An optimal consumption and investment problem with partial information
(Joint work with Shuenn-Jyi Sheu)
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1 Introduction

Merton (1971)

Maximization problem of the following expectation (HARA utility of consumption):

$$E \left[ \int_0^T \frac{1}{\gamma} e^{-\rho t} C_t^\gamma dt + Pe^{-\rho T} \right], \quad \rho > 0,$$

where $C_t$ is the rate of consumption at time $t$ and $P$ is the payment which the agent receives at time $T$.

- The interest rate is a constant.
- The price of the risky asset is governed by a geometric Brownian motion.
- HJB equation has an explicit solution.
Liu (2007) Maximization problem of the following expectation:

\[ E \left[ \int_0^T \frac{1}{\gamma} e^{-\rho t} C_t^\gamma dt + (1 - \alpha) e^{-\rho T} \frac{X_T^\gamma}{\gamma} \right], \quad \rho > 0, \]

where \( C_t \) is the rate of consumption at time \( t \) and \( X \) is the wealth at time \( T \).

- Complete or incomplete market for factor model with quadratic returns.
- Solving the HJB equation to obtain explicit solution.
  - In incomplete markets, explicit solutions can be obtained only if \( \alpha = 0 \).
• **Liu (2007)**
  - Full information (The investor selects his strategies with using all past information of risky stocks $S$ and economic factors $Y$)
  - An explicit solution (optimal strategy and optimal value) can be obtained in complete market.

Full information case seems to be not always realistic (economic factors $Y$ are to be considered implicit).

• **H – S.J. Sheu (2010)**
  - Partial information (The investor selects his strategies with using only all past information of risky stocks $S$)
  - An explicit solution can be obtained in incomplete market.
• **Linear Gaussian stochastic factor model**

bank account: \( dS_t^0 = rS_t^0 \, dt, \quad S_0^0 = s^0 \)

\( m \) risky stocks: \( dS_t^i = S_t^i \left\{ (a + AY_t)^i \, dt + \sum_{k=1}^{n+m} \Sigma^i_k dW^k_t \right\}, \quad S^i(0) = s^i, \quad i = 1, \ldots, m \)

\( n \) economic factors: \( dY_t = (b + BY_t)dt + \Lambda dW_t, \quad Y(0) = y \in \mathbb{R}^n. \)

• \( W_t = (W^k_t)_{k=1, \ldots, (n+m)} \) \( \cdots \) an \( m + n \)-dimensional standard Brownian motion process

• \( r \geq 0, \)

• \( a \in \mathbb{R}^m, b \in \mathbb{R}^n, \)

• \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times n}, \Sigma \in \mathbb{R}^{m \times (n+m)} \) and \( \Lambda \in \mathbb{R}^{n \times (n+m)}. \)

In this talk, we always assume that

(\( H \)) \quad \Sigma \Sigma^* > 0.
\[ \pi_t^0 \cdots \text{the proportion of wealth invested in the bank account at time } t \]
\[ \pi_t^i \cdots \text{the proportion of wealth in the } i\text{-th risky stock } S^i \text{ at time } t \]
\[ c_t = \frac{C_t}{X_t} \text{ (the rate of consumption at time } t) \]

\[ \pi_t : = (\pi_t^1, \cdots \pi_t^m) \Rightarrow \text{investment strategy } \]

Set

\[ \mathcal{G}_t : = \sigma(S_u; u \leq t). \]

Define the space \( \mathcal{H}_T \) of investment strategies as

\[ \mathcal{H}_T : = \{(c_t, \pi_t)_{t \in [0, T]} ; (c_t, \pi_t) \text{ is a } [0, T] \times \mathbb{R}^m\text{-valued } \mathcal{G}_t\text{-progressively measurable stochastic process such that } c_t + |h_t| \leq C(\omega), \ t \in [0, T], \ P - a.s \} . \]

Here \( C(\omega) \) is some constant.
By the self-financing condition, the investor’s wealth $X_{t}^{c,h}$ satisfies

$$\begin{align*}
\frac{dX_{t}^{c,h}}{X_{t}^{c,h}} &= (1 - h_{t}^{*}1) \frac{dS_{t}^{0}}{S_{t}^{0}} + \sum_{i=1}^{m} h_{t}^{i} \frac{dS_{t}^{i}}{S_{t}^{i}} - c_{t} dt, \\
X_{0}^{c,h} &= x,
\end{align*}$$

where $1 := (1, \cdots, 1)^{*}$.

Our goal is to select consumption and investment controls which maximize the finite time horizon discounted expected HARA utility of consumption and terminal wealth:

$$\begin{align*}
\text{(IC)} \quad V(0, x, y) := \sup_{(c, \pi) \in \mathcal{A}_{T}} J(x, y; c, h; T), \quad \rho > 0, \\
J(x, y; c, h; T) := E \left[ \int_{0}^{T} e^{-\rho t} \frac{1}{\gamma} (c_{t}X_{t}^{c,\pi})^{\gamma} dt + e^{-\rho T} \frac{1}{\gamma} (X_{T}^{c,\pi})^{\gamma} \right].
\end{align*}$$

Here $\mathcal{A}_{T}(\subset \mathcal{H}_{T})$ is the space of admissible strategies defined later.
Outline

1. To reformulate (IC) as a stochastic control problem with partial information,

2. To determine a corresponding stochastic control problem with full information.

3. Dynamic programming approach leads to HJB equation.

4. A suitable transformation $\implies$ HJB equation turns out to be a linear partial differential equation.

5. To obtain an explicit solution of the linear partial differential equation (HJB equation).

6. To show the verification theorem for $\gamma \in (0, 1)$ (To check that the maximizer in HJB equation is optimal) that $Z_R$. 
2 Reduction to a corresponding control problem with full information

Recall

\[
\left( X_{t}^{c,h} \right)^{\gamma} = x^{\gamma} \phi_{t}^{c,h},
\]

where \( \phi_{t}^{c,h} \), \( \Phi(y, h; \gamma) \) and \( \ell(y, c, h; \gamma) \) are defined by

\[
(2.1) \quad \phi_{t}^{c,h} := \exp \left[ \gamma \int_{0}^{t} \ell(Y_u, c_u, h_u; \gamma) du + \gamma \int_{0}^{t} h_u^{*} \Sigma dW_u - \frac{\gamma^2}{2} \int_{0}^{t} h_u^{*} \Sigma \Sigma^{*} h_u du \right],
\]

\[
\Phi(y, h; \gamma) := -\frac{1 - \gamma}{2} h^{*} \Sigma \Sigma^{*} h + r + h^{*} (Ay + a - r 1),
\]

\[
\ell(y, c, h; \gamma) := \Phi(y, h; \gamma) - c.
\]

Therefore, \((IC)\) is reduced to the problem maximizing

\[
(2.2) \quad J(x, y; c, h; T) = \frac{x^{\gamma}}{\gamma} E \left[ \int_{0}^{T} e^{-\rho t} c^{\gamma} \phi_{t}^{c,h} dt + e^{-\rho T} \phi_{T}^{c,h} \right].
\]
Now we shall reformulate (IC) as a partially observable stochastic problem. Set

\[ Z^i_t := \log S^i_t, \quad i = 1, \cdots m, \]

with \( Z_t = (Z^1_t, \cdots Z^m_t) \). Then \( Z \) solves

\[ dZ_t = (d + AY_t) dt + \Sigma dW_t, \tag{2.3} \]

with \( d = (d^i) := (a^i - \frac{1}{2}(\Sigma \Sigma^*)^{ii}) \).

Now let us introduce a new probability measure \( \hat{P} \) on \((\Omega, \mathcal{F})\) defined by

\[
\frac{d\hat{P}}{dP} \bigg|_{\mathcal{F}_T} = \eta_T := e^{-\int_0^t (d + AY_u)^*(\Sigma \Sigma^*)^{-1} \Sigma dW_u - \frac{1}{2} \int_0^t (d + AY_u)^*(\Sigma \Sigma^*)^{-1} (d + AY_u) du}. 
\]
Note that $E[\eta_T] = 1$. Hence, under $\hat{P}$ we have

$$dZ_t = \Sigma d\hat{W}_t,$$

$$dY_t = \{b + BY_t - \Lambda \Sigma^*(\Sigma \Sigma^*)^{-1}(d + AY_t)\} \, dt + \Lambda d\hat{W}_t,$$

where $\hat{W}_t := W_t + \int_0^t \Sigma^*(\Sigma \Sigma^*)^{-1}(d + AY_u) \, du$ is a Brownian motion process under $\hat{P}$. Then, under $\hat{P}$ we have

$$J(x, y; c, h; T) = \frac{x^\gamma}{\gamma} \hat{E} \left[ \int_0^T e^{-\rho t} c_t^\gamma \eta_t^{-1} \phi_t^{c,h} \, dt + e^{-\rho T} \eta_T^{-1} \phi_T^{c,h} \right]$$

$$= \frac{x^\gamma}{\gamma} \left[ \int_0^T e^{-\rho t} \hat{E} \left[ c_t^\gamma e^{-\gamma \int_0^t c_u \, du} \hat{E} \left[ \exp \left\{ \gamma \int_0^t \Phi(Y_u, h_u; \gamma) \, du \right\} \Psi_t | G_t \right] \right] dt \\
+ e^{-\rho T} \hat{E} \left[ e^{-\gamma \int_0^T c_u \, du} \hat{E} \left[ \exp \left\{ \gamma \int_0^T \Phi(Y_u, h_u; \gamma) \, du \right\} \Psi_T | G_T \right] \right] \right].$$
Here $\Psi_t$ is defined by

\begin{equation}
\Psi_t := e^{\int_0^t Q^*(Y_u, h_u) dZ_u - \frac{1}{2} \int_0^t Q^*(Y_u, h_u) \Sigma \Sigma^* Q^*(Y_u, h_u) du}
\end{equation}

(2.4)

\begin{equation}
Q(y, h) := (\Sigma \Sigma^*)^{-1}(d + Ay) + \gamma h.
\end{equation}

(2.5)

Setting

\begin{equation}
q^h(t)(\varphi(t)) := \hat{E} \left[ \exp \left\{ \gamma \int_0^t \Phi(Y_u, h_u; \gamma) du \right\} \Psi_t \varphi(t, Y_t) \bigg| \mathcal{G}_t \right],
\end{equation}

we have

\begin{equation}
J(x, y; c, h; T) = \frac{x^\gamma}{\gamma} \left[ \int_0^T e^{-\rho t} \hat{E} \left[ c_t^\gamma e^{-\gamma \int_0^t c_u du} q^h(t)(1) \right] dt 
+ e^{-\rho T} \hat{E} \left[ e^{-\gamma \int_0^T c_u du} q^h(T)(1) \right] \right].
\end{equation}

(2.6)
Using Proposition 2.1 of Nagai (2001), we see that $q^h(t)(\varphi(t))$ solves the following stochastic partial differential equation

(2.7)

\[ q^h(t)(\varphi(t)) = q^h(0)(\varphi(0)) \]
\[ + \int_0^t q(s) \left( \frac{\partial \varphi}{\partial t}(s, \cdot) + L\varphi(s, \cdot) + \gamma h_s^* \Sigma \Lambda^* D\varphi(s, \cdot) + \gamma \Phi_s(\cdot) \varphi(s, \cdot) \right) ds \]
\[ + \int_0^t q(s) (\varphi(s, \cdot) Q(\cdot, h_s)) dZ_s + \int_0^t q(s) \left( (D\varphi)^*(s, \cdot) \Lambda \Sigma^* (\Sigma \Sigma^*)^{-1} \right) dZ_s, \]

where $L\varphi := \frac{1}{2}(\Lambda \Lambda^*)^{ij} D_{ij} \varphi + (b + By)^i D_i \varphi$ and $\Phi_s(\cdot) := \Phi(\cdot, h_s; \gamma)$. 

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Furthermore, from Proposition 2.1 of Nagai (2001), (2.7) has an explicit solution:

\begin{equation}
q^h(t)(\varphi(t)) = \beta_t \int \varphi(t, \tilde{Y}_t + \Pi^{1/2}z) \frac{1}{(2\pi)^{n/2}} e^{-|z|^2} \, dz.
\end{equation}

Here \( \tilde{Y}_t = E[Y|G_t] \) solves the stochastic differential equation

\begin{equation}
d\tilde{Y}_t = \left\{ b + B\tilde{Y}_t - \lambda(\Pi)(A\tilde{Y}_t + d) \right\} dt + \lambda(\Pi)dZ_t, \quad \tilde{Y}_0 = y,
\end{equation}

where \( \lambda(\Pi) := (\Pi A^* + \Lambda \Sigma^*)(\Sigma \Sigma^*)^{-1} \). And \( \Pi_t = E[(Y_t - \tilde{Y}_t)(Y_t - \tilde{Y}_t)^*|G_t] \) solves the Riccati equation

\begin{equation}
\dot{\Pi}_t + (\Pi A^* + \Lambda \Sigma^*)(\Sigma \Sigma^*)^{-1}(A\Pi + \Sigma \Lambda^*) - \Lambda \Lambda^* - B\Pi - \Pi B^* = 0, \quad \Pi_0 = 0.
\end{equation}

Moreover, \( \beta_t \) is defined by

\begin{align*}
\beta_t := \exp \left\{ \gamma \int_0^t \Phi(\tilde{Y}_u, h_u; \gamma) \, du \right\} \tilde{\Psi}_t,
\end{align*}

\begin{align*}
\tilde{\Psi}_t := e^{\int_0^t Q^*(\tilde{Y}_u, h_u) \, dZ_u - \frac{1}{2} \int_0^t Q^*(\tilde{Y}_u, h_u) \Sigma \Sigma^* Q^*(\tilde{Y}_u, h_u) \, du}
\end{align*}
Setting
\[ \alpha_t = \alpha_t^{c,h} := e^{-\gamma \int_0^t c_u \, du} \beta_t = \exp \left\{ \gamma \int_0^t \ell(\tilde{Y}_u, c_u, h_u; \gamma) \, du \right\} \hat{\Psi}_t, \]
we have
\[ J(x, y; c, h; T) = \frac{x^\gamma}{\gamma} \hat{E} \left[ \int_0^T e^{-\rho t} c^\gamma \alpha_t^{c,h} \, dt + e^{-\rho T} \alpha_T^{c,h} \right]. \] (2.10)

Therefore, \((IC)\) is reduced to a stochastic control problem with full information, which is the problem maximizing the criterion (2.10) subject to a system process \(\tilde{Y}_t\) given by (2.9) on the filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}_t, \hat{P})\).  

3 HJB equation and its explicit solution

If a given strategy $h$ satisfies

$$\hat{E} \left[ \hat{\Psi}_T \right] = 1,$$

then we can define the probability measure $P^h$ defined by

$$\frac{d\hat{P}}{dP} \bigg|_{\mathcal{G}_T} = \hat{\Psi}_T.$$

Under $P^h$ $\hat{Y}_t$ solves

$$d\hat{Y}_t = \left\{ b + B\hat{Y}_t + \gamma \lambda(\Pi)\Sigma\Sigma^* h_t \right\} dt + \lambda(\Pi)dZ^h_t, \quad \hat{Y}_0 = y,$$

where $Z^h_t = Z_t - \int_0^t \Sigma\Sigma^* Q(\hat{Y}_u, h_u)du$ is a $\mathcal{G}_t$-martingale with

$$\langle Z^h \rangle_t = \Sigma\Sigma^* t.$$
Then the criterion (2.10) reads

\[
J(x, y; c, h; T) = \frac{x^\gamma}{\gamma} \hat{E}^h \left[ \int_0^T c^\gamma e^{\gamma \int_t^s \{\ell(\hat{Y}_u, c_u, h_u) - \rho\} du} \ d\gamma \left( t \right) + e^{\gamma \int_t^T \{\ell(\hat{Y}_u, c_u, h_u) - \rho\} du} \right].
\]

Then we introduce the value functions:

\[
V(t, x, y) = \sup_{(c,h) \in \mathcal{A}_{t,T}} \frac{x^\gamma}{\gamma} I(y; c, h; [t, T]),
\]

\[
I(y; c, h; [t, T]) := \hat{E}^h_{t,y} \left[ \int_t^T c^\gamma e^{\gamma \int_t^s \{\ell(\hat{Y}_u, c_u, h_u) - \rho\} du} \ ds + e^{\gamma \int_t^T \{\ell(\hat{Y}_u, c_u, h_u) - \rho\} du} \right].
\]

Here $\mathcal{A}_{t,T}$ is the restriction of $\mathcal{A}_T$ on the time interval $[t, T]$. Indeed, we consider

\[
v(t, y) = \sup_{(c,h) \in \mathcal{A}_{t,T}} \frac{1}{\gamma} I(y; c, h; [t, T]).
\]
Then, we decuce the following HJB equation for $v$:

$$
\begin{align*}
\frac{\partial v}{\partial t} + \frac{1}{2} tr \left[ \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* D^2 v \right] + (b + By)^* Dv + (-\rho + \gamma r)v \\
+ \gamma \sup_{h \in \mathbb{R}^m} \left[ -\frac{1 - \gamma}{2} h^* \Sigma \Sigma^* h + h^* \left\{ Ay + a - r1 + \lambda(\Pi) \Sigma \Sigma^* \frac{Dv}{v} \right\} \right] v \\
+ \gamma \sup_{c \geq 0} \left\{ -cv + \frac{c^\gamma}{\gamma} \right\} = 0,
\end{align*}
$$

$$v(T, y) = \frac{1}{\gamma}.$$
If we assume \( v(t, y) := \frac{1}{\gamma} u(t, y)^{1-\gamma} \), then \( u \) satisfies the following:

(3.3)
\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{2} \text{tr} \left[ \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* D^2 u \right] - \frac{\gamma}{2u} (Du)^* \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* Du \\
+ (b + By)^* Du + \frac{1}{1-\gamma} (-\rho + \gamma r) u \\
+ \frac{\gamma}{1-\gamma} \sup_{h \in \mathbb{R}^m} \left[ -\frac{1-\gamma}{2} h^* \Sigma \Sigma^* h + h^* \left\{ Ay + a - r \mathbf{1} + (1-\gamma) \lambda(\Pi) \Sigma \Sigma^* \frac{Du}{u} \right\} \right] v \\
+ \frac{\gamma}{1-\gamma} \sup_{c \geq 0} \left\{ -cu^{1-\gamma} + \frac{c^\gamma}{\gamma} \right\} u^\gamma = 0,
\end{align*}
\]

\( u(T, y) = 1. \)

Recall that the supremum in (3.3) is attained by

\[
(\tilde{c}(t, y), \tilde{\pi}(t, y)) := \left( u^{-1}, \frac{1}{1-\gamma} (\Sigma \Sigma^*)^{-1} \left\{ Ay + a - r \mathbf{1} + (1-\gamma) \lambda(\Pi) \Sigma \Sigma^* \frac{Du}{u} \right\} \right).\]
Then, (3.3) turns out to be

(3.4) \[ \frac{\partial u}{\partial t} + \mathcal{L}_t u + 1 = 0, \quad u(T, y) = 1, \]

where the operator \( \mathcal{L}_t \) on any function \( u \) is defined by

(3.5) \[
\mathcal{L}_t u := \frac{1}{2} tr \left[ \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* D^2 u \right] \\
+ \left[ b + \frac{\gamma}{1 - \gamma} \lambda(\Pi)(a - r\mathbf{1}) + \left\{ B + \frac{\gamma}{1 - \gamma} \lambda(\Pi) A \right\} y \right]^* D u \\
+ \frac{1}{2 - \gamma} \left\{ \frac{\gamma}{2(1 - \gamma)} (Ay + a - r\mathbf{1})^* (\Sigma \Sigma^*)^{-1} (Ay + a - r\mathbf{1}) + \gamma r - \rho \right\} u.
\]
Let us consider the following time inhomogeneous Riccati equation:

\[
\begin{cases}
\dot{U}(t) + U(t)K_2(t)U(t) + K_1(t)^*U(t) + U(t)K_1(t) + K_0 = 0, \\
U(s) = 0,
\end{cases}
\]

where

\[
K_2(t) := \frac{1}{1 - \gamma} \lambda(\Pi)\Sigma\Sigma^*\lambda(\Pi)^* \geq 0,
\]

\[
K_1(t) := B + \frac{\gamma}{1 - \gamma} \lambda(\Pi)A, \quad K_0 := \frac{\gamma}{1 - \gamma} A^*(\Sigma\Sigma^*)^{-1}A.
\]

We also use \(U(t) = U(t; s)(U(t; s; \gamma))\) for the dependence of \(U(t)\) on \(s \in [t, T]\) (and \(\gamma\)).
The term $g(t)$ is the solution of a linear differential equation:

\[
\begin{cases}
\dot{g}(t) + \{K_1(t) + K_2(t)U(t)\} g(t) + U(t)b \\
+ \frac{\gamma}{1 - \gamma} \left\{ A^* (\Sigma \Sigma^*)^{-1} + U(t)\lambda(\Pi) \right\} (a - r1) = 0, \\
q(s) = 0,
\end{cases}
\]

(3.8)

and $l(t)$ is the solution of

\[
\begin{cases}
\dot{l}(t) + \frac{1}{2} tr [\lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi) U(t)] + \frac{1}{2} g(t)^* \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* g(t) + b^* g(t) + \gamma r \\
+ \frac{\gamma}{2(1 - \gamma)} (a - r1 + \Sigma \Sigma^* \lambda(\Pi)^* g(t))^* (\Sigma \Sigma^*)^{-1} (a - r1 + \Sigma \Sigma^* \lambda(\Pi)^* g(t)) = 0, \\
l(s) = 0.
\end{cases}
\]

(3.9)
Now we consider the following equation:

\[
\frac{\partial w}{\partial t} + \mathcal{L}_t w = 0, \quad w(s, y) = 1.
\]

- \(\bar{w}(t; s, y) := e^{\frac{1}{1-\gamma} \left\{ \frac{1}{2} y U(t; s) y + g(t; s) y + l(t; s) - \rho(t - s) \right\} } \) solves (3.10).

Then we have the following.

**Theorem 1** If (3.6) has a solution, (3.4) has a solution:

\[
\bar{u}(t, y) := \int_t^T \bar{w}(t, s, y) ds + \bar{w}(t; T, y).
\]

Further, the value function \(\bar{V}(t, x, y)\) of (IC) is given by

\[
\bar{V}(t, x, y) = \frac{x^\gamma}{\gamma} \bar{u}(t, y)^{1-\gamma}.
\]
4 Verification theorem for $\gamma \in (0, 1)$

Define the space of admissible strategies:
\[ A_T := \left\{ h_t \in \mathcal{H}_T; \int_0^T |h_t|^2 dt < \infty, \; \hat{P} - a.s. \right\}. \]

Our objective is to obtain the following.

**Theorem 2** Let $\gamma \in (0, 1)$. Assume $(H)$, and that $(3.6)$ has a solution. Define
\[
\begin{align*}
\hat{c}(t, y) &:= \bar{u}(t, y)^{-1}, \\
\hat{h}(t, y) &:= \frac{1}{1 - \gamma} (\Sigma \Sigma^*)^{-1}\left\{ a - r1 + Ay + (1 - \gamma)(A\Pi + \Sigma\Lambda^*) \frac{D\bar{u}(t, y)}{\bar{u}(t, y)} \right\},
\end{align*}
\]
where $\bar{u}$ is given by $(3.11)$. Then,
\[
\left( \hat{c}, \hat{h} \right) := \left( \hat{c}(t, \hat{Y}_t), \hat{\pi}(t, \hat{Y}_t) \right) \in A_T
\]
is an optimal strategy for $(IC)$. Namely $V(0, x, y) = \overline{V}(0, x, y)$. 

Proof 1 First, for any \((c_t, h_t) \in A_T\), we shall show

\[
J(x, y; c, h, T) \leq \bar{V}(0, x, y),
\]

or equivalently

\[
(4.2) \quad \hat{E} \left[ \int_0^T e^{-\rho t} c_t \gamma c_t^{c,h} dt + e^{-\rho T} \alpha_T^{c,h} \right] \leq \bar{u}(0, y)^{1-\gamma}.
\]

Define the stopping time

\[
\tau_R := \inf\left\{ t > 0; |\hat{Y}_t| > R \right\} \land \inf\left\{ t > 0; \int_0^t |h_s|^2 ds > R \right\}.
\]
Then, we have

\[
\bar{u}(0, y)^{1-\gamma} = \hat{E} \left[ \int_0^{\bar{T} \land \tau_R} e^{-\rho t} c_t^\gamma \alpha_t^{c,h} dt + e^{-\rho (\bar{T} \land \tau_R)} \alpha_{T \land \tau_R}^{c,h} \bar{u}(T \land \tau_R, \hat{Y}_{T \land \tau_R})^{1-\gamma} \right] \\
- (1 - \gamma) \hat{E} \left[ \int_0^{\bar{T} \land \tau_R} e^{-\rho t} \alpha_t^{c,h} \bar{u}(t, \hat{Y}_t)^{-\gamma} \left( \frac{\partial \bar{u}}{\partial t} + \mathcal{L}_t^{c,h} \bar{u} \right)(t, \hat{Y}_t) dt \right].
\]

where \(\mathcal{L}_t^{c,h} \bar{u}\) is defined by

\[
\mathcal{L}_t^{c,h} \bar{u} := \frac{1}{2} tr \left[ \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* D^2 \bar{u} \right] - \frac{\gamma}{2u} (D\bar{u})^* \lambda(\Pi) \Sigma \Sigma^* \lambda(\Pi)^* D\bar{u} \\
+ (b + By)^* D\bar{u} + \frac{1}{1 - \gamma} (-\rho + \gamma r)\bar{u} + \frac{\gamma}{1 - \gamma} \left[ -\frac{1 - \gamma}{2} h^* \Sigma \Sigma^* h \right. \\
+h^* \left\{ Ay + a - r \mathbf{1} + (1 - \gamma)\lambda(\Pi) \Sigma \Sigma^* \frac{D\bar{u}}{\bar{u}} \right\} \bar{u} + \frac{\gamma}{1 - \gamma} \left( -c\bar{u}^{1-\gamma} + \frac{c^\gamma}{\gamma} \right) \bar{u} \gamma.
\]
Here we observe
\[
\frac{\partial \bar{u}}{\partial t} + \mathcal{L}^{c,h}_t \bar{u} \leq \frac{\partial \bar{u}}{\partial t} + \mathcal{L}_t \bar{u} + 1 = 0.
\]

Hence we see that
\[\bar{u}(0, y)^{1-\gamma} \geq \hat{E} \left[ \int_0^{T \wedge \tau_R} e^{-\rho t} c_t^\gamma \alpha_t^{c,h} \, dt + e^{-\rho (T \wedge \tau_R)} \alpha_{T \wedge \tau_R}^{c,h} \bar{u}(T \wedge \tau_R, \hat{Y}_{T \wedge \tau_R})^{1-\gamma} \right].\]

Using the monotone convergence theorem, for any fixed \(T > 0\), we have
\[\lim_{R \to \infty} \hat{E} \left[ \int_0^{T \wedge \tau_R} e^{-\rho t} c_t^\gamma \alpha_t^{c,h} \, dt \right] = \hat{E} \left[ \int_0^{T} e^{-\rho t} c_t^\gamma \alpha_t^{c,h} \, dt \right].\]
Now we want to show

\[ \lim_{R \to \infty} \hat{E} \left[ e^{-\rho (T \wedge \tau_R)} \alpha_{T \wedge \tau_R}^{c,h} \tilde{u}(T \wedge \tau_R, \hat{Y}_{T \wedge \tau_R})^{1-\gamma} \right] = \hat{E} \left[ e^{-\rho T} \alpha_{T}^{c,h} \right]. \]

Here we observe

\[ x^\gamma \hat{E} \left[ e^{-\rho (T \wedge \tau_R)} \alpha_{T \wedge \tau_R}^{c,h,(\gamma)} \tilde{u}_{\gamma}(T \wedge \tau_R, \hat{Y}_{T \wedge \tau_R})^{1-\gamma} \right] \]

\[ = E \left[ e^{-\rho (T \wedge \tau_R)} \left( X_{T \wedge \tau_R}^{c,h} \right)^\gamma \tilde{u}_{\gamma}(T \wedge \tau_R, \hat{Y}_{T \wedge \tau_R})^{1-\gamma} \right]. \]

Here, we use \( \alpha^{c,h} = \alpha^{c,h,(\gamma)} \) for the dependence of \( \alpha^{c,h} \) on \( \gamma \), and note that under \( P \hat{Y}_t \) solves

\[ d\hat{Y}_t = \left\{ b + B\hat{Y}_t - \lambda(\Pi)(A\hat{Y}_t + d) \right\} dt + \lambda(\Pi)dz_t, \quad \hat{Y}_0 = y, \]

where \( Z \) is given by (2.3).
Therefore, we now shall prove that there exists $C > 0$ which is a
independent of $R$ shch that

$$E \left[ e^{-\rho(T \land \tau_R)} \left( X_{T \land \tau_R}^{c,h} \right)^\theta \bar{u}_\gamma (T \land \tau_R, \hat{Y}_{T \land \tau_R})^{(1-\gamma)(1+\delta)} \right] \leq C,$$

where $\theta := (1 + \delta)\gamma$. Indeed, we need to prove

$$(4.8) \quad \hat{E} \left[ \alpha_{T \land \tau_R}^{c,h,(\theta)} \bar{u}_\gamma (T \land \tau_R, \hat{Y}_{T \land \tau_R})^{(1-\gamma)(1+\delta)} \right] \leq C.$$

Here we can observe the following:

- $\bar{u}_\gamma (t, y)^{(1-\gamma)(1+\delta)} \leq C_1 e^{\psi_\theta (t, y)},$

- $\alpha_{t}^{c,h,(\theta)} e^{\psi_\theta (t, y)} \leq e^{\psi_\theta (0, y)} \Gamma_t^h, \quad \hat{P} - a.s.$,
where \( \psi_\gamma(t, y) \) and \( \Gamma_t^h \) are defined by

\[
\psi_\gamma(t, y) := \frac{1}{2} y^* U(t; T; \gamma) y + g(t; T; \gamma)^* y + l(t; T; \gamma),
\]

\[
\Gamma_t^h := e^{\int_0^t \{Q_\theta(\hat{Y}_s, h_s) + \lambda(\Pi)^* D\psi_\theta(s, \hat{Y}_s)\}^* dZ_s}
\]

\[
e^{-\frac{1}{2} \int_0^t \{Q_\theta(\hat{Y}_s, h_s) + \lambda(\Pi)^* D\psi_\theta(s, \hat{Y}_s)\}^* \Sigma \Sigma^* \{Q_\theta(\hat{Y}_s, h_s) + \lambda(\Pi)^* D\psi_\theta(s, \hat{Y}_s)\} ds}.
\]

Here, we use \( Q(y, h) = Q_\theta(y, h) \) for the dependence of \( Q(y, h) \) on \( \theta \).

**Remark 1** \( \log x^\gamma + \psi_\gamma(0, y) \) is the optimal value of risk sensitive portfolio optimization problem on finite time horizon is to consider

\[
V_{RS} = \gamma \sup_{\pi \in \mathcal{A}_{RS}^T} \frac{1}{\gamma} \log E [(X_T^\pi)^\gamma],
\]

where \( X_t^h \) is the investor’s wealth and \( \mathcal{A}_{RS}^T \) is the space of admissible strategies.
Then, we see that

\[
\hat{E} \left[ \alpha_{T \wedge \tau_R}^{c,h,(\theta)} \bar{u}_\gamma(T \wedge \tau_R, \hat{Y}_{T \wedge \tau_R})^{(1-\gamma)(1+\delta)} \right] \leq C_1 \hat{E} \left[ \alpha_{T \wedge \tau_R}^{c,h,(\theta)} e^{\psi_\theta(T \wedge \tau_R, \hat{Y}_{T \wedge \tau_R})} \right] \\
\leq C_1 e^{\psi_\theta(0,y)}.
\]

We have proved (4.8) and (4.5). As \( R \to \infty \) in (4.3) we see that for any fixed \( T > 0 \) (4.9)

\[
(4.9) \quad \hat{E} \left[ \int_0^T e^{-\rho t} c_t^\gamma \alpha_t^{c,h} \, dt + e^{-\rho T} \alpha_T^{c,h} \right] \leq \bar{u}(0,y)^{1-\gamma}.
\]
We shall prove to

\[ V(0, x, y) = J(x, y; \hat{c}, \hat{h}, T) = \overline{V}(0, x, y). \]

or equivalently

\[ \bar{u}(0, y)^{1-\gamma} = E \left[ \int_0^T e^{-\rho t} c_t \gamma_t \alpha_t^{c, h} dt + e^{-\rho T} \alpha_T^{c, h} \right], \tag{4.10} \]

- \[ \left| \frac{D\bar{u}_\gamma(t, y)}{\bar{u}_\gamma(t, y)} \right| \leq K(1 + |y|) \Rightarrow (\hat{c}_t, \hat{h}_t) \in \mathcal{A}_T. \]

Note that \((\hat{c}_t, \hat{h}_t)\) is a maximizer in (3.3), we have

\[ \bar{u}(0, y)^{1-\gamma} = E \left[ \int_0^{T \wedge \hat{\tau}_R} e^{-\rho t} c_t \gamma_t \alpha_t^{c, h} dt + e^{-\rho (T \wedge \hat{\tau}_R)} \alpha_{T \wedge \hat{\tau}_R}^{c, h} \bar{u}(T \wedge \hat{\tau}_R, \hat{Y}_{T \wedge \hat{\tau}_R})^{1-\gamma} \right], \]

where \(\hat{\tau}_R := \inf \left\{ t > 0; |\hat{Y}_t| > R \right\} \). In a similar way as above, we obtain (4.10).