Recent results on the multiplicative renormalization method for orthogonal polynomials

Hui-Hsiung Kuo

Louisiana State University

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Outline

1. Introduction
   - Orthogonal polynomials
   - An example (key idea)
   - Another example (key idea)

2. Multiplicative Renormalization Method
   - OP-generating function
   - MRM procedure
   - Classical distributions

3. Characterization Theorems
   - Characterization problems
   - MRM-applicable measures
   - MRM-factors

4. References

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Recent results on MRM for orthogonal polynomials
\( \mu: \) a probab measure on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty, \quad \forall n \geq 1 \)

Apply the Gram-Schmidt orthogonalization process to get

\[ \{1, x, \ldots, x^n, \ldots\} \mapsto \{P_0(x), P_1(x), \ldots, P_n(x), \ldots\} \]

where \( P_n(x) \) is a poly of degree \( n \) with leading coefficient 1.

**Theorem**

*(Recursion formula)* \( \exists \) sequences \( \{\alpha_n\}, \{\omega_n\} \) such that

\[ xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \omega_n P_{n-1}, \quad n \geq 0, \]

where \( \omega_0 = 1 \) and \( P_{-1} = 0 \).

**Question:** Given \( \mu \), how can we derive \( \{P_n(x), \alpha_n, \omega_n\} \)?
\( \mu \): a probab measure on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty, \quad \forall n \geq 1 \)

Apply the **Gram-Schmidt orthogonalization process** to get

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$$E e^{tx} = e^{\frac{1}{2} \sigma^2 t^2}, \quad \psi(t,x) := \frac{e^{tx}}{E e^{tx}} = e^{tx - \frac{1}{2} \sigma^2 t^2}$$

Observation 1 $E[\psi(t,x)\psi(s,x)] = e^{\sigma^2 ts}$ is a function of $ts$.

Observation 2 We can expand $\psi(t,x)$ as a power series in $t$

$$\psi(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x)t^n$$

where $P_n(x)$ is a polynomial given by

$$P_n(x) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{2k} (2k - 1)!!(-\sigma^2)^k x^{n-2k}$$

Key Idea Observation 1 $\iff$ $P_n$'s are orthogonal (Hermite)
Let $\mu$ be Gaussian $N(0, \sigma^2)$. Then

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Let $\mu$ be Poisson with parameter $\lambda$. Then

$$E e^{\rho(t)x} = e^{-\lambda(1-e^{\rho(t)})}$$

$$\psi(t, x) := \frac{e^{\rho(t)x}}{E e^{\rho(t)x}} = e^{\rho(t)x + \lambda(1-e^{\rho(t)})}$$

Then we have

$$E \left[ \psi(t, x) \psi(s, x) \right] = e^{\lambda(e^{\rho(t)}-1)(e^{\rho(s)}-1)}$$

**Observation** $E \left[ \psi(t, x) \psi(s, x) \right]$ is a function of $ts$ if we take

$$e^{\rho(t)} - 1 = t, \quad i.e., \quad \rho(t) = \ln(1 + t)$$

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with $p_{x,0} = 1$, $p_{x,m} = x(x-1)(x-2)\cdots(x-m+1)$, $m \geq 1$.

Key Idea The above Observation $\implies$ $C_n$’s are orthogonal (Charlier polynomials)
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Let $\mu$ be a probab measure with infinite support and $\{P_n(x)\}$ the orthog polys from the Gram-Schmidt orthog process.

**Definition**

A function $\psi(t, x)$ is called an OP-generating function for $\mu$ if it has the series expansion in $t$

$$\psi(t, x) = \sum_{n=0}^{\infty} c_n P_n(x) t^n$$

where $c_n \neq 0$ for all $n$.

**Remark** $\psi(t, x)$ is called a generating function in the literature. It is a close-form function, e.g.,

$$\psi(t, x) = e^{tx - \frac{1}{2}\sigma^2 t^2} \text{ (Gaussian)}, \quad \psi(t, x) = e^{-\lambda t} (1 + t)^x \text{ (Poisson)}$$
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Fact \[ \| P_n \|^2 := \lambda_n = \omega_0 \omega_1 \cdots \omega_n, \ n \geq 0, \text{ or } \omega_n = \lambda_n/\lambda_{n-1} \]

**Theorem**

*If \( \psi(t, x) \) is an OP-generating function for \( \mu \), then*

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\int_{\mathbb{R}} \psi(t, x)^2 \, d\mu(x) = \sum_{n=0}^{\infty} c_n^2 \lambda_n t^{2n}
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*where \( c_{-1} = 0 \).*

**Conclusion** If we have an OP-generating function \( \psi(t, x) \), then we can find \( \{ P_n(x), \alpha_n, \omega_n \} \).

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Let \( h(x) \) be a “good” function. Define two functions

\[
\theta(t) = \int_{\mathbb{R}} h(tx) \, d\mu(x), \quad \tilde{\theta}(t, s) = \int_{\mathbb{R}} h(tx) h(sx) \, d\mu(x)
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**Theorem**

(Asai-Kubo-K, TJM 2003) Let \( \rho(t) \) be an analytic function at 0 with \( \rho(0) = 0 \) and \( \rho'(0) \neq 0 \). Then the multiplicative renormalization

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\psi(t, x) := \frac{h(\rho(t)x)}{\theta(\rho(t))}
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is an OP-generating function for \( \mu \) if and only if the function

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\Theta_\rho(t, s) := \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))}
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defined in some neighborhood of \((0, 0)\) is a function of \( ts \).
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*(Asai-Kubo-K, TJM 2003)* Let \( \rho(t) \) be an analytic function at 0 with \( \rho(0) = 0 \) and \( \rho'(0) \neq 0 \). Then the multiplicative renormalization

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Classical distributions with their OP-generating functions:

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<th>( h(x) )</th>
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<tr>
<td>Gaussian</td>
<td>( e^x )</td>
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<td>Poisson</td>
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<td>( e^{\lambda(e^t-1)} )</td>
<td>( \ln(1 + t) )</td>
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<td>gamma</td>
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<td>( \frac{1}{(1-t)^{\alpha}} )</td>
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<td>( \frac{1}{\sqrt{1-x}} )</td>
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<td>Stoch area</td>
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Outline

1. Introduction
   - Orthogonal polynomials
   - An example (key idea)
   - Another example (key idea)

2. Multiplicative Renormalization Method
   - OP-generating function
   - MRM procedure
   - Classical distributions

3. Characterization Theorems
   - Characterization problems
   - MRM-applicable measures
   - MRM-factors

4. References
• **First Characterization problem for** \( h(x) = e^x \)

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(Kubo, IDAQP 2004) *The class of all MRM-applicable probability measures for the function* \( h(x) = e^x \) *consists of translations and dilations of Gaussian, Poisson, gamma, Pascal, and Mexiner measures* \( M_{\kappa, \eta} \) *with parameter* \( \kappa > 0 \) *and* \( \eta \in \mathbb{R} \).

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• First Characterization problem for $h(x) = (1 - x)^{-1}$

Note that in the chart of classical distributions, there are two probab measures that are MRM-applicable for $h(x) = (1 - x)^{-1}$, namely,

(1) Arcsine

$$d\mu(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \, dx, \quad |x| < 1,$$

$$\psi(t, x) = \frac{1 - t^2}{1 - 2tx + t^2}.$$  

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$$d\mu(x) = \frac{2}{\pi} \sqrt{1 - x^2} \, dx, \quad |x| < 1,$$

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is MRM-applicable for $h(x) = (1 - x)^{-1}$ with OP-generating function given by

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Hui-Hsiung Kuo Recent results on MRM for orthogonal polynomials
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**Lemma**

Let \( \mu \) be MRM-applicable for \( h(x) = (1 - x)^{-1} \). Then \( \rho(t), \theta(\rho(t)), \) and \( \psi(t, x) \) must be given by

\[
\rho(t) = \frac{2t}{\alpha + 2\beta t + \gamma^2}, \quad \theta(\rho(t)) = \frac{1}{1 - (b + at)\rho(t)},
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Theorem

(Kubo-Namli-K, COSA 2007) For any $a > 0$ and $|b| \leq 1 - a$, the probability measure

$$d\mu_{a,b}(x) = \frac{a \sqrt{1 - x^2}}{\pi \left[ a^2 + b^2 - 2b(1 - a)x + (1 - 2a)x^2 \right]} \, dx, \ |x| < 1,$$

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Theorem

(Kubo-Namli-K, COSA 2007) *The class of all MRM-applicable probability measures for the function* \( h(x) = (1 - x)^{-1} \) *consists of translations and dilations of the probability measures of the form*

\[
d\mu(x) = W_0 \frac{\sqrt{1 - x^2}}{\pi (1 - px)(1 - qx)} 1_{(-1,1)}(x) \, dx + W_1 d\delta_{\frac{1}{p}}(x) + W_2 d\delta_{\frac{1}{q}}(x),
\]

*where* \( \delta_c \) *is the Dirac delta measure at* \( c \) *and* \( p, q, W_0, W_1, W_2 \) *are constants depending on two parameters* \( A > 0 \) *and* \( B \geq 0 \).
• **First Characterization problem for** \( h(x) = (1 - x)^{-1/2} \)

In the chart of classical distributions, the uniform distribution is MRM-applicable for the function \( h(x) = (1 - x)^{-1/2} \).

Are there other MRM-applicable probab measures for this function?

The computation for trying to find out the answer is rather complicated.

**Ideas** \( \varphi(t) := \theta(\rho(t)) \) satisfies the **Fundamental Equations**:

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\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = F_2(\rho(t), \rho'(t), t) = F_3(\rho(t), \rho'(t), t)
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which can be solved *(extremely complicated!)* to find possible forms of \( \rho(t) \). Then derive \( \varphi(t) \) and \( \theta(t) \), and finally \( \mu \).
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Theorem

(Kubo-Namli-K, COSA 2008) A probab measure $\mu$ (with infinite support) is MRM-applicable for $h(x) = (1 - x)^{-1/2}$ if and only if it is a uniform probab measure on an interval.

Remark For the function $h(x) = (1 - x)^{-1}$, the corresponding class has a lot of probab measures.
But for the function $h(x) = (1 - x)^{-1/2}$, uniform distribution on $[-1, 1]$ is the only one (up to translation and dilation).
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**Definition**

The beta distribution on $[-1, 1]$ is defined by

$$d\mu(x) = \frac{1}{\beta_{a,b}} (1 + x)^{a-1} (1 - x)^{b-1} \, dx,$$

where $\beta_{a,b} = \int_{-1}^{1} (1 + x)^{a-1} (1 - x)^{b-1} \, dx$.

Follow the same ideas as those for the case $h(x) = (1 - x)^{-1/2}$, except the computation is now much much more complicated.

**Theorem**

(Kubo-Namli-K, 2009) The class of continuous MRM-applicable probab measures for $h(x) = (1 - x)^{-2}$ consists of translations and dilations of the $\beta(\frac{5}{2}, \frac{5}{2})$, $\beta(\frac{5}{2}, \frac{3}{2})$, and $\beta(\frac{3}{2}, \frac{3}{2})$ distributions.
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• **First Characterization problem** for \( h(x) = (1 - x)^{-\kappa} \)

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(Kubo-Namli-K, 2009) Let \( \kappa > \frac{1}{2}, \kappa \neq 1 \). Then the class of continuous MRM-applicable probab measures for \( h(x) = (1 - x)^{-\kappa} \) consists of translations and dilations of the \( \beta(\kappa + \frac{1}{2}, \kappa + \frac{1}{2}), \beta(\kappa + \frac{1}{2}, \kappa - \frac{1}{2}), \text{ and } \beta(\kappa - \frac{1}{2}, \kappa - \frac{1}{2}) \) distributions.
Outline

1. Introduction
   - Orthogonal polynomials
   - An example (key idea)
   - Another example (key idea)

2. Multiplicative Renormalization Method
   - OP-generating function
   - MRM procedure
   - Classical distributions

3. Characterization Theorems
   - Characterization problems
   - MRM-applicable measures
   - MRM-factors

4. References

Hui-Hsiung Kuo
Recent results on MRM for orthogonal polynomials
Now, we address the **second characterization problem**, i.e., given an MRM-applicable probab measure $\mu$, find all MRM-factors for $\mu$.

**Notation** \((a)_n = a(a+1) \cdots (a+n-1), \ (a)_0 = 1\), rising factorial

**Definition**

A hypergeometric function is a function of the form

$$pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{1}{n!} x^n$$

**Notation** \(_0F_q(-; b_1, \ldots, b_q; x), \ pF_0(a_1, \ldots, a_p; -; x)\)

**Examples**

\(_0F_0(-; -; x) = e^x\)

\(_1F_0(\kappa; -; x) = (1 - x)^{-\kappa}\)

\(_1F_1(1; 2; x) = \frac{e^x - 1}{x}\)
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Theorem

(Kubo-K, 2010) All MRM-factors for the standard Gaussian distribution are given, up to scaling, by the functions:

\[ h(x) = e^x \quad \text{and} \quad \tilde{h}(x) = \binom{1}{2} F_1 \left( \frac{c}{2}; \frac{1}{2}; -x^2 \right) + \binom{c + 1}{2} F_1 \left( \frac{c + 1}{2}; \frac{3}{2}; -x^2 \right) x \]

where \( c \neq 0, -1, -2, -3, \ldots \)

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(Kubo-K, 2010) *MRM-factor for shifted Poisson distribution* \( \text{Poi}(\lambda) \) *is uniquely, up to scaling, given by* \( h(x) = e^x \).

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