Monodromies and isotopies of
Lagrangian tori

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Outline

1. Lagrangian tori and their classical invariants.

2. The isotopy problem and monodromy groups associated to self-isotopies.

3. Clifford torus and Chekanov torus are distinguished by invariants associated to their Hamiltonian monodromy groups.

4. The Lagrangian and smooth monodromy groups of a Clifford torus.

5. From smooth isotopy to Lagrangian isotopy.
Lagrangian torus in \((M^4, \omega)\).

An embedded torus \(L \hookrightarrow M\) is **Lagrangian** if

\[ \iota^* \omega = 0. \]

This implies that

\[ \omega = d\lambda \quad \text{near } L, \text{ and} \]
\[ \iota^* \lambda = 0. \]

- The **Liouville class**

\[ \alpha := [\iota^* \lambda] \in H^1(L, \mathbb{Z}) \]

is independent of the choice of \(\lambda\) provided that the map \(H^1(M, L; \mathbb{R}) \to H^1(M, \mathbb{R})\) is surjective.

- If moreover, \(c_1(M, \omega) = 0\), the **Maslov class**

\[ \mu \in H^1(L, \mathbb{Z}) \]

is defined as well.
The Maslov number.

Consider $\mathbb{R}^4$ with the symplectic form
\[
\omega = \sum_{j=1}^{2} dx_j \wedge dy_j.
\]

For a loop of Lagrangian subspaces
\[
L_t = g_t(L_0), \quad g_t \in U(2),
\]
the Maslov index is
\[
\mu(L_t) := \deg(t \in S^1 \to (\det g_t)^2).
\]
A Lagrangian torus $L$ is

- **exact** if $\alpha = 0$;
- **monotone** if $\alpha = c\mu$ for some $c > 0$.

\textbf{In $\mathbb{R}^4$:}

- (Gromov ’85) $\alpha \neq 0$, i.e., $L$ is not exact.
- (Polterovich ’91) $\mu \neq 0$ and has divisibility 2.
The linking class $\ell \in H^1(L, \mathbb{Z})$ for $L \hookrightarrow \mathbb{C}^2$.

Fix

- $J$: the standard complex structure on $\mathbb{C}^2$,
- $v$: a non-vanishing vector on $L$ homotopic to some non-vanishing vector field $v'$ which generates the kernel of a non-vanishing closed 1-form on $L$.

Define

$$\ell(\gamma) := \text{lk}(C_\gamma + \epsilon Jv, L), \quad \gamma \in H_1(L, \mathbb{Z}).$$

- (Eliashberg & Polterovich, ’94):
  The linking class $\ell \in H^1(L, \mathbb{Z})$ is 0 for any Lagrangian torus $L \subset \mathbb{R}^4$.

- (Polterovich, ’88):
  For each $\ell \in H^1(L, \mathbb{Z})$ there exists a totally real embedding $L \hookrightarrow \mathbb{C}^2$ whose linking class is equal to $\ell$. 
**Examples in** $\mathbb{R}^4$: 

1. **Clifford torus** $(a, b > 0)$

   $$T_{a,b} := \{ \pi |z_1|^2 = a, \pi |z_2|^2 = b \}$$

   $T_{b,b}$ is monotone, $c = b/2$.

2. (Chekanov '96)

   $$\rho : S^1 \times \mathbb{R} \xrightarrow{\text{diffeo}} E := \mathbb{R}_{x_1, x_2}^2 \setminus \{(0,0)\}$$

   $$\Psi := (\rho^*)^{-1} : (T^*S^1) \times \mathbb{R}^2 \to T^*E \subset \mathbb{R}^4$$

   $$T'_{a,b} := \Psi(\{\theta^* = a\} \times T_b)$$

   $T'_{0,b}$ is monotone with $c = b/2$, called a **Chekanov torus**.
Isotopy of Lagrangian tori.

$L_0, L_1$ are

- **smoothly** isotopic if they are connected by a smooth family of embedded tori $L_t$, i.e., if

$$\exists \phi_t \in \text{Diff}^c_0(M), \quad 0 \leq t \leq 1,$$
$$\phi_0 = id, \quad \phi_t(L_0) = L_t;$$

- **Lagrangian** isotopic if $L_t$ are Lagrangian;

- **Hamiltonian** isotopic if $\phi_t \in \text{Ham}^c(M, \omega)$.

(Eliashberg & Polterovich ’93)

**Lagrangian knot problem:**

Classify embedded Lagrangian surfaces up to various types of isotopies.
Some results in $\mathbb{R}^4$.

(Chekanov '96)

1. $T_{a,b} \overset{\text{ham}}{\sim} T_{a',b'}$ iff $(a', b') = (a, b)$ or $(b, a)$.

   (Use symplectic capacities [Ekland, Hofer, Viterbo].)

2. $T_{a,b}' \overset{\text{ham}}{\sim} T_{a,b+|a|}$ for $a \neq 0$.

3. $T_{0,b}' \overset{\text{lag}}{\sim} T_{b,b}$, but $T_{0,b}' \not\overset{\text{ham}}{\sim} T_{b,b}$.

   (Use symplectic capacities, displacement energy capacity [Hofer] and a versal deformation argument.)
Monodromy groups.

Let $\phi_t$ be either a Hamiltonian, a Lagrangian, or a smooth self-isotopy of $L$ with

$$\phi_0 = id, \quad \phi_1(L) = L.$$ 

We call $(\phi_1)_* \in \text{Isom}(H_1(L, \mathbb{Z}))$ the monodromy of $\phi_t$.

Define

$\mathcal{H} :=$ the Hamiltonian monodromy group of $L$;

$\mathcal{L} :=$ the Lagrangian monodromy group of $L$;

$\mathcal{S} :=$ the smooth monodromy group of $L$. 
For a monotone $L$.

$\mathcal{H}$ is a subgroup of

$$G_\mu := \{ g \in \text{Isom}(H_1(L, \mathbb{Z}) \mid \mu \circ g = \mu \} \cong \mathcal{D}_\infty.$$  

Let $\gamma_0 \in H_1(L, \mathbb{Z})$ be a generator of $\ker(\alpha)$. Then

$$G_\mu = \langle f, g \mid f^2 = e, \, fgfg = e \rangle,$$

- $f$ is a reflection along $\gamma_0$,
- $g$ is a simple Dehn twist along $\gamma_0$.

$$G_\mu = T \sqcup S$$

$T = \langle g \rangle, \quad S = fT.$

There are four types of $\mathcal{H}$:

$$\{e\}, \, \langle f \rangle \cong \mathbb{Z}_2, \, \langle g^k \rangle \cong \mathbb{Z}, \, \langle f, g^k \rangle \cong \mathcal{D}_\infty.$$
Two Invariants of $\mathcal{H}$.

- The **twist number**

  $$t(L) := |k| \text{ if } \mathcal{H} \cap T = \langle g^{|k|} \rangle.$$ 

- The **spectrum** $s(L) := \begin{cases} 0 & \text{if } \mathcal{H} \cap S = \emptyset, \\ \min\{m_f \mid f \in \mathcal{H} \cap S\}, & \end{cases}$

  where

  $$m_f := \mu(v_f)/2,$$

  $v_f \in \mathbb{Z}^2$ is the generator of the 1-eigenspace of $f$ with $\mu(v_f) > 0$. 
$\mathcal{H}(T_{b,b})$ v.s. $\mathcal{H}(T_{0,b}')$.

**Theorem 1 (Y.).** Let $b > 0$. Then

$$\mathcal{H}(T_{b,b}) \cong \mathbb{Z}_2 \cong \mathcal{H}(T_{0,b}')$$

as abstract groups and hence

$$t(T_{b,b}) = 0 = t(T_{0,b}').$$

However,

$$s(T_{b,b}) = 2, \quad s(T_{0,b}') = 1.$$ 

**Hence** $T_{b,b}$ and $T_{0,b}'$ are not Hamiltonian isotopic in $\mathbb{R}^4$.

Our approach provides a new way to distinguish $T_{b,b}$ from $T_{0,b}'$ up to Hamiltonian isotopy.
Sketch of the proof of Theorem 1.

1. "\( T_{a,b} \overset{\text{ham}}{\sim} T_{a',b'} \) iff \((a', b') = (a, b)\) or \((b, a)\)"

\[ \Rightarrow \mathcal{H}(T_{b,b}) \text{ and } \mathcal{H}(T'_{0,b}) \text{ contain no Dehn twists, hence have at most one involution.} \]

2. \( \mathcal{H}(T_{b,b}) \) is generated by the monodromy of the map \((z_1, z_2) \rightarrow (z_2, z_1)\). \( s = 2 \).

3. \( \mathcal{H}(T'_{0,b}) \) is generated by the monodromy of the map \((z_1, z_2) \rightarrow (-z_1, z_2)\) (for a suitable choice of \(\rho\)). \( s = 1 \).
The Lagrangian monodromy group $\mathcal{L}$.

**Theorem 2** (Y.). Assume that $L = T_{a,b}$ or $T'_{0,b}$. Then

$$\mathcal{L}(L) = G_\mu.$$  

**Sketch of the proof of Theorem 2:**

1. $\partial D^\omega_C(\epsilon)$ is Lagrangian for curve $C \subset \mathbb{R}^4$.

   Consider the basis $\{\gamma_1, \gamma_2\}$ for $H_1(H_1(T_{a,b}, \mathbb{Z}))$, where
   \[
   \gamma_1 = \{(ae^{it}, b) \mid t \in [0, 2\pi]\},
   \gamma_2 = \{(a, be^{it}) \mid t \in [0, 2\pi]\}.
   \]

2. Rotate the $x_1y_2$-plane by $180^\circ$, get monodromy $f_0 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

3. $\mathcal{L}(T_{b,b})$ is gen. by $f_0$ and $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. 


Framings of $N^\omega_C$.

Let $C \subset \mathbb{R}^4$ be any loop. Let $N^\omega_C$ denote its symplectic normal bundle. $N^\omega_C \cong C \times \mathbb{R}^2$.

To a non-vanishing section (i.e., a framing) $\sigma$ of $N^\omega_C$ one can associate an $S^1$-family of Lagrangian planes $\dot{C}(t) \wedge \sigma(t)$, $t \in S^1$.

The homotopy class of the framings is classified by the number $\frac{1}{2} \mu([\dot{C}(t) \wedge \sigma(t)])$.

**Example 3.** Let $C \subset L$ be a simple closed curve representing the class $\gamma \in H_1(L, \mathbb{Z})$ of a Lagrangian torus. Let $v$ be a non-vanishing section of $N^\omega_C \cap T_C L$. Then $v$ is a $\mu(\gamma)/2$-framing of $N^\omega_C$. 
Let $\sigma^m$ denote the framing of $N_C^{\omega}$ with $\mu = 2m$.

**Proposition 4.** Let $C_s, s \in [0, 1]$ be a smooth isotopy between loops $C_0$ and $C_1$. Write $C_s = \phi_s(C_0)$ where $\phi_s \in \text{Diff}_0(\mathbb{R}^4)$ with $\phi_0 = \text{id}$. Let $N_s^{\omega}$ and $\sigma_s^m$ denote the symplectic normal bundle and the $m$-framing of $C_s$ respectively.

1. Assume that $(\phi_1)_*N_0^{\omega} = N_1^{\omega}$. Then

$$\mu((\phi_1)_*\sigma_0^m) - \mu(\sigma_1^m) \in 4\mathbb{Z}.$$ 

2. If $\mu((\phi_1)_*\sigma_0^m) = \mu(\sigma_1^m) = 2m$ then up to a perturbation of $\phi_s$ we may assume that $(\phi_s)_*N_0^{\omega} = N_s^{\omega}$ and $(\phi_s)_*\sigma_0^m = \sigma_s^m$.

**Sketch of the proof:**

Use $\pi_1(SO(3)) \cong \mathbb{Z}_2$, $\pi_2(SO(3)) \cong 1$. 

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Smooth isotopy

Theorem 5 (Y.). Let $L_s = \phi_s(L_0)$, $0 \leq s \leq 1$, $\phi_0 = \text{id}$, be a smooth isotopy between two Lagrangian tori $L_0, L_1 \subset \mathbb{R}^4$. Then for any $\gamma \in H_1(L, \mathbb{Z})$,

$$\mu(\phi_1^*(\gamma)) - \mu(\gamma) \in 4\mathbb{Z}.$$ 

I.e.,

$$\phi_1^*\mu - \mu \in H^1(L, \mathbb{Z}) \quad \text{has divisibility } 4.$$ 

Thus $S(L)$ is a subgroup of

$$\mathcal{X} = \mathcal{X}_L := \{g \in \text{Isom}(H_1(L, \mathbb{Z})) \mid \mu_L \circ g - \mu_L \in 4 \cdot H^1(L, \mathbb{Z})\}.$$ 

Sketch of the proof:

Use $\ell_L = 0$ and Proposition 4.
Fix a basis \( \{ \gamma_1, \gamma_2 \} \) for \( H_1(L, \mathbb{Z}) \) with
\[
\mu(\gamma_1) = 2 = \mu(\gamma_2).
\]
Then
\[
\mathcal{X} = \mathcal{X}^o \sqcup \mathcal{X}^e \subset GL(2, \mathbb{Z}),
\]
\[
\mathcal{X}^o := \left\{ \begin{pmatrix} 1 + 2p & 2s \\ 2r & 1 + 2q \end{pmatrix} \in GL(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\},
\]
\[
\mathcal{X}^e := \left\{ \begin{pmatrix} 2r & 1 + 2q \\ 1 + 2p & 2s \end{pmatrix} \in GL(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\}.
\]
The smooth monodromy group $S$.

**Theorem 6 (Y.).** Let $L = T_{a,b}$.

Identify $\text{Isom}(H_1(L, \mathbb{Z}))$ with $\text{GL}(2, \mathbb{Z})$ w.r.t. the basis $\{\gamma_1, \gamma_2\}$. Then

$$S(L) = \mathcal{X}_L$$

is the subgroup generated by

$$f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \bar{r}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

$$S = \mathcal{X}^o \sqcup f_1 \mathcal{X}^o,$$

where

$$\mathcal{X}^o = \left\{ \begin{pmatrix} 1 + 2p & 2s \\ 2r & 1 + 2q \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\}.$$ 

We sketch the proof as follows.
Step 1:

**Proposition 7** (Y.). Each of the following four types of elements of $GL(2, \mathbb{Z}) \cong Isom(H_1(T_{a,b}, \mathbb{Z}))$ can be realized as the monodromy of some smooth self-isotopy of $T_{a,b}$:

1. a $k$-Dehn twist $\tau_1^k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ along $\gamma_1$ with $k \in 2\mathbb{Z} \setminus \{0\}$,

2. a $k$-Dehn twist $\tau_2^k := \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ along $\gamma_2$ with $k \in 2\mathbb{Z} \setminus \{0\}$,

3. the $\gamma_1$-reflection $\bar{r}_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

4. the $\gamma_2$-reflection $\bar{r}_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
By applying $f_1$ it is enough to prove (i) and (iii).

**Case (i):** Let $C := \{(0, be^{it}) \mid t \in S^1\}$.

Let $U$ be a tubular neighborhood of $C$ with $T \subset \partial U$. Let $U = U_C \cong B^3 \times S^1$ be with coor. $(\rho, \varphi, \theta, t)$ so that

$$T = \{ (\rho_0, \pi/2, \theta, t) \}.$$ 

Consider $\tilde{\phi} : U \to U$,

$$\tilde{\phi}(\rho, \varphi, \theta, t) = (\rho, \psi_t(\varphi, \theta), t) := (\rho, (\varphi, \theta + kt), t).$$

$k$ is even $\Rightarrow [\psi_t] \in \pi_1(SO(3))$ is trivial.

$$\exists \psi_{s,t} \in SO(3), s, t \in [0, 1] \times S^1, \psi_{0,t} = Id = \psi_{s,0}, \psi_{1,t} = \psi_t.$$

This induces an isotopy b/w $id$ and $\phi$:

$$\tilde{\phi}_s(\rho, (\varphi, \theta), t) := (\rho, \psi_{s,t}(\varphi, \theta), t).$$

Extend $\tilde{\phi}_s$ over $\mathbb{R}^4$ with compact support.
**Case (ii):** Parametrize $B^3$ by Cartesian coor. $(x_1, y_1, x_2)$ with $x_1^2 + y_1^2 + x_2^2 \leq 1$ so that

$$T = \{(x_1, y_1, 0, t) \mid x_1^2 + y_1^2 = 1\}.$$

$ar{r}_1$ is represented by the map

$$\phi(x_1, y_1, 0, t) = (-x_1, y_1, 0, t), \quad (x_1, y_1, 0, t) \in L.$$

Extend $\phi$ over $U$ to get $\tilde{\phi} : U \to U$,

$$\tilde{\phi}(x_1, y_1, x_2, t) = (\psi(x_1, y_1, x_2), t) := ((-x_1, y_1, -x_2), t).$$

The map $\psi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in SO(3)$ is isotopic to the identity map. Let $\psi_s$ be a smooth path in $SO(3)$ with $s \in [0, 1]$, $\psi_0 = Id$ and $\psi_1 = \psi$.

This induces an isotopy $\tilde{\phi}_s : U \to U$, $s \in [0, 1]$,

$$\tilde{\phi}_t((x_1, y_1, x_2), t) = (\psi_t(x_1, y_1, x_2), t).$$

Extend $\tilde{\phi}_s$ over $\mathbb{R}^4$ with compact support.
Let $\mathcal{R}$ be the subgroup generated by
\begin{equation*}
\mathcal{L} \text{ and } \tau_j^2, \bar{r}_j, j = 1, 2.
\end{equation*}
We have
\begin{equation*}
\mathcal{R} \subset \mathcal{S} \subset \mathcal{X}.
\end{equation*}
**Step 2:** $\mathcal{X} \subset \mathcal{R}$ indeed.

Let $\mathcal{E} \subset \mathcal{R}$ be generated by $\tau_j^2$, $j = 1, 2$.

- (Sanov, ’47)
  
  
  $$\mathcal{E} = \left\{ \begin{pmatrix} 1 + 4p & 2s \\ 2r & 1 + 4q \end{pmatrix} \in GL(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\}.$$  

  
  
  - Enough to show that for $h = \begin{pmatrix} 1 + 2p & 2s \\ 2r & 1 + 2q \end{pmatrix} \in \mathcal{X}^o$, there exists $g \in \mathcal{R}$ such that $gh \in \mathcal{E}$.

    1. $h \in \mathcal{E}$ if $p, q$ are both even,
    2. $\bar{r}_1\bar{r}_2 h \in \mathcal{E}$ if $p, q$ are both odd,
    3. $\bar{r}_1 h \in \mathcal{E}$ if $p$ odd, $q$ even,
    4. $\bar{r}_2 h \in \mathcal{E}$ if $p$ even, $q$ odd.

Hence $\mathcal{X} \subset \mathcal{R}$, therefore $\mathcal{R} = \mathcal{S} = \mathcal{X}$.  

Step 3:

\[ \tau_1^2 = \bar{r}_2 f_0, \quad \tau_2^2 = f_1 f_0 f_1 \bar{r}_1, \quad \bar{r}_2 = f_1 \bar{r}_1 f_1. \]

This completes the sketch of the proof of Theorem 6.
From smooth isotopy to Lagrangian isotopy.

Corollary 8. Let $L \subset \mathbb{R}^4$ be an embedded Lagrangian torus smoothly isotopic to a Clifford torus $T$. Then there exists a smooth isotopy $\phi_t \in \text{Diff}_c^c(\mathbb{R}^4)$ with $\phi_0 = \text{id}$, $\phi_1(L) = T$, such that $\phi_1^* \mu_T = \mu_L$.

We can improve the smooth isotopy $L_t$ so that it is indeed a Lagrangian isotopy outside a disc:

Lemma 9. Let $L_t = \phi_s(L_0)$, $s \in [0,1]$, be a smooth isotopy between a Lagrangian torus $L = L_0$ and a Clifford torus $T = L_1$ with $\phi_t \in \text{Diff}_c^c(\mathbb{R}^4)$, $\phi_0 = \text{id}$, and $\phi_1^* \mu_T = \mu_L$. Then there exists an smooth isotopy $L'_s = \phi'_s(L'_0)$ between $L = L'_0$ and $T = L'_1$ and a disc $D \subset L$ such that $L'_s \setminus \phi'_s(D)$ is Lagrangian for all $s \in [0,1]$. 

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Remarks.

• (Mohnke ’01, preprint)

Any Lagrangian torus in $\mathbb{R}^4$ is smoothly isotopic to the Clifford torus.

(Use pseudoholomorphic curve techniques due to Gromov (’85).)

• (Ivrii ’03, thesis)

Any Lagrangian torus in $\mathbb{R}^4$ is Lagrangian isotopic to the Clifford torus.

(Use methods of Symplectic Field Theory due to Eliashberg, Givental and Hofer.)