_b(F) := [\text{ch}_b(\nabla, F)] \in H^*_b(X), \]
and tangential Todd class of \( F \) is given by
\[ \text{Td}_b(F) = [\text{Td}_b(\nabla, F)] \in H^*_b(X). \]

Baouendi-Rothschild-Treves [1] proved that \( T^{1,0}X \) is a rigid complex vector bundle over \( X \). Thus, we can define tangential Todd class of \( T^{1,0}X \) and tangential Chern character of \( T^{1,0}X \).

Put \( \det \left( e^{\Theta(\nabla,F)}t + I \right) = \sum_{j=0}^r \hat{c}_j(\nabla,F)t^j \). It is easy to check that \( \hat{c}_j(\nabla,F) \in \Omega^*_0(D) \). Since for any matrix \( A \), we have \( \det A = e^{\text{Tr} \log A} \). By taking \( h(z) = \log(1 + z) \), we can check that for each \( j = 0,1,\ldots,r \), \( \hat{c}_j(\nabla,F) \) is a closed differential form on \( X \) and its tangential De-Rham cohomology class \( \hat{c}_j(X,F) \in H^*_b(X) \) does not depend on the choice of rigid connection \( D \). Put \( \hat{c}_j(F) = [\hat{c}_j(\nabla,F)] \in H^*_b(X) \). We call \( \hat{c}_j(F) \) \( j \)-th tangential Chern class of \( F \), and \( \hat{c}(F) = 1 + \sum_{j=1}^r \hat{c}_j(F) \in H^*_b(X) \) is called tangential total Chern class of \( F \).

### 2.4. BRT trivializations and rigid Hermitian metrics

In this work, we need the following result due to Baouendi-Rothschild-Treves [1].

**Theorem 2.8.** For every point \( x_0 \in X \), we can find local coordinates \( x = (x_1, \cdots, x_{2n+1}) = (z, \theta) = (z_1, \cdots, z_n, \theta), z_j = x_{2j-1} + ix_{2j}, j = 1, \cdots, n, x_{2n+1} = \theta \), defined in some small neighborhood \( D = \{(z, \theta) : |z| < \delta, -\varepsilon_0 < \theta < \varepsilon_0\} \) of \( x_0, \delta > 0, 0 < \varepsilon_0 < \pi \), such that \( (z(x_0), \theta(x_0)) = (0, 0) \) and
\begin{equation}
(2.12) \quad T = \frac{\partial}{\partial \theta}, \quad Z_j = \frac{\partial}{\partial z_j} - i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta} ; j = 1, \cdots, n
\end{equation}
where \( Z_j(x), j = 1, \cdots, n, \) form a basis of \( T^{1,0}_xX \), for each \( x \in D \) and \( \varphi(z) \in C^\infty(D, \mathbb{R}) \) independent of \( \theta \). We call \((D, (z, \theta), \varphi) \) BRT trivialization.

By using BRT trivialization, we get another way to define \( Tu, \forall u \in \Omega^{0,q}(X) \). Let \((D, (z, \theta), \varphi) \) be a BRT trivialization. It is clear that
\[
\{d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}, 1 \leq j_1 < \cdots < j_q \leq n\}
\]
is a basis for \( T_x^{0,q}X \), for every \( x \in D \). Let \( u \in \Omega^{0,q}(X) \). On \( D \), we write
\begin{equation}
(2.13) \quad u = \sum_{j_1 < \cdots < j_q} u_{j_1 \cdots j_q} d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.
\end{equation}
Then on \( D \) we can check that
\begin{equation}
(2.14) \quad Tu = \sum_{j_1 < \cdots < j_q} (Tu_{j_1 \cdots j_q}) d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.
\end{equation}
and $Tu$ is independent of the choice of BRT trivializations. Note that on BRT trivialization $(D, (z, \theta), \varphi)$, we have

\begin{equation}
(2.15) \quad \overline{\partial}_b = \sum_{j=1}^{n} dz_j \wedge \left( \frac{\partial}{\partial z_j} + i \frac{\partial \varphi}{\partial z_j \partial \theta} \right).
\end{equation}

Let $F$ be a rigid vector bundle over $X$. We need

**Definition 2.9.** Let $(\cdot | \cdot)_F$ be a Hermitian metric on $F$. We say that $(\cdot | \cdot)_F$ is a rigid Hermitian metric if for every rigid local frames $f_1, \ldots, f_r$ of $F$, we have $T(f_j | f_k)_F = 0$ for every $j, k = 1, 2, \ldots, r$.

We need

**Theorem 2.10.** There is a rigid Hermitian metric $(\cdot | \cdot)_F$ on $F$.

**Proof.** Fix $p \in X$ and let $(D, (z, \theta), \varphi)$ be a BRT trivialization defined in some neighbourhood of $p$ such that $(z(p), \theta(p)) = (0, 0)$. Suppose that $(z, \theta)$ is defined on $\{z \in \mathbb{C}^{n-1} : |z| < \delta \} \times \{\theta \in \mathbb{R} : |\theta| < \delta \}$, for some $\delta > 0$. Put

\[ A := \{ \lambda \in [-\pi, \pi] : \text{There is a local rigid trivializing frames } f = (f_1, \ldots, f_r) \text{ defined on the set } \{e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon, \theta \in [-\pi, \lambda + \varepsilon] \}, \text{for some } 0 < \varepsilon < \delta \}. \]

It is clear that $A$ is a non-empty open set in $[-\pi, \pi]$. We claim that $A$ is closed. Let $\lambda_0$ be a limit point of $A$. Consider the point $e^{-i\lambda_0} \circ (0, 0)$. For some $\varepsilon_1 > 0$, $\varepsilon_1$ small, there are local rigid trivializing frames $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_r)$ defined on $\{e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_1, \lambda_0 - \varepsilon_1 < \theta < \lambda_0 + \varepsilon_1 \}$. Since $\lambda_0$ is a limit point of $A$, we can find local rigid trivializing frames $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_r)$ defined on $\{e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_2, \theta \in [-\pi, \lambda_0 - \frac{\varepsilon_2}{2}] \}$, for some $\varepsilon_2 > 0$. Now, $\tilde{f} = g \tilde{f}$ on

\[ \left\{ e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_0, \theta \in (\lambda_0 - \varepsilon_1, \lambda_0 - \frac{\varepsilon_1}{2}) \right\} \]

for some rigid CR $r \times r$ matrix $g$, where $\varepsilon_0 = \min \{\varepsilon_1, \varepsilon_2\}$. Since $g$ is independent of $\theta$, $g$ is well-defined on $\{e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_0, \theta \in \mathbb{R} \}$. Put $f = \tilde{f}$ on $\{e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_0, \theta \in [-\pi, \lambda_0 - \frac{\varepsilon_0}{2}] \}$ and $f = g \tilde{f}$ on $\{e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_0, \theta \in [\lambda_0 - \frac{\varepsilon_0}{2}, \lambda_0 + \varepsilon_0] \}$. It is straightforward to check that $\tilde{f}$ is well-defined as local rigid trivializing frames on

\[ \left\{ e^{-i\theta} \circ (z, 0) ; |z| < \varepsilon_0, \theta \in [-\pi, \lambda_0 + \varepsilon_0] \right\} . \]

Thus, $\lambda_0 \in A$ and hence $A = [-\pi, \pi]$.

From the discussion above, we see that we can find local rigid trivializations $W_1, \ldots, W_N$ such that $X = \bigcup_{j=1}^{N} W_j$ and $W_t \subset \bigcup_{-\pi \leq \theta \leq \pi} e^{-i\theta} W_t$, $t = 1, \ldots, N$. Take any Hermitian metric $(\cdot | \cdot)_F$ on $F$. Let $(\cdot | \cdot)_F$ be the Hermitian metric on $F$ defined as follows. For each $j = 1, 2, \ldots, N$, let $h_j^1, \ldots, h_j^r$ be local rigid trivializing frames on $W_j$. Put $(h_j^s(x)|h_j^t(x))_F = \frac{1}{2\pi} \int_{-\pi}^{\pi} (h_j^s(e^{-iu} \circ x), h_j^t(e^{-iu} \circ x))_F du$, $s, t = 1, 2, \ldots, r$. Since $F$ is rigid, it is easy to see that $(\cdot | \cdot)_F$ is well-defined as a rigid Hermitian metric on $F$. \hfill \square

Now, we can prove

**Theorem 2.11.** There is a rigid connection on $F$.

**Proof.** From Theorem 2.10, there is a rigid Hermitian metric $(\cdot | \cdot)_F$ on $F$. Let $\nabla$ be the connection on $F$ induced by $(\cdot | \cdot)_F$. It is not difficult to check that $\nabla$ is rigid. \hfill \square
3. Hodge theory for $\square^{(q)}_{b,m}$

From now on, we take a rigid Hermitian metric $(\cdot|\cdot)_E$ on $E$ and take a rigid Hermitian metric $(\cdot|\cdot)$ on $\partial T^* X$ such that $T^{1,0} X \perp T^{0,1} X$ and $\langle T | T \rangle = 1$. This is always possible, see Theorem 2.10 and Theorem 9.2 in [20]. The Hermitian metric $(\cdot|\cdot)$ on $\partial T^* X$ induces by duality a Hermitian metric on $\partial T^* X$ and also on the bundles of $(0,q)$ forms $T^{0,q} X, q = 0, 1, \ldots, n$. We shall also denote all these induced metrics by $(\cdot|\cdot)$. For every $v \in T^{0,q} X$, we write $|v|^2 := (v|v)$. The Hermitian metrics on $T^{0,q} X$ and $E$ induce Hermitian metrics on $T^{0,q} X \otimes E, q = 0, 1, \ldots, n$. We shall denote these induced metrics by $(\cdot|\cdot)_E$. For $f \in \Omega^{0,q}(X,E)$, we denote the pointwise norm $|f(x)|^2_E := (f(x)|f(x))_E$. We denote by $d\nu_X = d\nu_X(x)$ the volume form on $X$ induced by the fixed Hermitian metric $(\cdot|\cdot)$ on $\partial T^* X$. Then we get natural global $L^2$ inner products $(\cdot|\cdot)_E, (\cdot|\cdot)$ on $\Omega^{0,q}(X,E)$ and $\Omega^{0,q}(X)$ respectively. We denote by $L^2(X, T^{0,q} X \otimes E)$ and $L^2(X, T^{0,q} X)$ the completions of $\Omega^{0,q}(X,E)$ and $\Omega^{0,q}(X)$ with respect to $(\cdot|\cdot)_E, (\cdot|\cdot)$ respectively. Similarly, for each $m \in \mathbb{Z}$, we denote by $L^2_m(X, T^{0,q} X \otimes E)$ and $L^2_m(X, T^{0,q} X)$ the completions of $\Omega^{0,q}_m(X,E)$ and $\Omega^{0,q}_m(X)$ with respect to $(\cdot|\cdot)_E$ and $(\cdot|\cdot)$ respectively. We extend $(\cdot|\cdot)_E$ and $(\cdot|\cdot)$ to $L^2(X, T^{0,q} X \otimes E)$ and $L^2(X, T^{0,q} X)$ in the standard way respectively. For $f \in \Omega^{0,q}(X,E)$, we denote $\|f\|^2_E := (f|f)_E$.

Similarly, for $f \in \Omega^{0,q}(X)$, we denote $\|f\|^2 := (f|f)$.

Let

$$\partial_b^* : \Omega^{0,q+1}(X,E) \to \Omega^{0,q}(X,E)$$

be the formal adjoint of $\partial_b$ with respect to $(\cdot|\cdot)_E$. Since $(\cdot|\cdot)_E$ is rigid, we can check that

$$T\partial_b^* = \partial_b^* T \text{ on } \Omega^{0,q}(X,E), q = 1, 2, \ldots, n,$$

$$(3.1) \quad \partial_b^* : \Omega^{0,q+1}_m(X,E) \to \Omega^{0,q}_m(X,E), \quad \forall m \in \mathbb{Z}.$$

Put

$$(3.2) \quad \square^{(q)}_b := \partial_b \partial_b^* + \partial_b^* \partial_b : \Omega^{0,q}(X,E) \to \Omega^{0,q}(X,E).$$

From (2.6) and (3.1), we have

$$T \square^{(q)}_b = \square^{(q)}_b T \text{ on } \Omega^{0,q}(X,E), q = 0, 1, \ldots, n,$$

$$(3.3) \quad \square^{(q)}_b : \Omega^{0,q}_m(X,E) \to \Omega^{0,q}_m(X,E), \quad \forall m \in \mathbb{Z}.$$

The following follows from Kohn’s $L^2$ estimates (see [5, Theorem 8.4.2])

**Theorem 3.1.** For every $s \in \mathbb{N}_0$, there is a constant $C_s > 0$ such that

$$(3.4) \quad \|u\|_{s+1} \leq C_s \left( \left\| \square^{(q)}_b u \right\|_s + \|Tu\|_s + \|u\|_s \right), \quad \forall u \in \Omega^{0,q}(X,E),$$

where $\|\cdot\|_s$ denotes the usual Sobolev norm of order $s$ on $X$.

We will write $\square^{(q)}_{b,m}$ to denote the restriction of $\square^{(q)}_b$ on the space $\Omega^{0,q}_m(X,E)$. From Theorem 3.1, we deduce

**Theorem 3.2.** Fix $m \in \mathbb{Z}$. For every $s \in \mathbb{N}_0$, there is a constant $C_s > 0$ such that

$$(3.5) \quad \|u\|_{s+1} \leq C_s \left( \left\| \square^{(q)}_{b,m} u \right\|_s + \|u\|_s \right), \quad \forall u \in \Omega^{0,q}_m(X,E),$$

where $\|\cdot\|_s$ denotes the usual Sobolev norm of order $s$ on $X$.

For every $m \in \mathbb{Z}$, we extend $\square^{(q)}_{b,m}$ to $L^2_m(X, T^{0,q} X \otimes E)$ by

$$(3.6) \quad \square^{(q)}_{b,m} : \text{Dom} \square^{(q)}_{b,m} \subset L^2_m(X, T^{0,q} X \otimes E) \to L^2_m(X, T^{0,q} X \otimes E),$$

where $\text{Dom} \square^{(q)}_{b,m} := \{ u \in L^2_m(X, T^{0,q} X \otimes E); \square^{(q)}_{b,m} u \in L^2_m(X, T^{0,q} X \otimes E) \}$, where for any $u \in L^2_m(X, T^{0,q} X \otimes E), \square^{(q)}_{b,m} u$ is defined in the sense of distribution.
Lemma 3.3. We have \( \operatorname{Dom} \Box^{(q)}_{b,m} = L^2_m(X, T^{4q} X \otimes E) \cap H^2(X, T^{4q} X \otimes E) \).

Proof. It is clear that \( L^2_m(X, T^{4q} X \otimes E) \cap H^2(X, T^{4q} X \otimes E) \subset \operatorname{Dom} \Box^{(q)}_{b,m} \). We only need to prove that \( \operatorname{Dom} \Box^{(q)}_{b,m} \subset L^2_m(X, T^{4q} X \otimes E) \cap H^2(X, T^{4q} X \otimes E) \). Let \( u \in \operatorname{Dom} \Box^{(q)}_{b,m} \). Put \( v = \Box^{(q)}_{b,m} u \in L^2_m(X, T^{4q} X \otimes E) \). We have \((\Box^{(q)}_{b,m} - T^2)u = v + m^2 u \in L^2_m(X, T^{4q} X \otimes E)\). Since \((\Box^{(q)}_{b,m} - T^2)\) is elliptic, we conclude that \( u \in H^2(X, T^{4q} X \otimes E) \). The lemma follows. \( \square \)

We can prove

Theorem 3.4. \( \Box^{(q)}_{b,m} : \operatorname{Dom} \Box^{(q)}_{b,m} \subset L^2_m(X, T^{4q} X \otimes E) \to L^2_m(X, T^{4q} X \otimes E) \) is self-adjoint.

Proof. Let \((\Box^{(q)}_{b,m})^* : \operatorname{Dom} (\Box^{(q)}_{b,m})^* \subset L^2_m(X, T^{4q} X \otimes E) \to L^2_m(X, T^{4q} X \otimes E)\) be the Hilbert space adjoint of \( \Box^{(q)}_{b,m} \). Let \( v \in \operatorname{Dom} (\Box^{(q)}_{b,m})^* \). Then, by the definition of the Hilbert space adjoint of \( \Box^{(q)}_{b,m} \), it is easy to see that \( \Box^{(q)}_{b,m} v \in L^2_m(X, T^{4q} X \otimes E) \) and hence \( v \in \operatorname{Dom} \Box^{(q)}_{b,m} \) and \( \Box^{(q)}_{b,m} v = (\Box^{(q)}_{b,m})^* v \).

From Lemma 3.3, we can check that
\[
(\Box^{(q)}_{b,m} g \mid f)_E = (g \mid (\Box^{(q)}_{b,m})^* f)_E, \quad \forall g, f \in \operatorname{Dom} \Box^{(q)}_{b,m}.
\]

From (3.7), we deduce that \( \operatorname{Dom} \Box^{(q)}_{b,m} \subset \operatorname{Dom} (\Box^{(q)}_{b,m})^* \) and \( \Box^{(q)}_{b,m} u = (\Box^{(q)}_{b,m})^* u \), for all \( u \in \operatorname{Dom} \Box^{(q)}_{b,m} \). The theorem follows. \( \square \)

Let \( \text{Spec} \Box^{(q)}_{b,m} \subset [0, \infty) \) denote the spectrum of \( \Box^{(q)}_{b,m} \). For any \( \lambda > 0 \), put
\[
\Pi_{m,\leq \lambda}^{(q)} := E_m^{(q)}([0, \lambda]),
\]
where \( E_m^{(q)} \) denotes the spectral measure for \( \Box^{(q)}_{b,m} \) (see Section 2 in Davies [7], for the precise meaning of spectral measure). We need

Theorem 3.5. \( \text{Spec} \Box^{(q)}_{b,m} \) is a discrete subset of \([0, \infty)\). Any \( \nu \in \text{Spec} \Box^{(q)}_{b,m} \) \( \nu \) is an eigenvalue of \( \Box^{(q)}_{b,m} \) and the eigenspace
\[
E_{m,\nu}^{(q)}(X, E) := \{ u \in \operatorname{Dom} \Box^{(q)}_{b,m} ; \Box^{(q)}_{b,m} u = \nu u \}
\]
is finite dimensional with \( E_{m,\nu}^{(q)}(X, E) \subset \Omega_m^{(q)}(X, E) \).

Proof. Fix \( \lambda > 0 \). We claim that \( \text{Spec} \Box^{(q)}_{b,m} \cap [0, \lambda] \) is discrete. If not, we can find \( f_j \in \text{Rang} E_m^{(q)}([0, \lambda]), j = 1, 2, \ldots, \) with \((f_j \mid f_k)_E = \delta_{j,k}, j, k = 1, 2, \ldots\). Note that
\[
\Box^{(q)}_{b,m} f_j \leq \lambda \| f_j \|_E, \quad j = 1, 2, \ldots.
\]
From (3.8), we have
\[
\Box^{(q)}_{b,m} f_j \leq \lambda \| f_j \|_E, \quad j = 1, 2, \ldots.
\]
Since \( \Box^{(q)}_{b,m} - T^2 \) is a second order elliptic operator, there is a constant \( C_m > 0 \) independent of \( j \) such that
\[
\| f_j \|_2 \leq C_m, \quad j = 1, 2, \ldots,
\]
where \( \| \cdot \|_2 \) denotes the usual Sobolev norm of order 2. From (3.10), we can apply Rellich’s theorem and find subsequence \( \{ f_{j_i} \}_{i=1}^{\infty}, 1 \leq j_1 < j_2 < \cdots \) such that \( f_{j_i} \to f \) in \( L^2_m(X, T^{4q} X \otimes E) \). Since \((f_j \mid f_k)_E = \delta_{j,k}, j, k = 1, 2, \ldots, \) we get a contradiction. Thus, \( \text{Spec} \Box^{(q)}_{b,m} \cap [0, \lambda] \) is discrete. Hence \( \text{Spec} \Box^{(q)}_{b,m} \) is a discrete subset of \([0, \infty)\).
Let $r \in \text{Spec} \, \Box^{(q)}_{b,m}$. Since $\text{Spec} \, \Box^{(q)}_{b,m}$ is discrete, $\Box^{(q)}_{b,m} - r$ has $L^2$ closed range. If $\Box^{(q)}_{b,m} - r$ is injective, then $\text{Range} (\Box^{(q)}_{b,m} - r) = L^2_m(X, T^{s,0,q}X \otimes E)$ and

$$(\Box^{(q)}_{b,m} - r)^{-1} : L^2_m(X, T^{s,0,q}X \otimes E) \to L^2_m(X, T^{s,0,q}X \otimes E)$$

is continuous. We get a contradiction. Hence $r$ is an eigenvalue of $\text{Spec} \, \Box^{(q)}_{b,m}$.

For any $\nu \in \text{Spec} \, \Box^{(q)}_{b,m}$, put

$$\mathcal{E}^{q}_{m,\nu}(X, E) := \{ u \in \text{Dom} \, \Box^{(q)}_{b,m} ; \, \Box^{(q)}_{b,m} u = \nu u \}.$$ 

We can repeat the argument before and conclude that $\mathcal{E}^{q}_{m,\nu}(X, E)$ is finite dimensional. Let $u \in \mathcal{E}^{q}_{m,\nu}(X, E)$. Then, $\Box^{(q)}_{b,m} u = \nu u$ and hence

$$(3.11) \quad (\Box^{(q)}_{b,m} - T^2)u = (\nu + m^2)u.$$ 

From (3.11), we deduce that $u \in \Omega^{0,\nu}_m(X, E)$. Thus, $\mathcal{E}^{q}_{m,\nu}(X, E) \subset \Omega^{0,q}_m(X, E)$. The theorem follows.

Let $N^{(q)}_m : L^2_m(X, T^{s,0,q}X \otimes E) \to \text{Dom} \, \Box^{(q)}_{b,m}$ be the partial inverse of $\Box^{(q)}_{b,m}$ and let

$$\Pi^{(q)}_m : L^2_m(X, T^{s,0,q}X \otimes E) \to \text{Ker} \, \Box^{(q)}_{b,m}$$

be the orthogonal projection. We have

$$(3.12) \quad \Box^{(q)}_{b,m} N^{(q)}_m + \Pi^{(q)}_m = I \quad \text{on} \, L^2_m(X, T^{s,0,q}X \otimes E),$$

$$N^{(q)}_m \Box^{(q)}_{b,m} + \Pi^{(q)}_m = I \quad \text{on} \, \text{Dom} \, \Box^{(q)}_{b,m}.$$ 

We need

**Lemma 3.6.** We have $N^{(q)}_m : \Omega^{0,q}_m(X, E) \to \Omega^{0,q}_m(X, E)$.

**Proof.** Let $u \in \Omega^{0,q}_m(X, E)$ and put $N^{(q)}_m u = v \in L^2_m(X, T^{s,0,q}X \otimes E)$. From (3.12), we have $(I - \Pi^{(q)}_m)u = \Box^{(q)}_{b,m} v$. Hence,

$$(3.13) \quad (\Box^{(q)}_{b,m} - T^2)v = (I - \Pi^{(q)}_m)u + m^2 v.$$ 

In view of Theorem 3.5, we see that

$$(3.14) \quad (I - \Pi^{(q)}_m)u \in \Omega^{0,q}_m(X, E).$$ 

Note that $\Box^{(q)}_{b,m} - T^2$ is elliptic. From this observation, (3.13) and (3.14), we can repeat the standard technique in elliptic partial differential operator and conclude that $v \in \Omega^{0,q}_m(X, E)$. The lemma follows.

Now, we can prove

**Theorem 3.7.** For every $q \in \{0, 1, 2, \ldots, n\}$ and every $m \in \mathbb{Z}$, we have

$$(3.15) \quad \text{Ker} \, \Box^{(q)}_{b,m} = \mathcal{E}^{q}_{m,0}(X, E) \cong H^{q}_{b,m}(X, E).$$ 

In particular, $H^{q}_{b,m}(X, E) < \infty$, for every $m \in \mathbb{Z}$ and every $q = 0, 1, 2, \ldots, n$.

**Proof.** Consider the map

$$\tau^{q}_m : \text{Ker} \, \overline{\partial}_{b,m} \cap \Omega^{0,q}_m(X, E) \to \text{Ker} \, \Box^{(q)}_{b,m},$$

$$u \to \Pi^{(q)}_m u.$$
It is clear that $\tau^q_m$ is surjective. Put $M'_m := \{ \partial_b,m u; \ u \in \Omega^{0,q-1}_m(X,E) \}$. We now prove that

$$\text{Ker } \tau^q_m = M'_m.$$  

It is clear that $M'_m \subset \text{Ker } \tau^q_m$. Let $u \in \text{Ker } \tau^q_m$. From (3.12), we have

$$u = \Box_n^{(q)} N_r^{(q)} u + \Pi_m^{(q)} u$$

(3.17)  

We claim that

$$\Box_n^{(q)} \psi_N^{(q)} u = 0.$$  

We have

$$\partial_b^* \partial_b N_m^{(q)} u = 0.$$  

From (3.19), the claim (3.18) follows. From (3.18) and (3.17), we deduce that

$$u = \partial_b^* \partial_b N_m^{(q)} u.$$  

From Lemma 3.6, we know that $\overline{\partial}_b N_m^{(q)} u \in \Omega_m^{0,q-1}(X,E)$. From this observation and (3.20), we get that $u \in M'_m$. (3.16) follows. From (3.16) and $\tau^q_m$ is surjective, the theorem follows. \hfill \Box

For every $m \in \mathbb{Z}$, put

$$\Omega_m^{0,+}(X,E) := \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is even}} \Omega_j^{0,j}(X,E),$$

$$\Omega_m^{0,-}(X,E) := \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is odd}} \Omega_j^{0,j}(X,E).$$

Similarly, put

$$\Omega_m^{0,+}(X,E) := \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is even}} \Omega_j^{0,j}(X,E),$$

$$\Omega_m^{0,-}(X,E) := \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is odd}} \Omega_j^{0,j}(X,E).$$

Let $L_{2,m}^{2,+}(X,E)$, $L_{2,m}^{2,-}(X,E)$, $L_{2,m}^{2,+}(X,E)$, $L_{2,m}^{2,-}(X,E)$ be the completions of $\Omega_m^{0,+}(X,E)$, $\Omega_m^{0,-}(X,E)$, $\Omega_m^{0,+}(X,E)$ and $\Omega_m^{0,-}(X,E)$ with respect to $(\cdot, \cdot)_E$ respectively. Let

$$D_{b,m} := \partial_b + \partial_b^* : \Omega_{b,m}^{0,+}(X,E) \to \Omega_{b,m}^{0,-}(X,E).$$

We extend $D_{b,m}$ to $L_{b,m}^{2,+}(X,E)$:

$$D_{b,m} : \text{Dom } D_{b,m} \subset L_{b,m}^{2,+}(X,E) \to L_{b,m}^{2,-}(X,E),$$

where \( \text{Dom } D_{b,m} = \left\{ u \in L_{b,m}^{2,+}(X,E); \ D_{b,m} u \in L_{b,m}^{2,-}(X,E) \right\} \). Let

$$D_{b,m}^* : \text{Dom } D_{b,m}^* \subset L_{b,m}^{2,-}(X,E) \to L_{b,m}^{2,+}(X,E)$$

be the Hilbert space adjoint of $D_{b,m}$ with respect to $(\cdot, \cdot)_E$. From Theorem 3.5 and Theorem 3.7, it is not difficult to see that (we omit the proof)

**Theorem 3.8.** We have

$$\text{Ker } D_{b,m} = \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is even}} \text{Ker } \Box_n^{(j)} \subset \Omega_m^{0,+}(X,E),$$

$$\text{Ker } D_{b,m}^* = \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is odd}} \text{Ker } \Box_n^{(j)} \subset \Omega_m^{0,-}(X,E).$$

Put $\text{ind } D_{b,m} := \dim \text{Ker } D_{b,m} - \dim \text{Ker } D_{b,m}^*$. Then,

$$\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E) = \text{ind } D_{b,m}. $$
4. Modified Kohn Laplacians

To calculate a local density of \( \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) \), we have to modify the standard Kohn Laplacian.

We recall briefly some basic definitions and properties of Clifford connection and Spin\(^c\) Dirac operator. For more details about Clifford connection and Spin\(^c\) Dirac operator, see Chapter 1 in [27] and [8].

Let \( B := (D, (z, \theta), \varphi) \) be a BRT trivialization. We may assume that \( D = U \times ]-\varepsilon, \varepsilon[ \), where \( \varepsilon > 0 \) and \( U \) is an open set of \( \mathbb{C}^n \). Let \( T^{1,0}U \) denote the holomorphic tangent bundle of \( U \) and put \( T^{0,1}U := \overline{T^{1,0}U} \).

Let \( \langle \cdot, \cdot \rangle \) be the Hermitian metric on \( TU \) induced by \( g^{TU} \), where \( g^{TU} \) be a BRT trivialization. We may assume that \( \langle \cdot, \cdot \rangle \) induces Hermitian metrics on \( T^{q,0}U \) bundle of \( (0, q) \) forms on \( U \), \( q = 0, 1, \ldots, n \), we shall also denote the Hermitian metrics by \( \langle \cdot, \cdot \rangle \). Let \( g^{TU} \) be the Riemannian metric on \( TU \) induced by \( \langle \cdot, \cdot \rangle \).

Let \( \{w_j\}_{j=1}^{n} \) be a local orthonormal frame of \( T^{1,0}U \) with respect to \( \langle \cdot, \cdot \rangle \) with dual frame \( \{w_j^*\}_{j=1}^{n} \).

Then,
\[
\langle \cdot, \cdot \rangle = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j) \quad \text{and} \quad \frac{1}{\sqrt{2}}(w_j - \bar{w}_j), \quad j = 1, 2, \ldots, n,
\]
form an orthonormal frame of \( TU \). We fix this notation throughout this section and use it without further notice.

For any \( v \in TU \) with decomposition \( v = v^{(1,0)} + v^{(0,1)} \in T^{1,0}U \oplus T^{0,1}U \), let \( \pi^{(1,0)*} \in T^{0,1}U \) be the metric dual of \( v^{(1,0)} \) with respect to \( \langle \cdot, \cdot \rangle \). That is,
\[
\pi^{(1,0)*}(u) = \langle v^{(1,0)}, u \rangle, \quad \forall u \in T^{0,1}U.
\]

Then
\[
\pi^{(1,0)*}(u) = \langle v^{(1,0)}, u \rangle, \quad \forall u \in T^{0,1}U.
\]

defines the Clifford action \( v \) on \( \Lambda(T^{q,0}U) := \otimes_{q=0}^{n} T^{q,0}U \), where \( \wedge \) and \( i \) denote the exterior and interior product respectively.

Let \( \nabla^{TU} \) be the Levi-Civita connection on \( TU \) with respect to \( g^{TU} \). Let \( \nabla^{\det} \) be the Chern connection on \( \det(T^{1,0}U) \) induced by \( \langle \cdot, \cdot \rangle \), where \( \det(T^{1,0}U) \) denotes the determinant line bundle of \( T^{1,0}U \). Let \( \Gamma^{TU} \in T^*U \otimes \text{End}TU \), \( \Gamma^{\det} \) be the connection forms of \( \nabla^{TU} \), \( \nabla^{\det} \) associated to the frames \( \{e_j\}_{j=1}^{2n} \) and \( w_1 \wedge \cdots \wedge w_n \). That is,
\[
\nabla^{TU}_{e_j} e_{j\ell} = \Gamma^{TU}(e_j)e_{j\ell}, \quad j, \ell = 1, 2, \ldots, 2n,
\]
\[
\nabla^{\det}(w_1 \wedge \cdots \wedge w_n) = \Gamma^{\det}w_1 \wedge \cdots \wedge w_n.
\]

The Clifford connection \( \nabla^{Cl} \) on \( \Lambda(T^{q,1}U) \) is defined for the frame
\[
\{\bar{w}^{j_1} \wedge \cdots \wedge \bar{w}^{j_q}; 1 \leq j_1 < \cdots < j_q \leq n\}
\]
by the local formula
\[
\nabla^{Cl} = d + \frac{1}{4} \sum_{j,\ell=1}^{2n} \langle \Gamma^{TU}c(e_j)c(e_{j\ell}) \rangle + \frac{1}{2} \Gamma^{\det}.
\]

By Proposition 1.3.1 in [27], \( \nabla^{Cl} \) defines a Hermitian connection on \( \Lambda(T^{q,1}X) \).

Since \( E \) is rigid, we can consider \( E \) as a holomorphic vector bundle over \( U \). We may assume that \( E \) is trivial on \( U \). We will use the same notations as before. Let \( \Omega^{b,q}(U, E) \) be the space of \( (0, q) \) forms on
$U$ with values in $E$, $q = 0, 1, 2, \ldots, n$. Put
\begin{align*}
\Omega^{0,+}(U, E) := & \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is even}} \Omega^{0,j}(U, E), \\
\Omega^{0,-}(U, E) := & \bigoplus_{j \in \{0,1,\ldots,n\}, j \text{ is odd}} \Omega^{0,j}(U, E), \\
\Omega^{0,\ast}(U, E) := & \Omega^{0,+}(U, E) \otimes L^m \oplus \Omega^{0,-}(U, E).
\end{align*}
Since the Hermitian fiber metric $(\cdot | \cdot)_E$ is rigid, we can consider $(\cdot | \cdot)_E$ as a Hermitian fiber metric on the holomorphic line bundle $E$ over $U$. Let $\nabla^E$ be the Chern connection on $E$ induced by $(\cdot | \cdot)_E$. We still denote by $\nabla^\mathcal{O}_E$ the connection on $\Lambda(T^{0,1}U) \otimes E$ induced by $\nabla^\mathcal{O}_E$ and $\nabla^E$.

**Definition 4.1.** The Spin\textsuperscript{c} Dirac operator $D_B$ is defined by
\begin{equation}
D_B = \sum_{j=1}^{2n} c(e_j) \nabla^\mathcal{O}_E : \Omega^{0,\ast}(U, E) \to \Omega^{0,\ast}(U, E),
\end{equation}
It is well-known (see Proposition 1.3.1 and equation (1.3.1) in [27]) that $D_B$ is formally self-adjoint and
\begin{align*}
D_B : \Omega^{0,+}(U, E) & \to \Omega^{0,-}(U, E), \\
D_B : \Omega^{0,-}(U, E) & \to \Omega^{0,+}(U, E).
\end{align*}
Let $\bar{\partial} : \Omega^{0,q}(U, E) \to \Omega^{0,q+1}(U), \ q = 0, 1, 2, \ldots, n - 1$, be the Cauchy-Riemann operator. Let $(\cdot, \cdot)$ be the $L^2$ inner product on $\Omega^{0,q}(U, E)$ induced by $(\cdot | \cdot)$ and $(\cdot | \cdot)_E, q = 0, 1, 2, \ldots, n$, and let $\bar{\partial}' : \Omega^{0,q+1}(U, E) \to \Omega^{0,q}(U, E), \ q = 0, 1, 2, \ldots, n - 1$, be the formal adjoint of $\partial$ with respect to $(\cdot, \cdot)$. By Theorem 1.4.5 in [27], we have
\begin{equation}
D_B = \partial + \partial' + A_B : \Omega^{0,+}(U, E) \to \Omega^{0,-}(U, E), \\
D_B = \partial + \partial' + A_B : \Omega^{0,-}(U, E) \to \Omega^{0,+}(U, E),
\end{equation}
where
\begin{align*}
A_B : \Omega^{0,+}(U, E) & \to \Omega^{0,-}(U, E), \\
A_B : \Omega^{0,-}(U, E) & \to \Omega^{0,+}(U, E),
\end{align*}
is a smooth zero order operator. It is clear that $A_B = A_B(z)$, that is $A_B$ is independent of $\theta$. Let $u \in \Omega^{0,q}(D, E), \ q = 0, 1, 2, \ldots, n$. Assume that $u = u(z)$, i.e. $u$ is independent of $\theta$. By using (2.13), from now on, we identify such $u$ with an element in $\Omega^{0,q}(U, E)$. We need

**Lemma 4.2.** Let $B = (D, (z, \theta), \varphi)$ and $\tilde{B} = (D, (w, \eta), \tilde{\varphi})$ be two BRT trivializations. We assume that $D = U \times ]-\varepsilon, \varepsilon[, \varepsilon > 0$ and $U$ is an open set of $\mathbb{C}^n$. Let $A_B, A_B : \Omega^{0,\pm}(U, E) \to \Omega^{0,\mp}(U, E)$ be the operators given by (4.7). Fix $m \in \mathbb{Z}$. Let $u \in \Omega^{0,\pm}(X, E)$. On $D$, we write $u = e^{-im\theta}v(z) = e^{-im\eta}\tilde{v}(w)$, $v(z), \tilde{v}(w) \in \Omega^{0,\pm}(U, E)$. Then,
\begin{equation}
e^{-im\theta}A_B(v(z)) = e^{-im\eta}A_B(\tilde{v}(w)) \text{ on } D.
\end{equation}

**Proof.** It is straightforward to check that
\begin{equation}
\begin{aligned}
w = (w_1, \ldots, w_n) = H(z) = (H_1(z), \ldots, H_n(z)), \\
\eta = \theta + G(z), \ G(z) \in C^\infty.
\end{aligned}
\end{equation}
Since $\bar{\partial}H(z) = 0$, it is not difficult to see that
\begin{equation}
\begin{aligned}
D_B = D_B \text{ on } \Omega^{0,\pm}(U, E), \\
A_B = A_B \text{ on } \Omega^{0,\pm}(U, E).
\end{aligned}
\end{equation}
Let \( u \in \Omega_{m}^{0,\pm}(X, E) \). On \( D \), write \( u = e^{-im\theta}v(z) = e^{-im\eta}\tilde{v}(w), \ v(z), \tilde{v}(w) \in \Omega^{0,\pm}(U, E) \). From (4.9), we can check that
\[
(4.11) \quad v(z) = e^{-imG(z)}\tilde{v}(w).
\]
From (4.11) and (4.10), we have
\[
e^{-im\theta}A_B(v(z)) = e^{-im\theta}A_B(e^{-imG(z)}\tilde{v}(w)) = e^{-im\theta}A^B_{\tilde{B}}(e^{-imG(z)}\tilde{v}(w)) = e^{-im\theta}A^B_{\tilde{B}}(\tilde{v}(w)).
\]
The lemma follows. \( \square \)

We can now introduce

**Definition 4.3.** For every \( m \in \mathbb{Z} \), let \( A_m : \Omega_{m}^{0,\pm}(X, E) \rightarrow \Omega_{m}^{0,\pm}(X, E) \) be the linear operator defined as follows. Let \( u \in \Omega_{m}^{0,\pm}(X, E) \). Then, \( v := Amu \) is an element in \( \Omega_{m}^{0,\pm}(X, E) \) such that for every BRT trivialization \( B := (D, (z, \theta), \varphi) \), \( D = \mathbb{D}_m \), \( m > 0 \), \( u \in \mathbb{D}_m \) (here and later, \( \mathbb{D}_m \) is an open set in \( \mathbb{C}^n \), we have \( v|_D = e^{-im\theta}A_B(\tilde{u})(z) \), where \( u = e^{-im\theta}\tilde{u}(z) \) on \( D, \tilde{u} \in \Omega^{0,\pm}(U, E) \), and \( A_B \) is given by (4.7).

In view of Lemma 4.2, we see that Definition 4.3 is well-defined.

Until further notice, we fix \( m \in \mathbb{Z} \). Consider
\[
(4.12) \quad \bar{D}_{b,m} = \bar{D}_b + \bar{D}_b^* + A_m : \Omega_{m}^{0,+}(X, E) \rightarrow \Omega_{m}^{0,-}(X, E),
\]
Let
\[
(4.13) \quad \bar{D}_{b,m}^* : \Omega_{m}^{0,-}(X, E) \rightarrow \Omega_{m}^{0,+}(X, E),
\]
be the formal adjoint of \( \bar{D}_{b,m} : \Omega_{m}^{0,+}(X, E) \rightarrow \Omega_{m}^{0,-}(X, E) \). Then,
\[
\bar{D}_{b,m} = \bar{D}_{b,m}^* : \Omega_{m}^{0,-}(X, E) \rightarrow \Omega_{m}^{0,+}(X, E).
\]
Put
\[
(4.14) \quad \tilde{D}_{b,m}^+ = \bar{D}_{b,m} \bar{D}_{b,m} : \Omega_{m}^{0,+}(X, E) \rightarrow \Omega_{m}^{0,+}(X, E),
\]
We extend \( \tilde{D}_{b,m}^+ \) and \( \tilde{D}_{b,m}^- \) to \( L_{m}^{2,+}(X, E) \) and \( L_{m}^{2,-}(X, E) \) by
\[
(4.15) \quad \tilde{D}_{b,m}^+ : \text{Dom} \tilde{D}_{b,m}^+ \subset L_{m}^{2,+}(X, E) \rightarrow L_{m}^{2,+}(X, E),
\]
where
\[
\text{Dom} \tilde{D}_{b,m}^+ := \{ u \in L_{m}^{2,+}(X, E); \tilde{D}_{b,m}^+ u \in L_{m}^{2,+}(X, E) \},
\]
For every \( s \in \mathbb{R} \), put
\[
H^{s,\pm}(X, E) := \bigoplus_{j \in \{0,1,2,\ldots\}} H^{s}(X, T^{s,0,j} X \otimes E),
\]

\[
H^{s,-}(X, E) := \bigoplus_{j \in \{0,1,2,\ldots\}} H^{s}(X, T^{s,0,j} X \otimes E).
\]

We can repeat the proofs of Lemma 3.3 and Theorem 3.4 and conclude that
\[
(4.16) \quad \text{Dom} \tilde{D}_{b,m}^+ = L_{m}^{2,+}(X, E) \cap H^{2,+}(X, E),
\]
\[
\text{Dom} \tilde{D}_{b,m}^- = L_{m}^{2,-}(X, E) \cap H^{2,-}(X, E),
\]
\( \tilde{D}_{b,m}^+ \) and \( \tilde{D}_{b,m}^- \) are self-adjoint.

Let \( \text{Spec} \tilde{D}_{b,m}^+ \subset [0, \infty[ \) and \( \text{Spec} \tilde{D}_{b,m}^- \subset [0, \infty[ \) denote the spectrum of \( \tilde{D}_{b,m}^+ \) and \( \tilde{D}_{b,m}^- \) respectively. We can repeat the proof of Theorem 3.5 with minor change and deduce that
Theorem 4.4. Spec $\tilde{\square}_{b,m}^+$ and Spec $\tilde{\square}_{b,m}^-$ are discrete subsets of $[0, \infty[$. For any $\mu \in \text{Spec } \tilde{\square}_{b,m}^+$, $\mu$ is an eigenvalue of $\tilde{\square}_{b,m}^+$ and the eigenspace

$$\tilde{E}_{m,\nu}(X, E) := \{ u \in \text{Dom } \tilde{\square}_{b,m}^+; \tilde{\square}_{b,m}^+ u = \nu u \}$$

is finite dimensional with $\tilde{E}_{m,\nu}(X, E) \subset \Omega_{m}^+ (X, E)$. Similarly, for any $\mu \in \text{Spec } \tilde{\square}_{b,m}^-$, $\mu$ is an eigenvalue of $\tilde{\square}_{b,m}^-$ and the eigenspace

$$\tilde{E}_{m,\nu}(X, E) := \{ u \in \text{Dom } \tilde{\square}_{b,m}^-; \tilde{\square}_{b,m}^- u = \nu u \}$$

is finite dimensional with $\tilde{E}_{m,\nu}(X, E) \subset \Omega_{m}^- (X, E)$.

We need

Theorem 4.5. We have Spec $\tilde{\square}_{b,m}^+ \cap [0, \infty[ = \text{Spec } \tilde{\square}_{b,m}^- \cap [0, \infty[$ and for every $\mu \in \text{Spec } \tilde{\square}_{b,m}^+$, $\mu \neq 0$, we have $\dim \tilde{E}_{m,\mu}(X, E) = \dim \tilde{E}_{m,-\mu}(X, E)$.

Proof: Let $\mu \in \text{Spec } \tilde{\square}_{b,m}^+, \mu \neq 0$. Then, there is a function $f \in \Omega_{m}^+(X, E)$, $f \neq 0$, such that $\tilde{\square}_{b,m}^+ f = \mu f$. Consider $\tilde{D}_{b,m} f = g \in \Omega_{m}^0 (X, E)$. Since $\mu \neq 0$, we can check that $g \neq 0$ and $\tilde{\square}_{b,m}^- g = \mu g$. Thus, $\mu \in \text{Spec } \tilde{\square}_{b,m}^-$. We have proved that

$$\dim \tilde{E}_{m,\mu}(X, E) = \dim \tilde{E}_{m,-\mu}(X, E).$$

Conversely, let $\nu \in \text{Spec } \tilde{\square}_{b,m}^-, \nu \neq 0$. Then, there is a function $h \in \Omega_{m}^-(X, E)$, $h \neq 0$, such that $\tilde{\square}_{b,m}^- h = \nu h$. Consider $\tilde{D}_{b,m}^* h = \ell \in \Omega_{m}^+ (X, E)$. Since $\nu \neq 0$, we can check that $\ell \neq 0$ and $\tilde{\square}_{b,m}^+ \ell = \nu \ell$. Thus, $\nu \in \text{Spec } \tilde{\square}_{b,m}^+$. We have proved that

$$\dim \tilde{E}_{m,\nu}(X, E) = \dim \tilde{E}_{m,-\nu}(X, E).$$

From (4.17) and (4.18), we deduce that Spec $\tilde{\square}_{b,m}^+ \cap [0, \infty[ = \text{Spec } \tilde{\square}_{b,m}^- \cap [0, \infty[$.

Let $\mu \in \text{Spec } \tilde{\square}_{b,m}^+, \mu \neq 0$. Consider the map:

$$F_\mu : \tilde{E}_{m,\mu}(X, E) \to \tilde{E}_{m,-\mu}(X, E),$$

$$f \to \tilde{D}_{b,m} f.$$

It is clear that $F_\mu$ is injective. Let $g \in \tilde{E}_{m,\mu}(X, E)$. Then, $\frac{1}{\mu} \tilde{D}_{b,m} g \in \tilde{E}_{m,\mu}(X, E)$, $\frac{1}{\mu} \tilde{D}_{b,m} g \neq 0$ and $F_\mu(\frac{1}{\mu} \tilde{D}_{b,m} g) = g$. Thus, $F_\mu$ is bijective. Hence, $\dim \tilde{E}_{m,\mu}(X, E) = \dim \tilde{E}_{m,-\mu}(X, E)$. The theorem follows.

We pause and introduce some notations. Let $L$ be a complex vector bundle over $X$ of rank $r$ with a Hermitian metric $\langle \cdot, \cdot \rangle_F$. Let $A(x, y) \in C^\infty (X \times X, F_y \otimes F_x)$. Let $\{ f_1, \ldots, f_r \}$ be an orthonormal frame of $F$ with respect to $\langle \cdot, \cdot \rangle_F$ on an open set $D \subset X$. On $D \times D$, we write $A(x, y) = (A_{j,k}(x, y))_{j,k=1}^r$, $A_{j,k}(x, y) \in C^\infty (D \times D)$, $j, k = 1, \ldots, r$, in the sense that fixing $y_0 \in D$, for every $u = \sum_{j=1}^r u_j \otimes f_j(y_0) \in F_{y_0}$, $u_j \in \mathbb{C}$, $j = 1, 2, \ldots, r$, we have

$$A(x, y_0) u = \sum_{j=1}^r (A_{j,1}(x, y_0) u_1 + \cdots + A_{j,r}(x, y_0) u_r) \otimes f_j(x) \in F_x.$$
For every $\nu \in \text{Spec} \tilde{\Gamma}_{b,m}^+$ and $\mu \in \text{Spec} \tilde{\Gamma}_{b,m}^-$, let
\begin{align}
P_{m,\nu}^+ : L^{2,+}(X, E) &\rightarrow \tilde{\Theta}_{m,\nu}^+(X, E), \\
P_{m,\mu}^- : L^{2,+}(X, E) &\rightarrow \tilde{\Theta}_{m,\mu}^-(X, E),
\end{align}
be the orthogonal projections with respect to $\cdot | \cdot$$_E$ and let $P_{m,\nu}^+(x, y) \in \mathcal{C}^\infty(\mathcal{X} \times \mathcal{X}, (T^{s^0,+}_y \mathcal{X} \otimes E_y) \boxtimes (T^{s^0,+}_x \mathcal{X} \otimes E_x))$, $P_{m,\mu}^-(x, y) \in \mathcal{C}^\infty(\mathcal{X} \times \mathcal{X}, (T^{s^0,-}_y \mathcal{X} \otimes E_y) \boxtimes (T^{s^0,-}_x \mathcal{X} \otimes E_x))$ be the distribution kernels of $P_{m,\nu}^+$ and $P_{m,\mu}^-$ respectively. The heat kernels of $\tilde{\Gamma}_{b,m}^+$ and $\tilde{\Gamma}_{b,m}^-$ are given by
\begin{align}
e^{-t\tilde{\Gamma}_{b,m}^+}(x, y) &= P_{m,0}^+(x, y) + \sum_{\nu \in \text{Spec} \tilde{\Gamma}_{b,m}^+, \nu > 0} e^{-\nu t} P_{m,\nu}^+(x, y), \\
e^{-t\tilde{\Gamma}_{b,m}^-}(x, y) &= P_{m,0}^-(x, y) + \sum_{\mu \in \text{Spec} \tilde{\Gamma}_{b,m}^-, \mu > 0} e^{-\mu t} P_{m,\mu}^-(x, y).
\end{align}
Let $e^{-t\tilde{\Gamma}_{b,m}^+} : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}_m(X, E)$ and $e^{-t\tilde{\Gamma}_{b,m}^-} : \Omega^{0,-}(X, E) \rightarrow \Omega^{0,-}_m(X, E)$ be the continuous operators given by
\begin{align}
e^{-t\tilde{\Gamma}_{b,m}^+} : \Omega^{0,+}(X, E) &\rightarrow \Omega^{0,+}_m(X, E), \\
u &\rightarrow \int e^{-t\tilde{\Gamma}_{b,m}^+(x, y)}u(y)dv_X(y), \\
(\partial_t + \tilde{\Gamma}_{b,m}^+) &\rightarrow \Omega^{0,-}_m(X, E), \\
u &\rightarrow \int e^{-t\tilde{\Gamma}_{b,m}^-+(x, y)}u(y)dv_X(y).
\end{align}
Let
\begin{align}
Q_{m}^+ : L^{2,+}(X, E) &\rightarrow L^{2,+}_m(X, E), \\
Q_{m}^- : L^{2,-}(X, E) &\rightarrow L^{2,-}_m(X, E)
\end{align}
be the orthogonal projections. From elementary Fourier analysis and (2.5), it is easy to see that
\begin{align}
Q_{m}^+ : \Omega^{0,+}(X, E) &\rightarrow \Omega^{0,+}_m(X, E), \\
Q_{m}^- : \Omega^{0,-}(X, E) &\rightarrow \Omega^{0,-}_m(X, E),
\end{align}
and
\begin{align}
Q_{m}^+ u &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{-i\theta} \circ x)e^{im\theta}d\theta, \quad \forall u \in \Omega^{0,+}(X, E), \\
Q_{m}^- u &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{-i\theta} \circ x)e^{im\theta}d\theta, \quad \forall u \in \Omega^{0,-}(X, E).
\end{align}
From (4.21), (4.22) and (4.23), it is straightforward to check that
\begin{align}
\frac{\partial}{\partial t} + \tilde{\Gamma}_{b,m}^+(e^{-t\tilde{\Gamma}_{b,m}^+} u) &= 0, \quad \forall u \in \Omega^{0,+}(X, E), \quad \forall t > 0, \\
\lim_{t \rightarrow 0^+} e^{-t\tilde{\Gamma}_{b,m}^+} u &= Q_{m}^+ u, \quad \forall u \in \Omega^{0,+}(X, E),
\end{align}
and
\begin{align}
\frac{\partial}{\partial t} + \tilde{\Gamma}_{b,m}^-(e^{-t\tilde{\Gamma}_{b,m}^-} u) &= 0, \quad \forall u \in \Omega^{0,-}(X, E), \quad \forall t > 0, \\
\lim_{t \rightarrow 0^+} e^{-t\tilde{\Gamma}_{b,m}^-} u &= Q_{m}^- u, \quad \forall u \in \Omega^{0,-}(X, E).
\end{align}
For every \( \nu \in \text{Spec} \, \tilde{\Delta}^+_{b,m} \) and \( \mu \in \text{Spec} \, \tilde{\Delta}^-_{b,m} \), let \( \{ f_1^\nu, \ldots, f_{d_\nu}^\nu \} \) and \( \{ g_1^\mu, \ldots, g_{d_\mu}^\mu \} \) be orthonormal frames for \( \tilde{\mathcal{E}}^+_{m,\nu}(X, E) \) and \( \tilde{\mathcal{E}}^-_{m,\mu}(X, E) \) respectively, where \( d_\nu = \dim \tilde{\mathcal{E}}^+_{m,\nu}(X, E) \), \( d_\mu = \dim \tilde{\mathcal{E}}^-_{m,\mu}(X, E) \).

Define
\[
\text{Tr} \, P^+_{m,\nu}(x, x) := \sum_{j=1}^{d_\nu} \left| f_j^\nu(x) \right|^2 E \in C^\infty(X),
\]
(4.29)
\[
\text{Tr} \, P^-_{m,\mu}(x, x) := \sum_{j=1}^{d_\mu} \left| g_j^\mu(x) \right|^2 E \in C^\infty(X).
\]

Let \( e_1(x), \ldots, e_d(x) \) and \( f_1(x), \ldots, f_d(x) \) be orthonormal frames of \( T_x^{0,+} X \otimes E_x \) and \( T_x^{0,-} X \otimes E_x \) respectively. We can check that
\[
\text{Tr} \, P^+_{m,\nu}(x, x) = \sum_{j=1}^{d_\nu} \langle P^+_{m,\nu}(x, x) e_j(x) | e_j(x) \rangle_E,
\]
\[
\text{Tr} \, P^-_{m,\mu}(x, x) = \sum_{j=1}^{d_\mu} \langle P^-_{m,\mu}(x, x) f_j(x) | f_j(x) \rangle_E.
\]

Note that
\[
\dim \tilde{\mathcal{E}}^+_{m,\nu}(X, E) = \int_X \text{Tr} \, P^+_{m,\nu}(x, x) dv_X(x),
\]
(4.30)
\[
\dim \tilde{\mathcal{E}}^-_{m,\mu}(X, E) = \int_X \text{Tr} \, P^-_{m,\mu}(x, x) dv_X(x).
\]

Put
\[
\text{Tr} \, e^{-t \tilde{\Delta}^+_{b,m}}(x, x) := \text{Tr} \, P^+_{m,0}(x, x) + \sum_{\nu \in \text{Spec} \, \tilde{\Delta}^+_{b,m}, \nu > 0} e^{-\nu t} \text{Tr} \, P^+_{m,\nu}(x, x),
\]
(4.31)
\[
\text{Tr} \, e^{-t \tilde{\Delta}^-_{b,m}}(x, x) := \text{Tr} \, P^-_{m,0}(x, x) + \sum_{\nu \in \text{Spec} \, \tilde{\Delta}^-_{b,m}, \nu > 0} e^{-\nu t} \text{Tr} \, P^-_{m,\nu}(x, x).
\]

From (4.30) and (4.31), we deduce that for every \( t > 0 \),
\[
\int_X \text{Tr} \, e^{-t \tilde{\Delta}^+_{b,m}}(x, x) dv_X(x) = \dim \tilde{\mathcal{E}}^+_{m,0}(X, E) + \sum_{\nu \in \text{Spec} \, \tilde{\Delta}^+_{b,m}, \nu > 0} e^{-\nu t} \dim \tilde{\mathcal{E}}^+_{m,\nu}(X, E),
\]
(4.32)
\[
\int_X \text{Tr} \, e^{-t \tilde{\Delta}^-_{b,m}}(x, x) dv_X(x) = \dim \tilde{\mathcal{E}}^-_{m,0}(X, E) + \sum_{\nu \in \text{Spec} \, \tilde{\Delta}^-_{b,m}, \nu > 0} e^{-\nu t} \dim \tilde{\mathcal{E}}^-_{m,\nu}(X, E).
\]

From Theorem 4.5 and (4.32), we conclude that
\[
\int_X \left( \text{Tr} \, e^{-t \tilde{\Delta}^-_{b,m}}(x, x) - \text{Tr} \, e^{-t \tilde{\Delta}^+_{b,m}}(x, x) \right) dv_X(x) = \dim \text{Ker} \, \tilde{\Delta}^+_{b,m} - \dim \text{Ker} \, \tilde{\Delta}^-_{b,m}.
\]
(4.33)

We extend \( \tilde{D}_{b,m} \) to \( L^2_{m,+}(X, E) \):
\[
\tilde{D}_{b,m} : \text{Dom} \, \tilde{D}_{b,m} \subset L^2_{m,+}(X, E) \to L^2_{m,-}(X, E),
\]
where \( \text{Dom} \, \tilde{D}_{b,m} = \{ u \in L^2_{m,+}(X, E); \, D_{b,m} u \in L^2_{m,-}(X, E) \} \). We also write \( \tilde{D}_{b,m}^* \) to denote the Hilbert space adjoint of \( \tilde{D}_{b,m} \). It is not difficult to see that
\[
\text{Ker} \, \tilde{D}_{b,m} = \text{Ker} \, \tilde{\Delta}^+_{b,m} \subset \Omega^0_{m,+}(X, E),
\]
(4.34)
\[
\text{Ker} \, \tilde{D}_{b,m}^* = \text{Ker} \, \tilde{\Delta}^-_{b,m} \subset \Omega^0_{m,-}(X, E).
\]
Put \( \text{ind } \tilde{D}_{b,m} := \dim \ker \tilde{D}_{b,m} - \dim \ker \tilde{D}_{b,m}^* \). From (4.34) and (4.33), we deduce

**Theorem 4.6.** For every \( t > 0 \), we have

\[
\text{ind } \tilde{D}_{b,m} = \int_X \left( \text{Tr} e^{-t\tilde{D}_{b,m}^*} - \text{Tr} e^{-t\tilde{D}_{b,m}} \right) dv_X(x).
\]

We have

**Theorem 4.7.** We have \( \text{ind } D_{b,m} = \text{ind } \tilde{D}_{b,m} \).

To prove Theorem 4.7, we need some preparations. For every \( t \in [0, 1] \), put \( L_t = \tilde{D}_{b,m} + \tilde{D}_{b,m}^* + tA_m : \Omega_{m}^{0,+}(X, E) \to \Omega_{m}^{1,-}(X, E) \). We extend \( L_t \) to \( L_{m}^{1,+}(X, E) \) and \( L_{m}^{2,-}(X, E) \). Then, \( \text{Dom } \text{L}_t \subset L_{m}^{2,+}(X, E) \to L_{m}^{2,-}(X, E) \), where \( \text{Dom } \text{L}_t = \{ u \in L_{m}^{2,+}(X, E); L_tu \in L_{m}^{2,-}(X, E) \} \). We also write \( L_t^* \) to denote the Hilbert space adjoint of \( L_t \). It is clear that \( L_0 = \tilde{D}_{b,m} \) and \( L_1 = \tilde{D}_{b,m} \). Let \( H_{m}^{1,+}(X, E) \) be the completion of \( \Omega_{m}^{1,+}(X, E) \) with respect to the Hermitian inner product

\[
Q(u, v) = (u \mid v)_E + (\partial_b u \mid \partial_b v)_E + (\partial_b u^* \mid \partial_b v^*)_E.
\]

Then, \( H_{m}^{1,+}(X, E) \) is a Hilbert space and

\[
H_{m}^{1,+}(X, E) = \text{Dom } L_t, \quad \forall t \in \mathbb{R}.
\]

Consider \( H_0 := H_{m}^{1,+}(X, E) \oplus \ker L_0^* \) and \( H_1 = L_{m}^{2,-}(X, E) \oplus \ker L_0 \). Let \( (\cdot \mid \cdot)_{H_0} \) and \( (\cdot \mid \cdot)_{H_1} \) be inner products on \( H_0 \) and \( H_1 \) respectively given by

\[
((f_1, g_1) \mid (f_2, g_2))_{H_0} = Q(f_1, f_2) + (g_1 | g_2)_E, \quad (f_1, g_1, (f_2, g_2) \in H_{m}^{1,+}(X, E) \oplus \ker L_0^*,
\]

\[
((\tilde{f}_1, \tilde{g}_1) \mid (\tilde{f}_2, \tilde{g}_2))_{H_1} = (\tilde{f}_1 \mid \tilde{f}_2)_E + (\tilde{g}_1 \mid \tilde{g}_2)_E, \quad (\tilde{f}_1, \tilde{g}_1, (\tilde{f}_2, \tilde{g}_2) \in L_{m}^{2,-}(X, E) \oplus \ker L_0.
\]

With \((\cdot \mid \cdot)_{H_0}\) and \((\cdot \mid \cdot)_{H_1}\), \( H_0 \) and \( H_1 \) are Hilbert spaces. Let \( \| \cdot \|_{H_0} \) and \( \| \cdot \|_{H_1} \) denote the corresponding norms. Let \( P_{\ker L_0^*} \) denote the orthogonal projection onto \( \ker L_0^* \) with respect to \((\cdot \mid \cdot)_{H_0}\). Let \( A_t : H_0 \to H_1 \) be the linear map defined as follows. Let \((u, v) \in H_0 = H_{m}^{1,+}(X, E) \oplus \ker L_0^* \). Then,

\[
A_t(u, v) = (L_t u + v, P_{\ker L_0^*} u) \in H_1 = L_{m}^{2,-}(X, E) \oplus \ker L_0.
\]

**Lemma 4.8.** There is a \( r > 0 \) such that \( A_t : H_0 \to H_1 \) is invertible, for every \( 0 \leq t \leq r \).

**Proof.** We first claim that

\[
A_0 \text{ is invertible.}
\]

If \( A_0(u, v) = 0 \), for some \((u, v) \in H_0 \). Then,

\[
L_0 u = -v \in \ker L_0^*
\]

and

\[
P_{\ker L_0^*} u = 0.
\]

From (4.38), we have

\[
(L_0 u | L_0 u)_E = -(L_0 u | v)_E = -(v | L_0^* v)_E = 0.
\]

Thus, \( u \in \ker L_0 \). From this and (4.33), we obtain \( u = 0 \) and hence \( v = 0 \). We have proved that \( A_0 \) is injective. We now prove that \( A_0 \) is surjective. Let \((a, b) \in H_1 = L_{m}^{2,-}(X, E) \oplus \ker L_0 \). Since \( L_0 : \text{Dom } L_0 \to L_{m}^{2,-}(X, E) \) has \( L_0 \)-closed range, we have

\[
a = L_0 \alpha + \beta, \quad \alpha \in H_{m}^{1,+}(X, E), \quad \alpha \perp \ker L_0, \quad \beta \in (\text{Rang } L_0)^\perp = \ker L_0^*.
\]

From (4.41), we can check that

\[
A_0(\alpha + b, \beta) = (L_0(\alpha + b) + \beta, P_{\ker L_0}(\alpha + b)) = (L_0 \alpha + \beta, b) = (a, b).
\]
Thus, $A_0$ is surjective. The claim (4.37) follows.

Let $A_0^{-1} : H_1 \to H_0$ be the inverse of $A_0$. From open mapping theorem, $A_0^{-1}$ is continuous. It is clear that $A_t = A_0 + R_t$, where $R_t : H_0 \to H_1$ is continuous and there is a constant $c > 0$ such that $\|R_t u\|_{H_1} \leq ct \|u\|_{H_0}$, for every $u \in H_0$. Put

$$H_t = I - A_0^{-1} R_t + (A_0^{-1} R_t)^2 - (A_0^{-1} R_t)^3 + \cdots ,$$
$$\tilde{H}_t = I - R_t A_0^{-1} + (R_t A_0^{-1})^2 - (R_t A_0^{-1})^3 + \cdots .$$

Since $A_0^{-1}$ is continuous, $H_t : H_0 \to H_0$ and $\tilde{H}_t : H_1 \to H_1$ are well-defined as continuous maps, for $t \geq 0$, $t$ small. Moreover, we can check that $A_t \circ (H_t \circ A_0^{-1}) = I$ on $H_1$ and $(A_0^{-1} \circ \tilde{H}_t) \circ A_t = I$ on $H_0$,

for $t \geq 0$, $t$ small. Thus, there is a $r > 0$ such that $A_t : H_0 \to H_1$ is invertible, for every $0 \leq t \leq r$. The lemma follows. □

For every $t \in [0, 1]$, let

$$L_t^* : \text{Dom } L_t^* \subset L_m^2(X, E) \to L_m^2(X, E)$$

be the $L^2$ space adjoint of $L_t$ with respect to $(\cdot, \cdot)_E$. It is clear that for every $t \in [0, 1]$, $\text{Ker } L_t \subset \Omega_m^0(X, E)$, $\text{Ker } L_t^* \subset \Omega_m^0(X, E)$ and $\dim \text{Ker } L_t < \infty$, $\dim \text{Ker } L_t^* < \infty$. Put $\text{ind } L_t := \dim \text{Ker } L_t - \dim \text{Ker } L_t^*$. We have

**Lemma 4.9.** There is a $r_0 > 0$ such that $\text{ind } L_t = \text{ind } L_0$, for every $0 \leq t \leq r_0$.

**Proof.** Let $r > 0$ be as in Lemma 4.8. We first claim that

(4.42) $\text{ind } L_0 \leq \text{ind } L_t$, $\forall 0 \leq t \leq r$.

Fix $0 \leq t \leq r$. Let

$$B : \text{Ker } L_t^* \oplus \text{Ker } L_0 \to \text{Ker } L_t \oplus \text{Ker } L_0^*$$

be the linear operator defined as follows. Let $(a, b) \in \text{Ker } L_t^* \oplus \text{Ker } L_0$. By Lemma 4.8,

$$A_t : H_m^1(X, E) \oplus \text{Ker } L_0^* \to L_m^2(X, E) \oplus \text{Ker } L_0$$

is invertible. There is a unique $(u, v) \in H_m^1(X, E) \oplus \text{Ker } L_0^*$ such that $A_t(u, v) = (a, b)$. Let $P_{\text{Ker } L_t^*} : L_m^2(X, E) \to \text{Ker } L_t^*$ be the orthogonal projection with respect to $(\cdot, \cdot)_E$. Define $B(a, b) := (P_{\text{Ker } L_t^*} u, v) \in \text{Ker } L_t \oplus \text{Ker } L_0^*$. We now show that $B$ is injective. Assume $B(a, b) = (0, 0)$ for some $(a, b) \in \text{Ker } L_t^* \oplus \text{Ker } L_0$. Let $(u, v) \in H_m^1(X, E) \oplus \text{Ker } L_0^*$ with $A_t(u, v) = (a, b)$. Then, $B(a, b) = (P_{\text{Ker } L_t^*} u, v) = (0, 0)$. Thus, $P_{\text{Ker } L_t^*} u = 0$ and $v = 0$. From the definition of $A_t$, we have

$$A_t(u, v) = A_t(u, 0) = (L_t u, P_{\text{Ker } L_0} u) = (a, b) \in \text{Ker } L_t^* \oplus \text{Ker } L_0.$$

Thus, $a = L_t u \in \text{Ker } L_t^*$. Hence, $(a | a)_E = (a | L_t u)_E = (L_t^* a | u)_E = 0$. We conclude that $u \in \text{Ker } L_t$ and $u = P_{\text{Ker } L_t} u = 0$. We obtain $(a, b) = (L_t u, P_{\text{Ker } L_0} u) = (0, 0)$ and hence $B$ is injective. Since $B$ is injective, we have

(4.43) $\dim \text{Ker } L_t^* + \dim \text{Ker } L_0 \leq \dim \text{Ker } L_t + \dim \text{Ker } L_0^*$.

From (4.43), the claim (4.42) follows.

Similarly, we can repeat the procedure above and deduce that there is a $r_1 > 0$ such that

(4.44) $\text{ind } L_0^* \leq \text{ind } L_t^*$, $\forall 0 \leq t \leq r_1$.

It is not difficult to check that $\text{ind } L_t^* = -\text{ind } L_t$, for every $t \in [0, 1]$. From this observation and (4.44), we conclude that

(4.45) $\text{ind } L_0 \geq \text{ind } L_t$, $\forall 0 \leq t \leq r_1$.

From (4.45) and (4.42), the lemma follows. □
Proof of Theorem 4.7. Let

\[ I_0 := \{ r \in [0, 1] ; \text{there is a } \varepsilon > 0 \text{ such that } \text{ind } L_t = \text{ind } L_0, \forall t \in (r - \varepsilon, r + \varepsilon) \cap [0, 1] \}. \]

From the definition of \( I_0 \) and Lemma 4.9, it is clear that \( I_0 \) is a non-empty open set in \([0, 1]\). Let \( r_j \in I_0 \), \( j = 1, 2, \ldots \), with \( \lim_{j \to \infty} r_j = r_\infty \in [0, 1] \). We can repeat the proof of Lemma 4.9 and deduce that there is an \( \varepsilon_0 > 0 \) such that

\[ (4.46) \quad \text{ind } L_t = \text{ind } L_{r_\infty}, \forall t \in (r_\infty - \varepsilon_0, r_\infty + \varepsilon_0). \]

Since for \( j \) large, \( r_j \in (r_\infty - \varepsilon_0, r_\infty + \varepsilon_0) \), we have

\[ (4.47) \quad \text{ind } L_{r_j} = \text{ind } L_0, \quad \text{for } j \text{ large.} \]

Note that \( (4.46) \) and \( (4.47) \), we deduce that

\[ \text{ind } L_t = \text{ind } L_0, \forall t \in (r_\infty - \varepsilon_0, r_\infty + \varepsilon_0). \]

Thus, \( r_\infty \in I_0 \) and hence \( I_0 \) is closed in \([0, 1]\). We conclude that \( I_0 = [0, 1] \). Theorem 4.7 follows. \( \square \)

From Theorem 4.7, Theorem 4.6 and (3.25), we get

**Theorem 4.10.** Fix \( m \in \mathbb{Z} \). For every \( t > 0 \), we have

\[ (4.48) \quad \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \int_X \left( \text{Tr} e^{-\frac{t}{\varepsilon} b,m(x, x)} - \text{Tr} e^{-\frac{t}{\varepsilon} b,m(x, x)} \right) dv_X(x). \]

**Remark 4.11.** Let \( e^{-\frac{t}{\varepsilon} b,m} \) and \( e^{-\frac{t}{\varepsilon} b,m} \) be the heat operators of \( \square_{b,m}^+ \) and \( \square_{b,m}^- \) respectively. As Theorem 4.10, we have

\[ \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \int_X \left( \text{Tr} e^{-\frac{t}{\varepsilon} b,m(x, x)} - \text{Tr} e^{-\frac{t}{\varepsilon} b,m(x, x)} \right) dv_X(x). \]

When \( X \) is not CR Kähler, it is difficult to calculate \( \text{Tr} e^{-\frac{t}{\varepsilon} b,m(x, x)} - \text{Tr} e^{-\frac{t}{\varepsilon} b,m(x, x)} \) and therefore we consider modified Kohn Laplacians \( \square_{b,m}^+, \square_{b,m}^- \). It should be noticed that when \( X \) is CR Kähler, \( \square_{b,m}^+, \square_{b,m}^- \) and \( \square_{b,m}^-, \square_{b,m}^- \) are closed in \( \Omega^{0,q}(U, E \otimes L^m) \).

## 5. Asymptotic expansions for the heat kernels of the modified Kohn Laplacians

### 5.1 Heat kernels of the modified Kodaira Laplacians on BRT trivializations

From now on, we fix \( m \in \mathbb{Z} \). Let \( B := (D, (z, \theta), \varphi) \) be a BRT trivialization. We may assume that \( D = U \times \mathbb{C} \), where \( \varepsilon > 0 \) and \( U \) is an open set of \( \mathbb{C}^n \). Since \( E \) is rigid, in Section 4, we can consider \( E \) as a holomorphic vector bundle over \( U \). We may assume that \( E \) is trivial on \( U \). We will use the same notations as in Section 4. Consider \( L \to U \) be a trivial line bundle with non-trivial Hermitian fiber metric \( |\cdot|_h^L = e^{-2\varphi} \). Let \( (L^m, h_L^m) \to U \) be the \( m \)-th power of \( (L, h_L^m) \). Let \( \Omega^{0,q}(U, E \otimes L^m) \) be the space of \( (0, q) \) forms on \( U \) with values in \( E \otimes L^m, q = 0, 1, 2, \ldots, n \). Put

\[ \Omega^{0,+}(U, E \otimes L^m) := \oplus_{j\in\{0,1,\ldots,n\}} \text{even} \Omega^{0,j}(U, E \otimes L^m), \]

\[ \Omega^{0,-}(U, E \otimes L^m) := \oplus_{j\in\{0,1,\ldots,n\}} \text{odd} \Omega^{0,j}(U, E \otimes L^m). \]

Since \( L \) is trivial, from now on, we identify \( \Omega^{0,q}(U, E) \) with \( \Omega^{0,q}(U, E \otimes L^m) \), \( q = 0, 1, 2, \ldots, n \). Since the Hermitian fiber metric \( \langle \cdot, \cdot \rangle \) is rigid, we can consider \( \langle \cdot, \cdot \rangle \) as a Hermitian fiber metric on the holomorphic vector bundle \( E \) over \( U \). Let \( \langle \cdot, \cdot \rangle \) be the Hermitian metric on \( CTU \) given by

\[ \langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \rangle = \langle \frac{\partial}{\partial z_j} - i \frac{\partial}{\partial \theta} (\varphi(z) \frac{\partial}{\partial \theta} - i \frac{\partial}{\partial \theta} (\varphi(z) \frac{\partial}{\partial \theta}), j, k = 1, 2, \ldots, n. \]

\( \langle \cdot, \cdot \rangle \) induces Hermitian metrics on \( T^{0,q}U \) bundle of \( (0, q) \) forms on \( U, q = 0, 1, \ldots, n \). We shall also denote the Hermitian metrics by \( \langle \cdot, \cdot \rangle \) The Hermitian metrics on \( T^{0,q}U \) and \( E \) induce Hermitian
metrics on $T^{*a,q}U \otimes E$, $q = 0, 1, \ldots, n$. We shall also denote these induced metrics by $\langle \cdot | \cdot \rangle_E$. Let $(\cdot, \cdot)$ be the $L^2$ inner product on $\Omega^{0,q}(U, E)$ induced by $\langle \cdot, \cdot \rangle_E$, $q = 0, 1, 2, \ldots, n$. Similarly, let $(\cdot, \cdot)_m$ be the $L^2$ inner product on $\Omega^{0,q}(U, E \otimes L^m)$ induced by $\langle \cdot, \cdot \rangle_E$ and $h^L_m$, $q = 0, 1, 2, \ldots, n$.

Let

$$\overline{\partial} : \Omega^{0,q}(U, E \otimes L^m) \to \Omega^{0,q+1}(U, E \otimes L^m), \quad q = 0, 1, 2, \ldots, n-1,$$

be the Cauchy-Riemann operator. Let

$$\overline{\partial}^* : \Omega^{0,q+1}(U, E \otimes L^m) \to \Omega^{0,q}(U, E \otimes L^m), \quad q = 0, 1, 2, \ldots, n-1,$$

be the formal adjoint of $\overline{\partial}$ with respect to $(\cdot, \cdot)_m$. Put

$$D_{B,m} := \overline{\partial} + \overline{\partial}^* + A_B : \Omega^{0,+}(U, E \otimes L^m) \to \Omega^{0,-}(U, E \otimes L^m),$$

where $A_B : \Omega^{0,+}(U, E \otimes L^m) \to \Omega^{0,-}(U, E \otimes L^m)$ is as in (4.7). Let

$$D_{B,m}^* : \Omega^{0,-}(U, E \otimes L^m) \to \Omega^{0,+}(U, E \otimes L^m)$$

be the formal adjoint of $D_{B,m}$ with respect to $(\cdot, \cdot)_m$. Put

$$\square^\pm_{B,m} := D_{B,m}^* D_{B,m} : \Omega^{0,+}(U, E \otimes L^m) \to \Omega^{0,+}(U, E \otimes L^m),$$

$$\square^0_{B,m} := D_{B,m} D_{B,m}^* : \Omega^{0,-}(U, E \otimes L^m) \to \Omega^{0,-}(U, E \otimes L^m).$$

We need

**Lemma 5.1.** Let $u \in \Omega^0_m(X, E)$, $v \in \Omega^0_m(X, E)$. On $D$, we write $u(z, \theta) = e^{-im\theta}u(z), v(z, \theta) = e^{-im\theta}v(z), \tilde{u}(z) \in \Omega^0+(U, E), \tilde{v}(z) \in \Omega^0-(U, E)$. Then,

$$e^{-im\theta} \tilde{\square}_m^+ (e^{im\theta} u) = e^{im\theta} \tilde{\square}_m^+ (u),$$

$$e^{-im\theta} \tilde{\square}_m^- (e^{im\theta} v) = e^{im\theta} \tilde{\square}_m^- (v).$$

**Proof.** From (2.15), it is easy to see that for every $g \in \Omega^{0,q}(U, E \otimes L^m)$, we have

$$e^{im\theta} \tilde{\square}_m (e^{-im\theta} g(z)) = \overline{\partial}_m g(z),$$

where

$$\overline{\partial}_m = \overline{\partial} + m \sum_{j=1}^n \frac{\partial \varphi(z)}{\partial z_j} dz_j \wedge : \Omega^{0,q}(U, E \times L^m) \to \Omega^{0,q+1}(U, E \otimes L^m).$$

From (5.5) and (5.6), we have for every $h \in \Omega^{0,q+1}(U, E \otimes L^m),

$$e^{im\theta} \tilde{\square}_m (e^{-im\theta} h(z)) = \overline{\partial}_m h(z),$$

where

$$\overline{\partial}_m^* = \overline{\partial}^* + m \sum_{j=1}^n \frac{\partial \varphi(z)}{\partial z_j} (dz_j \wedge) : \Omega^{0,q+1}(U, E \otimes L^m) \to \Omega^{0,q}(U, E \otimes L^m).$$

Here $\overline{\partial}^* : \Omega^{0,q+1}(U, E \otimes L^m) \to \Omega^{0,q}(U, E \otimes L^m)$ is the formal adjoint of $\overline{\partial}$ with respect to $(\cdot, \cdot)$ and $(dz_j \wedge)^* : T^{*q,0+1}U \to T^{*0,q}U$ is the adjoint of $(dz_j \wedge)$. That is, $\langle dz_j \wedge \alpha, \beta \rangle = \langle \alpha, (dz_j \wedge)^* \beta \rangle$, for every $\alpha \in T^{*q,0}U, \beta \in T^{*0,q+1}U$. Moreover, it is straightforward to check that

$$\overline{\partial}^* m \sum_{j=1}^n \frac{\partial \varphi(z)}{\partial z_j} (dz_j \wedge) = \overline{\partial}_m + m \sum_{j=1}^n \frac{\partial \varphi(z)}{\partial z_j} (dz_j \wedge)^*.$$
for every \( h \in \Omega^{0,q}(U, E \otimes L^m) \). Let \( u \in \Omega^0_m(X, E) \). On \( D \), we write \( u(z, \theta) = e^{-im\theta} \tilde{u}(z), \tilde{u}(z) \in \Omega^0(U, E) \). From (5.10) and (5.11), we conclude that
\[
e^{-m\varphi(\overline{\partial} + \overline{\partial}^m)}(e^{m\varphi} \tilde{u}) = e^{im\theta}(\overline{\partial}_b + \overline{\partial}_b^m)(u).
\]
Moreover, from the definition of the zero order operator \( A_m : \Omega^0_m(X, E) \to \Omega^0_m(X, E) \) (see Definition 4.3), it is clear that
\[
e^{-m\varphi} A_B(e^{m\varphi} \tilde{u}) = e^{im\theta} A_m(u).
\]
From (5.11), (5.12) and (5.13), we get
\[
e^{-m\varphi} D_B(m, e^{m\varphi} \tilde{u}) = e^{im\theta} \overline{D}_b(m)(u)
\]
and hence
\[
e^{-m\varphi} \overline{\square}_{b}(m, e^{m\varphi} \tilde{u}) = e^{im\theta} \overline{\square}_{b,m}(u).
\]
Let \( v \in \Omega^0_m(X, E) \). On \( D \), we write \( v(z, \theta) = e^{-im\theta} \tilde{v}(z), \tilde{v}(z) \in \Omega^0(U, E) \). We can repeat the procedure above and deduce that
\[
e^{-m\varphi} \overline{\square}_b(m, e^{m\varphi} \tilde{v}) = e^{im\theta} \overline{\square}_{b,m}(v).
\]
The lemma follows.

We introduce some notations. Let \( M \) be a \( C^\infty \) orientable paracompact manifold and let \( F \) be a vector bundle over \( M \). We need

**Definition 5.2.** Let \( A(t, x) \in C^\infty(\mathbb{R}_+ \times M, F) \). We write
\[
A(t, x) \sim t^k b_{-k}(x, t) + t^{k+1} b_{-k+1}(x, t) + t^{k+2} b_{-k+2}(x, t) + \cdots \text{ as } t \to 0^+,
\]
where \( k \in \mathbb{Z} \), if for every compact set \( K \subseteq M \), every \( m \in \mathbb{N} \) and every \( N_0 > 0 \), there are \( C_{m,K,N_0} > 0 \) and \( \varepsilon_0 > 0 \) independent of \( t \) such that
\[
\left| A(t, x) - \sum_{j=0}^{N_0} t^{k+j} b_{-k+j}(x, t) \right|_{C^m(K)} \leq C_{m,K,N_0} t^{N_0+1}, \quad 0 < t < \varepsilon_0.
\]
Put
\[
T^{a_0,0^+} U := \bigoplus_{j=0}^{\infty} T^{a_0,j} U, \quad T^{a_0,0^-} U := \bigoplus_{j=0}^{\infty} T^{a_0,j} U.
\]
Let \( z, w \in U \) and let \( T(z, w) \in (T^{a_0,0^+} U \otimes E_w) \boxplus (T^{a_0,0^-} U \otimes E_z) \). We write \( |T(z, w)| \) to denote the standard pointwise matrix norm of \( T(z, w) \) induced by \( (\cdot, \cdot) \) and \( (\cdot | \cdot)_E \). Let \( \Omega^{0,+}_0(U, E) \) and \( \Omega^{0,-}_0(U, E) \) be the subspaces of \( \Omega^{0,+}(U, E) \) and \( \Omega^{0,+}(U, E) \) whose elements have compact support in \( U \) respectively. Let \( dv_U \) be the volume form on \( U \) induced by \( (\cdot, \cdot) \). Let \( u \in \Omega^{a_0}(U, E) \). We define the integral \( \int T(z, w)u(w)dv_U(w) \) as in (4.19). Let \( G(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{a_0,0^+} U \otimes E_w) \boxplus (T^{a_0,0^-} U \otimes E_z)) \). We write \( G(t) \) to denote the continuous operator
\[
G(t) : \Omega^{a_0}_0(U, E) \to \Omega^{a_0}_0(U, E),
\]
\[
u \to \int G(t, z, w)u(w)dv_U(w)
\]
and we write \( G'(t) \) to denote the continuous operator
\[
G'(t) : \Omega^{a_0}_0(U, E) \to \Omega^{a_0}_0(U, E),
\]
\[
u \to \int \partial G(t, z, w)u(w)dv_U(w).
\]
We consider the heat operators of $\Box_{B,m}^+$ and $\Box_{B,m}^-$. By using the Dirichlet heat kernel construction (see [19]), we deduce the following two theorems

**Theorem 5.3.** There is $A_{B,+}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{t^*0,+}_w U \otimes E_w) \boxplus (T^{t^*0,+}_z U \otimes E_z))$ such that

$$\lim_{t \to 0^+} A_{B,+}(t) = I \text{ in } \mathcal{D}'(U, T^{t^*0,+}_U \otimes E),$$

and $A_{B,+}(t, z, w)$ satisfies the following: (I) For every compact set $K \subset U$ and every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0^d$, every $\gamma \in \mathbb{N}_0$, there are constants $C_{\gamma,\alpha_1,\alpha_2,\beta_1,\beta_2,K} > 0$ and $\varepsilon_0 > 0$ independent of $t$ such that

$$\left| \frac{\partial^\alpha g}{\partial z^\alpha \partial w^\gamma} A_{B,+}(t, z, w) \right| \leq C_{\gamma,\alpha_1,\alpha_2,\beta_1,\beta_2,K} e^{-\varepsilon_0 \frac{|z-w|^2}{t}}, \quad (t, z, w) \in \mathbb{R}_+ \times K \times K.$$  

(II) Let $g \in \mathcal{O}_{\mathbb{N}^d_0}^0(U, E)$. For every $\alpha_1, \alpha_2 \in \mathbb{N}_0^d$ and every compact set $K \subset U$, there is a $C_{\alpha_1,\alpha_2,K} > 0$ independent of $t$ such that

$$\sup \left\{ \left| \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial w^{\gamma}} (A_{B,+}(t) g)(z) ; z \in K \right| \right\} \leq C_{\alpha_1,\alpha_2,K} \sum_{\beta_1,\beta_2 \in \mathbb{N}_0^d, |\beta_1| + |\beta_2| \leq |\alpha_1| + |\alpha_2|} \sup \left\{ \left| \frac{\partial^{\beta_1}}{\partial z^{\beta_1}} \frac{\partial^{\beta_2}}{\partial w^{\gamma}} g(z) ; z \in U \right| \right\}.$$  

(III) $A_{B,+}(t, z, w)$ admits an asymptotic expansion:

$$A_{B,+}(t, z, w) = e^{-\frac{h_+(z,w)}{t}} K_{B,+}(t, z, w),$$

where $h_+(z, w) \in C^\infty(U \times U, (T^{t^*0,-}_w U \otimes E_w) \boxminus (T^{t^*0,-}_z U \otimes E_z))$ with $h_+(z, z) = 0$ for every $z \in U$ and for every compact set $K \subset U$, there is a constant $C_K > 1$ such that $\frac{1}{K} |z-w|^2 \leq h_+(z, w) \leq C_K |z-w|^2.$

**Theorem 5.4.** There is $A_{B,-}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{t^*0,-}_w U \otimes E_w) \boxplus (T^{t^*0,-}_z U \otimes E_z))$ such that

$$\lim_{t \to 0^+} A_{B,-}(t) = u \text{ in } \mathcal{D}'(U, T^{t^*0,-}_U \otimes E),$$

and $A_{B,-}(t, z, w)$ satisfies the following: (I) For every compact set $K \subset U$ and every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0^d$, every $\gamma \in \mathbb{N}_0$, there are constants $C_{\gamma,\alpha_1,\alpha_2,\beta_1,\beta_2,K} > 0$ and $\varepsilon_0 > 0$ independent of $t$ such that

$$\left| \frac{\partial^\alpha g}{\partial z^\alpha \partial w^\gamma} A_{B,-}(t, z, w) \right| \leq C_{\gamma,\alpha_1,\alpha_2,\beta_1,\beta_2,K} e^{-\varepsilon_0 \frac{|z-w|^2}{t}}, \quad (t, z, w) \in \mathbb{R}_+ \times K \times K.$$  

(II) Let $g \in \mathcal{O}_{\mathbb{N}^d_0}^0(U, E)$. For every $\alpha_1, \alpha_2 \in \mathbb{N}_0^d$ and every compact set $K \subset U$, there is a $C_{\alpha_1,\alpha_2,K} > 0$ independent of $t$ such that

$$\sup \left\{ \left| \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial w^{\gamma}} (A_{B,-}(t) g)(z) ; z \in K \right| \right\} \leq C_{\alpha_1,\alpha_2,K} \sum_{\beta_1,\beta_2 \in \mathbb{N}_0^d, |\beta_1| + |\beta_2| \leq |\alpha_1| + |\alpha_2|} \sup \left\{ \left| \frac{\partial^{\beta_1}}{\partial z^{\beta_1}} \frac{\partial^{\beta_2}}{\partial w^{\gamma}} g(z) ; z \in U \right| \right\}.$$  

(III) $A_{B,-}(t, z, w)$ admits an asymptotic expansion:

$$A_{B,-}(t, z, w) = e^{-\frac{h_-(z,w)}{t}} K_{B,-}(t, z, w),$$

where $h_-(z, w) \in C^\infty(U \times U, (T^{t^*0,-}_w U \otimes E_w) \boxminus (T^{t^*0,-}_z U \otimes E_z))$, $s = n, n-1, n-2, \ldots$.
where \( h_{\pm}(z, w) \in C^\infty(U \times U, \mathbb{R}_+) \) with \( h_{\pm}(z, z) = 0 \) for every \( z \in U \) and for every compact set \( K \subseteq U \), there is a constant \( C_K > 1 \) such that \( \frac{1}{C_K} |z - w|^2 \leq h_{\pm}(z, w) \leq C_K |z - w|^2 \).

Let \( e_1(z), \ldots, e_d(z) \) and \( f_1(z), \ldots, f_d(z) \) be orthonormal frames of \( T_z^{0,1}U \otimes E_z \) and \( T_z^{0,0}U \otimes E_z \) with respect to \( \langle \cdot, \cdot \rangle_E \) respectively. Fix \( j = n, n-1, \ldots \). Let

\[
b_j^+(z, z) \in C^\infty(U, \text{End}(T_z^{0,1}U \otimes E_z)), \quad b_j^-(z, z) \in C^\infty(U, \text{End}(T_z^{0,0}U \otimes E_z))
\]

be as in (5.17) and (5.21) respectively. Define

\[
(5.22) \quad \text{Tr} b_j^+(z, z) = \sum_{s=1}^{d} (b_j^+(z, z)e_s(z) | e_s(z))_E, \quad \text{Tr} b_j^-(z, z) = \sum_{s=1}^{d} (b_j^-(z, z)f_s(z) | f_s(z))_E.
\]

From Theorem 1.3.5 and Theorem 1.4.5 in [27], we have Lichnerowicz formulas for \( \tilde{\square}_{B,m}^+ \) and \( \tilde{\square}_{B,m}^- \). Thus, we can apply the rescaling technique in [2] and [8] and calculate \( \text{Tr} b_j^+(z) - \text{Tr} b_j^-(z) \), \( j = n, n-1, \ldots, 0 \). The calculation is the same as in the standard case and therefore we omit the detail. To state the result precisely, we introduce some notations. Let \( \nabla^{T_U} \) be the Levi-Civita connection on \( CT_U \) with respect to \( \langle \cdot, \cdot \rangle_E \). Let \( P_{T^{1,0}U} \) be the natural projection from \( CT_U \) onto \( T^{1,0}U \). Then, \( \nabla^{T^{1,0}U} := P_{T^{1,0}U} \nabla^{T_U} \) is a connection on \( T^{1,0}U \). Let \( \nabla^{E \otimes L^m} \) be the connection on \( E \otimes L^m \to U \) induce by \( \langle \cdot, \cdot \rangle_E \) and \( h_{L^m} \). Let \( \Theta(\nabla^{T^{1,0}U}, T^{1,0}U) \subseteq C^\infty(U, \Lambda^2(CT^{1,0}U) \otimes \text{End}(T^{1,0}U)) \) and \( \Theta(\nabla^{E \otimes L^m}, E \otimes L^m) \subseteq C^\infty(U, \Lambda^2(CT^*U) \otimes \text{End}(E \otimes L^m)) \) be the curvatures induced by \( \nabla^{T^{1,0}U} \) and \( \nabla^{E \otimes L^m} \) respectively. As in complex geometry, put

\[
\text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right) = e^{\text{Tr}(\frac{i}{2\pi} \Theta(\nabla^{T^{1,0}U}, T^{1,0}U))}, \quad h(z) = \log(\frac{z}{1 - e^{-z}}),
\]

\[
\text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) = \text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right), \quad \tilde{h}(z) = e^z.
\]

We have

\[
(5.23) \quad \left( \text{Tr} b_j^+(z, z) - \text{Tr} b_j^-(z, z) \right) = 0, \quad \forall z \in U, \quad j = n, n-1, \ldots, 1,
\]

\[
\left( \text{Tr} b_0^+(z, z) - \text{Tr} b_0^-(z, z) \right) d\nu_U(z) = \left[ \text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) \right]_{2n}(z), \quad \forall z \in U,
\]

where \( \left[ \text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) \right]_{2n} \) denotes the \( 2n \) forms part of

\[
\text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right).
\]

Let \( \nabla^{T_X} \) be the Levi-Civita connection on \( TX \) with respect to \( \langle \cdot, \cdot \rangle_E \). Let \( P_{T^{1,0}X} \) be the natural projection from \( CT_X \) onto \( T^{1,0}X \). Then, \( \nabla^{T^{1,0}X} := P_{T^{1,0}X} \nabla^{T_X} \) is a connection on \( T^{1,0}X \). Let \( \nabla^E \) be the connection on \( E \) induced by \( \langle \cdot, \cdot \rangle_E \). Since \( \langle \cdot, \cdot \rangle_E \) and \( \langle \cdot, \cdot \rangle_E \) are rigid, we can check that \( \nabla^{T^{1,0}X} \) and \( \nabla^E \) are rigid. Moreover, it is straightforward to check that

\[
(5.24) \quad \text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right)(z) = \text{Td}_b \left( \nabla^{T^{1,0}X}, T^{1,0}X \right)(z, \theta), \quad \forall (z, \theta) \in D,
\]

\[
\text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right)(z) = \left( \text{ch}_b \left( \nabla^E, E \right) \wedge e^{-m \frac{\text{d} \omega_b}{2\pi}} \right)(z, \theta), \quad \forall (z, \theta) \in D,
\]

and

\[
\left[ \text{Td} \left( \nabla^{T^{1,0}U}, T^{1,0}U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) \right]_{2n}(z) \wedge d\theta
\]

\[
= \left[ \text{Td}_b \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge \text{ch}_b \left( \nabla^E, E \right) \wedge e^{-m \frac{\text{d} \omega_b}{2\pi}} \wedge \omega_b \right]_{2n+1}(z, \theta), \quad \forall (z, \theta) \in D,
\]

where \( \left[ \text{Td}_b \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge \text{ch}_b \left( \nabla^E, E \right) \wedge e^{-m \frac{\text{d} \omega_b}{2\pi}} \wedge \omega_b \right]_{2n+1} \) denotes the \( 2n + 1 \) forms part of

\[
\text{Td}_b \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge \text{ch}_b \left( \nabla^E, E \right) \wedge e^{-m \frac{\text{d} \omega_b}{2\pi}} \wedge \omega_b.
\]
From (5.23), (5.25) and note that \(dv_U \wedge d\theta = dv_X\) on \(D\), we get

**Theorem 5.5.** With the notations above, we have

\[
\begin{align*}
\left( \text{Tr} b_j^+(z, z) - \text{Tr} b_j^-(z, z) \right) &= 0, \quad \forall z \in U, \quad j = n, n - 1, \ldots, 1, \\
\left( \text{Tr} b_0^+(z, z) - \text{Tr} b_0^-(z, z) \right) dv_X(z, \theta) \\
&= \left[ Td_b \left( \nabla^{T_1,0} X, T_{1,0} X \right) \wedge c_h \left( \nabla^E, E \right) \wedge e^{-m d_{uv} \frac{dv}{\pi r}} \wedge \omega_0 \right]_{2n+1} (z, \theta), \quad \forall (z, \theta) \in D.
\end{align*}
\]

(5.26)

5.2. Heat kernels of the modified Kohn Laplacians. From now on, we fix \(m \in \mathbb{Z}\). Assume that \(X = D_1 \cup D_2 \cup \cdots \cup D_N\), where \(B_j := \left( D_j, (z, \theta), \varphi_j \right)\) is a BRT trivialization, for each \(j\). We may assume that for each \(j\), \(D_j = U_j \times [-\delta_j, \delta_j] \subset \mathbb{C}^n \times \mathbb{R}\), \(\delta_j > 0\), \(\delta_j > 0\), \(U_j\) is an open set in \(\mathbb{C}^n\). Let \(\chi_j \in C_0^\infty(D_j), j = 1, 2, \ldots, N\), with \(\sum_{j=1}^N \chi_j = 1\) on \(X\). Fix \(j = 1, 2, \ldots, N\). Put

\[
K_j = \left\{ z \in U_j; \text{there is a } \theta \in [-\delta_j, \delta_j] \text{ such that } \chi_j(z, \theta) \neq 0 \right\}.
\]

Let \(\tau_j(z) \in C_0^\infty(U_j)\) with \(\tau_j \equiv 1\) on some neighborhood \(W_j\) of \(K_j\). Let \(\sigma_j \in C_0^\infty(]-\delta_j, \delta_j[)\) with \(\int \sigma_j(\theta) d\theta = 1\). Let \(A_{B_j, +}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U_j \times U_j, (T_{w}^{0,1}U_j \otimes E_w) \boxtimes (T_{w}^{0,1}U_j \otimes E_z))\) be as in Theorem 5.3. Put

\[
H_j(t, x, y) = \chi_j(x) e^{-m \varphi_j(z) - \imath m \theta} A_{B_j, +}(t, z, w) e^{m \varphi_j(w) + \imath m \eta} \tau_j(w) \sigma_j(\eta),
\]

where \(x = (z, \theta), y = (w, \eta) \in \mathbb{C}^n \times \mathbb{R}\). Let \(H_j(t)\) be the continuous operator

\[
H_j(t) : \Omega^{0, +}(X, E) \to \Omega^{0, +}(X, E),
\]

(5.28)

\[
u \mapsto \chi_j(x) e^{-m \varphi(z) - \imath m \theta} A_{B_j, +}(t, z, w) e^{m \varphi_j(w) + \imath m \eta} \tau_j(w) \sigma_j(\eta) u(y) dv_X(y).
\]

Consider

\[
\Gamma(t) := \sum_{j=1}^N Q_m \circ H_j(t) \circ Q_m : \Omega^{0, +}(X, E) \to \Omega^{0, +}_m(X, E)
\]

and let \(\Gamma(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T_y^{0,1}X \otimes E_y) \boxtimes (T_z^{0,1}X \otimes E_z))\) be the distribution kernel of \(\Gamma(t)\). Let \(h_{j, +}(t, w), A_{B_j, +}(t, z, w), K_{B_j, +} \) and \(b_{j, +}^+ \), \(s = n, n - 1, \ldots, \) be as in (5.17). Let \(\delta_j \in C_0^\infty(]-\delta_j, \delta_j[)\) such that \(\delta_j = 1\) on some neighborhood of \(\text{Supp } \sigma_j\) and \(\delta_j(\theta) = 1\) if \((z, \theta) \in \text{Supp } \chi_j\). Put

\[
\begin{align*}
\hat{h}_{j, +}(x, y) &= \delta_j(\theta) h_{j, +}(z, w) \sigma_j(\eta) \in C_0^\infty(D_j), \quad x = (z, \theta), \quad y = (w, \eta), \\
\hat{H}_j(t, x, y) &= \chi_j(x) e^{-m \varphi_j(z) - \imath m \theta} K_{B_j, +}(t, z, w) e^{m \varphi_j(w) + \imath m \eta} \tau_j(w) \sigma_j(\eta), \\
\hat{b}_{j, +}^+(x, y) &= \chi_j(x) e^{-m \varphi_j(z) - \imath m \theta} b_{j, +}^+(z, w) e^{m \varphi_j(w) + \imath m \eta} \tau_j(w) \sigma_j(\eta), \quad s = n, n - 1, \ldots.
\end{align*}
\]

(5.30)

It is not difficult to check that

\[
\Gamma(t, x, y) = \frac{1}{(2\pi)^N} \sum_{j=1}^N \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_j(t, e^{-i\theta} \circ x, e^{-iu} \circ y) e^{im(\theta - u)} d\theta du
\]

(5.31)

\[
= \frac{1}{(2\pi)^2} \sum_{j=1}^N \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\hat{h}_{j, +}(e^{-i\theta} \circ x, e^{-iu} \circ y)/t} \hat{H}_j(t, e^{-i\theta} \circ x, e^{-iu} \circ y) e^{im(\theta - u)} d\theta du
\]

\[
\sim t^{-n}a_n^+t(t, x, y) + t^{-n+1}a_{n-1}^+(t, x, y) + \cdots \text{ as } t \to 0^+.
\]
where

\begin{equation}
\begin{aligned}
a^+_\ell(t, x, y) &= \frac{1}{(2\pi)^2} \sum_{j=1}^{N} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i(t \theta - y \phi)} b^+_j (e^{-i \theta} \circ x, e^{-i \phi} \circ y) e^{i m(\theta - u)} d\theta du, \\
&\quad \text{for } s = n, n - 1, n - 2, \ldots.
\end{aligned}
\end{equation}

Lemma 5.6. We have

\begin{equation}
\lim_{t \to 0^+} \Gamma(t) u = Q_m u \text{ on } \mathcal{D}'(X, T^{*0,+}X \otimes E),
\end{equation}

for every \( u \in \Omega^{0,+}(X, T^{*0,+}X \otimes E) \).

Proof. Let \( u \in \Omega^{0,+}(X, E) \). Fix \( j = 1, 2, \ldots, N \). On \( D_j \), set \( Q_m u = e^{-im\eta} v_j(w), \) \( v_j(w) \in \Omega^{0,+}(U_j, E) \). From (5.28), we have

\begin{equation}
\begin{aligned}
&\lim_{t \to 0^+} H_j(t) Q_m u \\
&= \lim_{t \to 0^+} \int \chi_j(x) e^{-m \phi_j(z) - im \theta} A_{B_j,+}(t, z, w) e^{m \phi_j(w) + im \eta} \tau_j(w) \sigma_j(\eta) e^{-im \eta} v_j(w) dv_{U_j}(w) d\eta \\
&= \lim_{t \to 0^+} \int \chi_j(x) e^{-m \phi_j(z) - im \theta} A_{B_j,+}(t, z, w) e^{m \phi_j(w)} \tau_j(w) v_j(w) dv_{U_j}(w) \\
&= \chi_j(x) e^{-m \phi_j(z) - im \theta} e^{m \phi_j(w)} \tau_j(z) v_j(z) \\
&= \chi e^{-im \theta} v_j = \chi_j Q_m u.
\end{aligned}
\end{equation}

Thus,

\begin{equation}
\lim_{t \to 0^+} \Gamma(t) u = \lim_{t \to 0^+} \sum_{j=1}^{N} Q_m \circ H_j(t) \circ Q_m u = \sum_{j=1}^{N} Q_m \chi_j Q_m u = Q_m u.
\end{equation}

The lemma follows. \( \square \)

We need

Lemma 5.7. We have

\begin{equation}
\Gamma'(t) u + \Gamma(t) \tilde{\Box}^+_{b, m} u = R(t) u, \quad \forall u \in \Omega^{0,+}(X, E),
\end{equation}

where \( R(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E) \) is a continuous operator with distribution kernel \( R(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+}X \otimes E) \boxtimes (T^{*0,+}X \otimes E)) \) and \( R(t, x, y) \) satisfies the following: For every \( \ell \in \mathbb{N}_0 \), there is a \( \varepsilon_0 > 0 \) independent of \( t \) such that

\begin{equation}
\|R(t, x, y)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\varepsilon_0 t}, \quad \forall t \in \mathbb{R}_+.
\end{equation}
Proof. Let \( u \in \Omega^0_0(X, E) \). Fix \( j = 1, 2, \ldots, N \). On \( D_j \), set \( u = e^{-in\theta}v_j(w), \ v_j(w) \in \Omega^{0,+}(U_j, E) \). From (5.4), (5.14), we can check that
\[
\begin{align*}
H_j^*(t)u + H_j(t)\hat{\Box}^+_b,m u &= \int \chi_j(x)e^{-\varphi_j(x)}(z)\cdot\mathcal{V}A^b_j,+(t, z, w)e^{\varphi_j(w)+im\tau_j(w)}\sigma_j(\eta)u(y)dv_X(y) \\
&+ \int \chi_j(x)e^{-\varphi_j(x)}(z)\cdot\mathcal{V}A^b_j,+(t, z, w)e^{\varphi_j(w)+im\tau_j(w)}\sigma_j(\eta)(\hat{\Box}^+_b,m u)(y)dv_X(y) \\
&= \int \chi_j(x)e^{-\varphi_j(x)}(z)\cdot\mathcal{V}A^b_j,+(t, z, w)\tau_j(w)e^{\varphi_j(w)}v_j(w)dv_U_j(w) \\
&+ \int \chi_j(x)e^{-\varphi_j(x)}(z)\cdot\mathcal{V}A^b_j,+(t, z, w)\tau_j(w)(\hat{\Box}^+_b,m (e^{\varphi_j} v_j))(y)dv_X(y) \\
&= \int \chi_j(x)R_j(t, x, w)v_j(w)dv_U_j(w) = \int \chi_j(x)R_j(t, x, w)e^{im\tau_j(\eta)}u(y)dv_X(y),
\end{align*}
\]
where \( R_j(t, x, w) \in C_0^\infty(\mathbb{R}_+ \times D_j \times U_j, (T^{0,+}X \otimes E) \boxtimes (T^{0,+}X \otimes E)) \). Since \( \tau_j(z) = 1 \) if \((z, \theta)\) in some small neighborhood of \( \chi_j \), we can check that \( R_j(t, x, w) = 0 \) if \((x, w)\) in some small neighborhood of \((z, z)\). From this observation and (5.15), we conclude that for every \( \ell \in \mathbb{N}_0 \), there is a \( \varepsilon > 0 \) independent of \( t \) such that
\[
\|R_j(t, x, w)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\varepsilon \ell} t, \ \forall t \in \mathbb{R}_+.
\]
Put \( \tilde{R}(t, x, y) := \sum_{j=1}^N R_j(t, x, w)e^{im\varphi_j(\eta)} \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{0,+}X \otimes E) \boxtimes (T^{0,+}X \otimes E)) \) and set
\[
R(t, x, y) = \frac{1}{(2\pi)^2} \sum_{j=1}^N \int_{-\pi}^\pi \int_{-\pi}^\pi \tilde{R}(t, e^{-i\theta} \circ x, e^{-iu} \circ y)dv_X(d\theta dv_u).
\]
Let \( R(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E) \) be the continuous operator with distribution kernel \( R(t, x, y) \). From (5.35), it is easy to see that \( R(t, x, y) \) satisfies (5.33). Moreover, from (5.34), we see that \( \Gamma'(t)u + \Gamma(t)\hat{\Box}^+_b,m u = R(t)u, \ \forall u \in \Omega^0_0(X, E) \). The lemma follows.

Let \( \Gamma^*(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E) \) and \( R^*(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E) \) be the adjoints of \( \Gamma(t) \) and \( R(t) \) with respect to \((\cdot, \cdot)_E\) respectively. From Lemma 5.6 and Lemma 5.7, we deduce that

**Theorem 5.8.** With the notations above, we have
\[
\lim_{t \rightarrow 0^+} \Gamma^*(t)u = Q_m u \text{ on } \mathcal{D}'(X, T^{0,+}X \otimes E),
\]
for every \( u \in \Omega^{0,+}(X, T^{0,+}X \otimes E) \), and
\[
\frac{\partial \Gamma^*(t)}{\partial t}u + \hat{\Box}_b,m^+ \Gamma^*(t)u = R^*(t)u, \ \forall u \in \Omega^{0,+}(X, E),
\]
where \( R^*(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}_m(X, E) \) is a continuous operator with distribution kernel \( R^*(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{0,+}_yX \otimes E_y) \boxtimes (T^{0,+}_xX \otimes E_x)) \) and \( R^*(t, x, y) \) satisfies the following: For every \( \ell \in \mathbb{N}_0 \), there is a \( \varepsilon_0 > 0 \) independent of \( t \) such that
\[
\|R^*(t, x, y)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\varepsilon_0 \ell} t, \ \forall t \in \mathbb{R}_+.
\]
We introduce some notations. Let $A(t, B(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ be continuous operators with distribution kernels $A(t, x, y), B(t, x, y) \in C^{\infty}(\mathbb{R}_{+} \times X \times X, (T_{y}^{0,+}X \otimes E_{y}) \boxplus (T_{x}^{0,+}X \otimes E_{x})).$ Let $(A \sharp B)(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ be the continuous operator with distribution kernel

$$(A \sharp B)(t, x, y) := \int_{0}^{t} \int_{X} A(t - s, x, z)B(s, z, y)dv_{X}(z)ds \in C^{\infty}(\mathbb{R}_{+} \times X \times X, (T_{y}^{0,+}X \otimes E_{y}) \boxplus (T_{x}^{0,+}X \otimes E_{x})).$$

Let $C(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ be a continuous operator with distribution kernel $C(t, x, y) \in C^{\infty}(\mathbb{R}_{+} \times X \times X, (T_{y}^{0,+}X \otimes E_{y}) \boxplus (T_{x}^{0,+}X \otimes E_{x})).$ It is straightforward to check that $((A \sharp B)\sharp C)(t) = (A \sharp (B \sharp C))(t).$ We write

$$(A \sharp B \sharp C)(t) := (A \sharp (B \sharp C))(t).$$

Similarly, let $A_{j}(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ be a continuous operator with distribution kernel $A_{j}(t, x, y) \in C^{\infty}(\mathbb{R}_{+} \times X \times X, (T_{y}^{0,+}X \otimes E_{y}) \boxplus (T_{x}^{0,+}X \otimes E_{x})), j = 1, 2, \ldots, N.$ We write

$$(A_{1} \sharp A_{2} \sharp \cdots \sharp A_{N})(t) = (A_{1} \sharp (A_{2} \sharp A_{3} \sharp \cdots \sharp A_{N}))(t).$$

We need

**Theorem 5.9.** Fix $\ell \in \mathbb{N}, \ell \geq 2.$ Then, there is a $\epsilon > 0$ such that the sequence

$$(5.37) \quad \Lambda(t) := \Gamma^{*}(t) - (\Gamma^{*}R^{*})t + (\Gamma^{*}R^{*}R^{*})t - \cdots$$

converges in $C^{\ell}((0, \epsilon) \times X \times X, (T^{0,+}X \otimes E) \boxplus (T^{0,+}X \otimes E))$ and

$$(5.38) \quad \Lambda(t) : \Omega^{0,+}(X, E) \to C^{\ell}(X, T^{0,+}X \otimes E)) \bigcap L_{m}^{2}(X, E),$$

$$\lim_{t \to 0^{+}} \Lambda(t)u = Q_{m}u \text{ on } \mathcal{D}'(X, T^{0,+}X \otimes E), \forall u \in \Omega^{0,+}(X, T^{0,+}X \otimes E),$$

$$\Lambda'(t)u + \square_{b,m}^{+}\Lambda(t)u = 0, \forall u \in \Omega^{0,+}(X, E).$$

Let $\Lambda(t, x, y) \in C^{\ell}((0, \epsilon) \times X \times X, (T_{y}^{0,+}X \otimes E_{y}) \boxplus (T_{x}^{0,+}X \otimes E_{x}))$ be the distribution kernel of $\Lambda(t).$ Then, there is a $\epsilon_{0} > 0$ independent of $t$ such that

$$(5.39) \quad ||\Lambda(t, x, y) - \Gamma^{*}(t, x, y)||_{C^{\ell}(X \times X)} \leq e^{-\frac{\epsilon_{0}}{T}}, \forall t \in (0, \epsilon_{0}).$$

**Proof:** Fix $\ell \in \mathbb{N}, \ell \geq 2.$ From (5.36), it is not difficult to see that there are $1 > \delta_{0}, \delta_{1} > 0$ such that

$$(5.40) \quad ||R^{*}u(t, x, y)||_{C^{\ell}(X \times X)} \leq \min \left\{ \frac{1}{2} \int_{X} \frac{1}{dv_{X}}e^{\frac{\epsilon_{0}}{t}}, \forall t \in (0, \delta_{0}). \right\}$$

From (5.40), we can check that for all $t \in (0, \delta_{0}),$

$$(5.41) \quad ||R^{*}R^{*}R^{*}u(t, x, y)||_{C^{\ell}(X \times X)} \leq \frac{1}{2}e^{-\frac{\epsilon_{0}}{T}}, \quad ||R^{*}R^{*}R^{*}R^{*}u(t, x, y)||_{C^{\ell}(X \times X)} \leq \frac{1}{2^{2}}e^{-\frac{\epsilon_{0}}{4T}}, \quad \cdots$$

$$(5.42) \quad ||\Gamma^{*}R^{*}R^{*}R^{*}u(t, x, y)||_{C^{\ell}(X \times X)} \leq \frac{C_{1}}{2^{2}}e^{-\frac{\epsilon_{0}}{4T}}, \quad ||\Gamma^{*}R^{*}R^{*}R^{*}R^{*}u(t, x, y)||_{C^{\ell}(X \times X)} \leq \frac{C_{1}}{2^{3}}e^{-\frac{\epsilon_{0}}{8T}}, \quad \cdots,$$

where $C_{1} > 0$ is a constant. From (5.42), it is easy to check that the sequence $(5.37)$ converges in $C^{\ell}((0, \epsilon) \times X \times X, (T^{0,+}X \otimes E) \boxplus (T^{0,+}X \otimes E))$ and (5.38), (5.39) hold. \hfill \Box

Now, we can prove
Theorem 5.10. For every $\ell \in \mathbb{N}$, $\ell \geq 2$, there are $\epsilon_0 > 0$ and $\epsilon > 0$ independent of $t$ such that

$$\left\| e^{-t \mathcal{E}^+_b,m}(x,y) - \Gamma(t,x,y) \right\|_{C^\ell(X \times X)} \leq e^{-\epsilon t}, \quad \forall t \in (0, \epsilon),$$

where $e^{-t \mathcal{E}^+_b,m}(x,y)$ is as in (4.21). In particular,

$$e^{-t \mathcal{E}^+_b,m}(x,y) \sim t^{-n} a^+_0(t,x,y) + t^{-n+1} a^+_{n-1}(t,x,y) + \cdots + a^+_0(t,x,y) + t a^+_{-1}(t,x,y) + \cdots \text{ as } t \to 0^+,$$

$$a^+_{s}(t,x,y) = \frac{1}{(2\pi)^2} \sum_{j=1}^{N} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-b_{j,0}(e^{-i\theta} \circ x, e^{-i\theta} \circ y)} b_{j,s}(e^{-i\theta} \circ x, e^{-i\theta} \circ y) e^{im(\theta-u)} d\theta du$$

$$\in C^\infty(\mathbb{R}_+ \times X \times X, (T^*y^{0,+} X \otimes E_y) \boxtimes (T^*x^{0,+} X \otimes E_x)), \quad s = n, n-1, n-2, \ldots.$$

Proof Fix $\ell \in \mathbb{N}$, $\ell \geq 2$ and let $\Lambda(t) : \Omega^{0,+}(X,E) \to C^\ell(0,\epsilon) \times X \times X, (T^*y^{0,+} X \otimes E_y) \boxtimes (T^*x^{0,+} X \otimes E_x)$ be as in Theorem 5.9 with distribution kernel $\Lambda(t,x,y) \subset C^\ell(0,\epsilon) \times X \times X, (T^*y^{0,+} X \otimes E_y) \boxtimes (T^*x^{0,+} X \otimes E_x)$, where $\epsilon > 0$ is a constant dependent on $\ell$. Let $f, g \in \Omega^{0,+}(X,E)$. From (4.27) and (5.38), it is not difficult to see that for every $0 < t < \epsilon$,

$$0 = \int_{0}^{t} \frac{d}{ds} \left( \left( \Lambda(t-s)f \left| e^{-s \mathcal{E}^+_b,m} g \right) \right) \right) ds$$

$$= (Q_m f \left| e^{-s \mathcal{E}^+_b,m} g \right) - \left( \Lambda(t)f \right) g E$$

$$= (f \left| e^{-s \mathcal{E}^+_b,m} g \right) - \left( \Lambda(t)f \right) g E$$

$$= (e^{-s \mathcal{E}^+_b,m} f \left| g \right) - \left( \Lambda(t)f \right) g E.$$  

From (5.45), we conclude that $e^{-t \mathcal{E}^+_b,m}(x,y) = \Lambda(t,x,y)$ in $C^\ell(0,\epsilon) \times X \times X, (T^*y^{0,+} X \otimes E_y) \boxtimes (T^*x^{0,+} X \otimes E_x)).$ From this, (5.39) and note that $e^{-t \mathcal{E}^+_b,m}$ is self-adjoint, the theorem follows. $\square$

We can repeat the proof of Theorem 5.10 and deduce that

Theorem 5.11. We have

$$e^{-t \mathcal{E}^-_b,m}(x,y) \sim t^{-n} a^-_0(t,x,y) + t^{-n+1} a^-_{n-1}(t,x,y) + \cdots + a^-_0(t,x,y) + t a^-_{-1}(t,x,y) + \cdots \text{ as } t \to 0^-,$$

$$a^\pm_{s}(t,x,y) = \frac{1}{(2\pi)^2} \sum_{j=1}^{N} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-b_{j,0}(e^{-i\theta} \circ x, e^{-i\theta} \circ y)} b_{j,s}(e^{-i\theta} \circ x, e^{-i\theta} \circ y) e^{im(\theta-u)} d\theta du$$

$$\in C^\infty(\mathbb{R}_+ \times X \times X, (T^*y^{0,-} X \otimes E_y) \boxtimes (T^*x^{0,-} X \otimes E_x)), \quad s = n, n-1, n-2, \ldots.$$

Let $e_1(x), \ldots, e_d(x)$ and $f_1(x), \ldots, f_d(x)$ be orthonormal frames of $T^*_x X \otimes E_x$ and $T^*_x X \otimes E_x$ with respect to $\langle \cdot | \cdot \rangle_E$ respectively. Fix $\ell = n, n-1, \ldots$, let

$$a^\pm_{\ell}(t,x,x) \in C^\infty(\mathbb{R}_+ \times X, \text{End}(T^*_x X \otimes E_x)), \quad a^\pm_{\ell}(t,x,x) \in C^\infty(\mathbb{R}_+ \times X, \text{End}(T^*_x U \otimes E_x))$$

be as in (5.44) and (5.46) respectively. Define

$$\text{Tr} a^+_{\ell}(t,x,x) = \sum_{s=1}^{d} \langle a^+_s(t,x,x)e_s(x) | e_s(x) \rangle_E, \quad \text{Tr} a^-_{\ell}(t,x,x) = \sum_{s=1}^{d} \langle a^-_s(t,x,x)f_s(x) | f_s(x) \rangle_E.$$  

From Theorem 4.10, (5.44) and (5.46), we conclude that

Theorem 5.12. We have

$$\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \lim_{t \to 0^+} \int_{X} \int_{0}^{t} \left( \text{Tr} a^+_\ell(t,x,x) - \text{Tr} a^-_\ell(t,x,x) \right) dv_X(x).$$


6. Local density for the Euler characteristic $\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E)$

Put $X_\ell = \{ x \in X; \text{the period of } x \text{ is } \frac{2\pi}{p} \}$ and let $p = \min \{ \ell \in \mathbb{N}; X_\ell \neq \emptyset \}$. It is easy to see that $X_p$ is an open subset of $X$. The goal of this section is to prove the following

**Theorem 6.1.** Let $x_0 \in \overline{X}_p$. Then, for every $t > 0$,

$$
\sum_{\ell=0}^{n} t^{-\ell} \left( \text{Tr } a^+_\ell(t, x_0, x_0) - \text{Tr } a^-_\ell(t, x_0, x_0) \right) d\nu_X(x_0)
$$

(6.1)

$$
= \sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}} m \left[ T_b \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge ch_b \left( \nabla E, E \right) \wedge e^{-\frac{\delta_m}{2\pi} \wedge \omega_0} \right]_{2n+1}(x_0),
$$

where $a^+_\ell(t, x, x) \in C^\infty(\mathbb{R}_+ \times X, \text{End } (T_x^{1,0}X \otimes E_x))$, $a^-_\ell(t, x, x) \in C^\infty(\mathbb{R}_+ \times X, \text{End } (T_x^{0,1}X \otimes E_x))$ are as in (5.44) and (5.46).

**Proof.** Fix $x_0 \in X_p$. Let $B_j = (D_j, (z, \theta), (\varphi_j), x_j(x), \sigma_j$ and $\tau_j, j = 1, 2, \ldots, N$, be as in the beginning of Section 5.2. We will use the same notations as in Section 5.2. Fix $j = 1, 2, \ldots, N$. Assume that $e^{-i\theta_0} \circ x_0 \in D_j$ for some $\theta_0 \in [-\pi, \pi]$ and suppose that $e^{-i\theta_0} \circ x_0 = (z_0, 0) \in D_j$. Let $\gamma > 0$ be a small constant so that $\{ z \in \mathbb{C}^n; |z - z_0| < \gamma \} \subset W_j$ and the canonical coordinates $(z, \theta)$ can be defined on

$$
D_0 := \left\{ (z, \theta) \in \mathbb{C}^n \times \mathbb{R}; |z - z_0| < \gamma, \theta \in \left[ -\frac{\pi}{p}, \frac{\pi}{p} \right] \right\}
$$

(this is always possible, see Lemma 1.17 in [22]), where $W_j \subset U_j$ is as in the beginning of Section 5.2. Let $\psi(x) = \psi(z, \theta) \in C^\infty_0(D_0)$ with $\int \psi(x) d\nu_X(x) = 1$. For every $\varepsilon > 0$, put $\psi_\varepsilon(x) := \varepsilon^{-(2n+1)} \psi(z_0 + \frac{z - z_0}{\varepsilon}, \frac{\theta}{\varepsilon})$. Then,

$$
f(e^{-i\theta_0} \circ x_0) = \lim_{\varepsilon \to 0^+} \int f(x) \psi_\varepsilon(x) d\nu_X(x), \ f \in C^\infty(X).
$$

From (6.3), we have

$$
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_j(t, e^{-i\theta} \circ x_0, e^{-iu} \circ x_0) e^{im(\theta-u)} d\theta du
$$

(6.4)

$$
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_j(t, e^{-i\theta} \circ x_0, e^{-iu} \circ e^{-i\theta_0} \circ x_0) e^{im(\theta-u)} d\theta du
$$

$$
= \int_{-\pi}^{\pi} \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int H_j(t, e^{-i\theta} \circ x, e^{-iu} \circ y) e^{im(\theta-u)} \psi_\varepsilon(x) \psi_\delta(y) d\nu_X(x) d\nu_X(y) d\theta du.
$$

Fix $\theta, u \in [-\pi, \pi]$, from (5.27), we have

$$
\lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int H_j(t, e^{-i\theta} \circ x, e^{-iu} \circ y) e^{im(\theta-u)} \psi_\varepsilon(x) \psi_\delta(y) d\nu_X(x) d\nu_X(y)
$$

(6.5)

$$
= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int H_j(t, x, y) e^{im(\theta-u)} \psi_\varepsilon(e^{i\theta} \circ x) \psi_\delta(e^{iu} \circ y) d\nu_X(x) d\nu_X(y)
$$

$$
= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int H_j(t, (z, \xi), (w, \eta)) e^{im(\theta-u)} e^{-\frac{z - z_0}{\varepsilon} \frac{\xi - \theta}{\varepsilon}}
$$

$$
\times \delta^{(2n+1)} \psi(z_0 + \frac{w - z_0}{\delta}, \frac{\eta - u}{\delta}) d\nu_X(z, \xi) d\nu_X(w, \eta)
$$

$$
= H_j(t, (z_0, \theta), (z_0, u)) e^{im(\theta-u)}.
$$
where \( x = (z, \xi) \) and \( y = (w, \eta) \). From (5.27), (6.4) and (6.5), we have

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_j(t, e^{-i\theta} \circ x_0, e^{-iu} \circ x_0) e^{im(\theta-u)} d\theta d\mu
\]

\[
= \left( \sum_{s=1}^{p} e^{im \frac{2\pi s}{p} (s-1)} \right) \left( \sum_{k=1}^{p} e^{-im \frac{2\pi k}{p} (k-1)} \right) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_j(t, (z_0, \theta), (z_0, u)) e^{im(\theta-u)} d\theta d\mu
\]

\[
= \left( \sum_{s=1}^{p} e^{im \frac{2\pi s}{p} (s-1)} \right) \left( \sum_{k=1}^{p} e^{-im \frac{2\pi k}{p} (k-1)} \right) \int_{-\pi}^{\pi} \frac{\chi_j(z_0, \theta) K_{B_j,B_j}(t, z_0, z_0) d\theta}{p}
\]

\[
= \left( \sum_{s=1}^{p} e^{im \frac{2\pi s}{p} (s-1)} \right) \left( \sum_{k=1}^{p} e^{-im \frac{2\pi k}{p} (k-1)} \right) \int_{-\pi}^{\pi} \frac{\chi_j(e^{-i\theta} \circ x_0) K_{B_j,B_j}(t, z_0, z_0) d\theta}{p}.
\]

From (5.30), (5.31), (5.32) and (6.6), we conclude that

\[
\sum_{\ell=0}^{n} t^{-\ell} \text{Tr} a_\ell^+(t, x_0, x_0)
\]

\[
= \frac{1}{(2\pi)^2} \left( \sum_{s=1}^{p} e^{im \frac{2\pi s}{p} (s-1)} \right) \left( \sum_{k=1}^{p} e^{-im \frac{2\pi k}{p} (k-1)} \right) \sum_{\ell=0}^{n} t^{-\ell} \int_{-\pi}^{\pi} \left( \sum_{j=1}^{N} \text{Tr} b_{j,\ell}^+(z_0, z_0) \chi_j(e^{-i\theta} \circ x_0) \right) d\theta.
\]

We can repeat the proof of (6.7) and get that

\[
\sum_{\ell=0}^{n} t^{-\ell} \text{Tr} a_\ell^-(t, x_0, x_0)
\]

\[
= \frac{1}{(2\pi)^2} \left( \sum_{s=1}^{p} e^{im \frac{2\pi s}{p} (s-1)} \right) \left( \sum_{k=1}^{p} e^{-im \frac{2\pi k}{p} (k-1)} \right) \sum_{\ell=0}^{n} t^{-\ell} \int_{-\pi}^{\pi} \left( \sum_{j=1}^{N} \text{Tr} b_{j,\ell}^-(z_0, z_0) \chi_j(e^{-i\theta} \circ x_0) \right) d\theta.
\]

From (6.7), (6.8), Theorem 5.5 and notice that \( \sum_{j=1}^{N} \chi_j(e^{-i\theta} \circ x_0) = 1 \), for every \( \theta \in [-\frac{\pi}{p}, \frac{\pi}{p}] \), we have

\[
\sum_{\ell=0}^{n} t^{-\ell} \left( \text{Tr} a_{\ell}^+(t, x_0, x_0) - \text{Tr} a_{\ell}^-(t, x_0, x_0) \right) d\nu_X(x_0)
\]

\[
= \frac{1}{(2\pi)^2} \left( \sum_{j=1}^{p} e^{im \frac{2\pi j}{p} (j-1)} \right) \left( \sum_{k=1}^{p} e^{-im \frac{2\pi k}{p} (k-1)} \right) \frac{2\pi}{p}
\]

\[
\times \left[ T_{db} \left( \nabla^{1.0} X, T_{1.0} X \right) \land c_{db} \left( \nabla E, E \right) \land e^{-m \frac{d\omega}{2\pi}} \land \omega \right]_{2n+1}(x_0)
\]

\[
d\nu_X(x_0)
\]

We have proved that (6.1) holds for every \( x \in \mathcal{X}_p \). Since the left side and the right side of (6.1) are continuous functions on \( X \), we conclude that (6.1) holds for every \( x \in \mathcal{X}_p \) and the theorem follows. \( \square \)

Now, we can prove

**Theorem 6.2.** Recall that we work with the assumption that \( X \) is connected. Then, \( \mathcal{X}_p \) is dense in \( X \).

**Proof.** Assume that \( \mathcal{X}_p \neq X \). For \( \ell > p \), \( \ell \in \mathbb{N} \), put \( \mathcal{X}_\ell = \{ x \in X \setminus \mathcal{X}_p ; \text{the period of } x \text{ is } \frac{2\pi}{\ell} \} \). Assume that

\[
X = \mathcal{X}_p \cup \mathcal{X}_{\ell_1} \cup \mathcal{X}_{\ell_2} \cup \cdots \cup \mathcal{X}_{\ell_N},
\]

where \( p < \ell_1 < \ell_2 < \cdots < \ell_N \). Since \( \mathcal{X}_p \) and \( \mathcal{X}_{\ell_2} \cup \cdots \cup \mathcal{X}_{\ell_N} \) are closed sets, \( \mathcal{X}_{\ell_1} \) is an open set. We claim that

\[
\mathcal{X}_{\ell_1} \cap \mathcal{X}_p \neq \emptyset.
\]
where $\bar{X}_{\ell}$ denotes the closure of $X_{\ell}$ in $X$. If $X_{\ell} \cap \bar{X}_p \neq \emptyset$. Let $x_0 \in \bar{X}_{\ell} \cap \bar{X}_p$ It is straightforward to see that we can find rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $CTX$ and rigid Hermitian metric $\langle \cdot | \cdot \rangle_E$ on $E$ such that

$$\frac{T_{d_{\bar{X}}}(\nabla T^{1.0}X, T^{1.0}X) \wedge ch_{\bar{X}}(\nabla^E, E) \wedge e^{-\frac{dn}{2\pi} \wedge \omega}}{dv_{X}(x_0)} \neq 0. \quad (6.10)$$

Since $x_0 \in \bar{X}_{\ell}$ and $\bar{X}_{\ell}$ is open, we can repeat the proof of Theorem 6.1 and conclude that

$$\sum_{\ell=0}^{n} t^{-\ell} \left( \frac{1}{m} \left( T_{d_{\bar{X}}}(\nabla T^{1.0}X, T^{1.0}X) \wedge ch_{\bar{X}}(\nabla^E, E) \wedge e^{-\frac{dn}{2\pi} \wedge \omega} \right) \right) dv_{X}(x_0) \quad (6.11)$$

From (6.10), (6.11) and (6.1), we get a contradiction. The claim (6.9) follows.

Assume that $\bar{X}_p \cup \bar{X}_{\ell} \neq X$. For $\ell > \ell_1$, $\ell \in \mathbb{N}$, put

$$X_{\ell} = \left\{ x \in X \setminus (\bar{X}_p \cup \bar{X}_{\ell}) ; \text{ the period of } x \text{ is } \frac{2\pi}{2} \right\}.$$

Assume that $X = \bar{X}_p \cup \bar{X}_{\ell_1} \cup X_{k_1} \cup X_{k_2} \cup \cdots \cup X_{k_M}$, where $p < \ell_1 < k_1 < k_2 < \cdots < k_M$. We can repeat the proof of (6.9) with minor change and get that

$$\bar{X}_{k_1} \cap (\bar{X}_p \cup \bar{X}_{\ell_1}) = \emptyset.$$

Continuing in this way, we can find closed sets $C_1, \ldots, C_K$, with $\bar{X}_p \cap (\bigcup_{j=1}^{K} C_j) = \emptyset$ such that

$$X = \bar{X}_p \cup C_1 \cup \cdots \cup C_K.$$

Hence, $\bar{X}_p$ is an open set and also a closed set. Since $X$ is connected, we get a contradiction. Thus, $X = \bar{X}_p$. \hfill $\Box$

**Proof of Theorem 1.3.** From Theorem 5.12, Theorem 6.1 and Theorem 6.2, we get Theorem 1.3. \hfill $\Box$

**References**


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