On existence theorems of nonlinear partial differential systems

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Plane of talk

Part 0: History of solvability of linear PDE;

Part 1: Nonlinearity: Case $\mathbb{C}$;

Part 2: Nonlinearity: Case $\mathbb{R}^n$;
Part O: History of Local solvability of linear PDE

Timeline of resolution of Nirenberg-Treves conjecture

- **Lewy’s example in** $\mathbb{R}^3$ *(1957, Ann. of Math)* and *Mizohata’s in* $\mathbb{R}^2$ *(1963)*
- Hormander’s work *(1960, Ann of Math)*
- M. Beals and C. Fefferman: Sufficiency of condition $(\Psi)$ for differential equations *(1973, Ann. of Math)*
Part 1: Dimension two–Mizohata Equation

In 1963, Mizohata considered, in $\mathbb{R}^2$,

$$\frac{\partial u}{\partial x} + ix \frac{\partial u}{\partial y} = F(x, y).$$

$\exists$ a smooth function $F$ for which the above equation has no solutions near any $(0, y)$. Complex form

$$\bar{\partial}u + \frac{1 - \Re z}{1 + \Re z} \partial u = \frac{1}{1 + \Re z} F(z, \bar{z}).$$

$$\bar{\partial}u = a(z, u) \partial u + f(z, u)$$

$|a(0)| = 1$

We prove if $|a(0)|$ is small, then local solution exists.
High order Cauchy-Riemann operator-local solvability

**Theorem A:** There is a small number $\delta$ only dependent on $m, \alpha$ such that if

$$a(0) = 0$$
$$|\partial_{\eta_m} a(0)| + |ar{\partial}_{\eta_m} a(0)| < \delta$$
$$|\partial_{\eta_m} \partial_{\eta_m} a(0)| + |\partial_{\eta_m} \bar{\partial}_{\eta_m} a(0)| + |\bar{\partial}_{\eta_m} \bar{\partial}_{\eta_m} a(0)| < \delta$$

Then the system: $u = (u^1, ..., u^n), \mu + \nu = m$

$$\partial^\mu \bar{\partial}^\nu u = a(z, u, D^1 u, ..., D^{m-1} u, D^m u)$$

has solutions of class $C^{m+\alpha}(\{|z| \leq R\})$ of vanishing order $m$ at $0$ for sufficiently small values of $R$ for any $\alpha$ ($0 < \alpha < 1$).

Note: 1) In Mizohata equation $|\partial a(0)| = 1$.
2) J-holomorphic curve: $\bar{\partial} u = a(u)_{n \times n} \bar{\partial} u; a(0) = 0$
3) **Theorem:** $\delta$ must be less than 0.5.

By the counterexample of Pogorelov on $C^3$ surface isometric embedding in $\mathbb{R}^3$. 
Theorem A': Let \( a_i(z, \eta_0, \eta_1, \ldots, \eta_m); i = 1, \ldots, n \) be functions of class \( C^2 \). Assume that, \( a = (a^1, \ldots, a^n) \),

\[
a(0) = \partial_{\eta_m} a(0) = \bar{\partial}_{\eta_m} a(0) = 0,
\]

\[
\partial_{\eta_m} \partial_{\eta_m} a(0) = \partial_{\eta_m} \bar{\partial}_{\eta_m} a(0) = \bar{\partial}_{\eta_m} \bar{\partial}_{\eta_m} a(0) = 0.
\]

Then the system: \( u = (u^1, \ldots, u^n), \mu + \nu = m \)

\[
\partial^\mu \bar{\partial}^\nu u = a(z, u, D^1 u, \ldots, D^{m-1} u, D^m u)
\]

has solutions of class \( C^{m+\alpha}(|z| \leq R) \) of vanishing order \( m \) at 0 for sufficiently small values of \( R \) for any \( \alpha \) (\( 0 < \alpha < 1 \)).
Local solvability with jets

**Theorem B.** Let $a(z, \eta_0, \eta_1, \ldots, \eta_{m-1}) = (a^1, \ldots, a^n)$ be a map from $\mathbb{C}^M$ to $\mathbb{C}^n$ of class $C^{1,\alpha}$ that is independent of $\eta_m$. Let $p^i(z, \bar{z})$ be any polynomial of degree at most $m-1$ ($i = 1, \ldots, n$). Then the system: $u = (u^1, \ldots, u^n)$

$$\partial^\mu \bar{\partial}^\nu u = a(z, u, D^1 u, \ldots, D^{m-1} u)$$

has solutions of class $C^{m+\alpha}$ for sufficiently small values of $R$. The solutions can be chosen near the origin

$$u^i = p^i(z, \bar{z}) + O(|z|^m).$$
Theorem C. There is a (small) constant \( \tau \) depending only on \( m, \alpha, R \) if

\[
a(0) = 0 \text{ and } |\nabla a(0)| < \tau
\]

then

\[
\partial^\mu \bar{\partial}^\nu u = a(u, \mathcal{D}^1 u, ..., \mathcal{D}^m u)
\]

has solutions of class \( C^{m+\alpha} \) in \( D = \{|z| \leq R\} \) with vanishing order 2 at the origin.
Ordinary differential system in $\bar{\partial}$ operator

**Theorem D.** Let $F(z, \eta_0, \eta_1, ..., \eta_{m-1}) = (F^1, ..., F^n)$ be any function of $C^{1,\alpha}$ smooth in its variables with $z \in \mathbb{C}, \eta_j \in \mathbb{C}^n$. The system \( f = (f^1, ..., f^n) \)

\[
\frac{\partial^m}{\partial \bar{z}^m} f(z) = F(z, f(z), \frac{\partial}{\partial \bar{z}} f(z), ..., \frac{\partial^{m-1}}{\partial \bar{z}^{m-1}} f(z))
\]

\[
\frac{\partial^{i+j}}{\partial^i \partial \bar{z}^j} f(0) = a_{ij} \in \mathbb{C}^n, \ i + j \leq m - 1
\]

has solutions of class $C^{m+\alpha}$ near the origin.

**Remarks:** 1) solutions not unique. 2) Unique continuation property holds. 3) If $F$ is independent of $z$ and vanishes to first order at 0, then semi-global solutions exist.
Local solvability of nonlinear PDE systems in $\mathbb{R}^2$

A trivial example of no solutions: $\exp(\Delta u) = 0$.

A non trivial example by M. Khuri (2007, Comm. in PDE)

$$(u_{xx} + a)(u_{yy} + c) - (u_{xy} + b)^2 = f(x, y, u, \nabla u)$$

for some $a, b, c, f$ has no $C^3$ local solutions near $(0, 0)$. 
m-Laplace local solvability

**Theorem E.** Let $A = (A_1, A_2, ..., A_N)$ be any function of class $C^{1,\alpha}$ in its variables in $\mathbb{R}^M$. Let $p(z) = (p^1, ..., p^N)$ be any polynomial of degree $\leq 2m - 1$. Then the system of $m$-Laplace,

$$u(z) = (u^1, ..., u^N) : \{z \in \mathbb{C} : |z| \leq R\} \to \mathbb{R}^N,$$

$$\Delta^m u = A(z, u, \nabla u, ..., \nabla^{2m-1} u)$$

has solutions of class $C^{2m+\alpha}$ for sufficiently small values of $R$ s.t. $u = p(z) + O(|z|^{2m})$ near the origin.

**Osserman’s theorem says:** $R \approx 0$ in general; If $\Delta u = e^{2u}; u(0) = a, \nabla u(0) = b$ (by Theorem A), then $R \leq 2e^{-a}$. $\implies R \to 0$ as $a \to \infty$. 
Local existence of harmonic map to Riemannian manifold

An immediate corollary:

**Theorem F.** Let $S$ be a Riemann surface and $N$ be a Riemannian manifold of class $C^3$. Let $z_0 \in S$, and $p \in N$ and $v \in T_p N$. Then there is a local harmonic map $\phi$ of class $C^2$ from a neighborhood of $z_0$ to $N$ such that $\phi(z_0) = p$ and $d\phi(z_0)(\frac{\partial}{\partial x}) = v$.

In local coordinates a harmonic map $f : D \to N$ of class $C^2$ is a solution of equations

$$\frac{\partial^2 f^i}{\partial z \partial \bar{z}} + \Gamma^i_{jk}(f(z)) \frac{\partial f^j}{\partial z} \frac{\partial f^k}{\partial \bar{z}} = 0 \text{ for } i = 1, \ldots, \dim N.$$

$$f(0) = p, \ df(0)(\frac{\partial}{\partial x}) = v$$
Kobayashi metric on a Riemannian manifold

Theorem F makes it possible to define Kobayashi metric on any Riemannian manifold.

**Definition of Kobayashi metric** Let \( N \) be any (smooth) Riemannian manifold; let \( p \in N \), and \( v \in T_p N \). We define the Kobayashi metric on \( TN \)

\[
K_N(p, v) = \inf \left\{ \frac{1}{R} |f : D \to N \text{ har. map} : f(0) = p, df(0)\left(\frac{\partial}{\partial x}\right) = v \right\}
\]

\[
D = D(R) = \{z \in \mathbb{C} : |z| < R\}; z = x + iy
\]

**Theorem:** \( K_N(p, v) \) is well-defined on \( TN \).

**Research in progress:**
- Upper-semi continuity of Kobayashi metric
- Kobayashi distance and Kobayashi hyperbolicity
- These new concepts vs existing ones.
m-Laplace-semi-global solvability

When the system is autonomous, we can find global solutions.

**Theorem G.** Let \( A = (A_1, A_2, \ldots, A_N) \) be any function of class \( C^2 \) in its variables in \( \mathbb{R}^M \). Assume that

\[
A(0) = 0, \quad \nabla A(0) = 0. \quad (\text{no linear terms})
\]

Then the autonomous system of m-Laplace,

\[
u : \{ z \in \mathbb{C} : |z| \leq R \} \to \mathbb{R}^N,
\]

\[
\Delta^m u = A(u, \nabla u, \ldots, \nabla^{2m-1} u, \nabla^{2m} u)
\]

has solutions of class \( C^{2m+\alpha} \) in \( \{ z \in \mathbb{C} : |z| \leq R \} \) for any given \( R \) with vanishing order of \( 2m \) at the origin.
The Hölder space $C^\alpha(D); \| f \| = |f|_D + (2R)^\alpha H_\alpha[f]$. We define a function $\| \cdot \|^{(m)}$ on the standard space $C^{m+\alpha}(D)$.

$$\| f \|^{(m)} = \max\{\| \partial^i \bar{\partial}^j f \| : \forall i, j, i + j = m\}$$

We define a new space

$$C_0^{m+\alpha}(D) = \{ f \in C^{m+\alpha}(D) : \partial^i \bar{\partial}^j f(0) = 0, \forall i + j \leq m - 1\}$$

**Theorem H:** The space $(C_0^{m+\alpha}(D), \| \cdot \|^{(m)})$ is a Banach space.
High order singular integral

The classical Green operator: \( D = \{ z \in \mathbb{C} : |z| \leq R \} \)

\[
Tf(z) = -\frac{1}{2\pi i} \int_{D} \frac{f(\zeta)d\bar{\zeta} \wedge d\zeta}{\zeta - z}
\]

Well-known property:

\[
\bar{\partial} Tf = f, \text{ for } f \in C^\alpha(D)
\]

We define, for \( f \in C^{m+\alpha}(D) \), an high order singular operator

\[
m+2 Tf(z) = -\frac{(m + 1)!}{2\pi i} \int_{D} \frac{f(\zeta) - P_m(\zeta, z)}{(\zeta - z)^{m+2}} d\bar{\zeta} \wedge d\zeta,
\]

\[
P_m(\zeta, z) = \sum_{l=0}^{m} \frac{1}{l!} \sum_{i+j=l} \partial^i \bar{\partial}^j f(z)(\zeta - z)^i(\bar{\zeta} - \bar{z})^j.
\]
High order singular integral

\[ m = 0 \implies: \]

\[ 2Tf(z) = \frac{-1}{2\pi i} \int_D \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\zeta \wedge d\zeta \]

\[ \partial Tf = 2Tf \]

\[ \bar{\partial} Tf = f \]

Our main theorems:

**Theorem I:** If \( f \in C^{m+\alpha}(D) \), then \( m+2 Tf \in C^\alpha(D) \). Furthermore,

\[ \partial^{m+1} Tf = m+2 Tf \]

\[ \|m+2 Tf\| \leq C(m, \alpha)\|f\|^{(m)} \]

**Theorem J:** If \( h \in C^\alpha(D) \) and \( \mu + \nu = m \), then

\[ \|T^\nu \bar{T}^\mu h\|^{(m)} \leq C(m, \alpha)\|h\| \]
Mapping of Banach space

\[ \mathbf{B}(R) = ([C_0^{m+\alpha}(D)]^n, \| \cdot \|(^{m})) \]

\[ \mathbf{A}(R, \gamma) = \{ f \in \mathbf{B}(R) \| f \|^{(m)} \leq \gamma \} \]

\[ u^i = \psi^i + T^\nu \bar{T}^\mu a^i(z, u, D^1u, ..., D^{m-1}u, D^mu) \]

where \( \psi = (\psi^1, ..., \psi^n) \) is such that \( \partial^\mu \bar{\partial}^\nu \partial \psi = 0 \), and vanishes of order \( m \). Let

\[ \omega^i(f) = T^\nu \bar{T}^\mu a^i(z, f, D^1f, ..., D^{m-1}f, D^mf) \]

Define

\[ \Theta^i(f)(\zeta) = \omega^i(f)(\zeta) - \sum_{p=0}^{m-1} \frac{1}{p!} \sum_{k+l=p} [\partial^k \bar{\partial}^l \omega^i(f)](0) \zeta^k \bar{\zeta}^l \]

\[ - \frac{1}{m!} \sum_{k+l=m} [\partial^k \bar{\partial}^l \omega^i(f)](0) \zeta^k \bar{\zeta}^l + \frac{1}{\mu! \nu!} [\partial^\mu \bar{\partial}^\nu \omega^i(f)](0) \zeta^\mu \bar{\zeta}^\nu. \]
Mapping of Banach space, continued

We have the mapping:

$$\Theta : B(R) \rightarrow B(R)$$

$$\Theta(f) = (\Theta^1(f), ..., \Theta^n(f))$$.

A general estimate:

**Theorem K** Let $$\Theta : B(R) \rightarrow B(R)$$ be defined as above. Then if $$f, g \in A(R, \gamma)$$ then

$$\|\Theta(f) - \Theta(g)\|^{(m)} \leq \delta(R, \gamma)\|f - g\|^{(m)}$$

$$\|\Theta(f)\|^{(m)} \leq \eta(R, \gamma)$$

Where $$\delta(R, \gamma), \eta(R, \gamma)$$ also depend on Hölder and Lipschitz constants of $$a$$. 
Proofs

We consider equation
\[ u = \psi + \Theta(u) \]
with \( \|\psi\|^{(m)} \leq \frac{\gamma}{2} \). Apply Fixed point theorem by determining \( R, \gamma \) so that
\[ \delta(R, \gamma) < \frac{3}{4} \]
\[ \eta(R, \gamma) < \frac{\gamma}{2} \]
Theorem M Let \( a(x, p, q, r) = (a_1(x, p, q, r), \ldots, a_N(x, p, q, r)) \) be of class \( C^k_{\text{loc}} \) \((2 \leq k \leq \infty, 0 < \alpha < 1)\), where \( x \in \mathbb{R}^n \), \( p \in \mathbb{R}^N \), \( q \in \mathbb{R}^n \otimes \mathbb{R}^N \), and \( r \in \text{Sym}(n) \otimes \mathbb{R}^N \). \( \exists \) a constant \( \delta = \delta(n, N, \alpha) \) such that if

\[
|a(0)| = 0,  \\
|\nabla_r a(0)| + |\nabla^2_r a(0)| < \delta,
\]

then system: \( u(x) =: \{|x| \leq R\} \rightarrow \mathbb{R}^N \),

\[
\Delta u(x) = a(x, u(x), \nabla u(x), \nabla^2 u(x))
\]

has solutions of \( C^{k+2+\alpha} \) of vanishing order 2 at the origin for \( R \approx 0 \).
Higher Dimension: Local solvability of Poisson Type in $\mathbb{R}^n$

Theorem N Let $a(x, p, q) = (a_1(x, p, q), ..., a_N(x, p, q))$ be of class $C_{loc}^{k+\alpha}$ ($1 \leq k \leq \infty$, $0 < \alpha < 1$), where $x \in \mathbb{R}^n$, $p \in \mathbb{R}^N$, and $q \in \mathbb{R}^n \otimes \mathbb{R}^N$. Then the following system: $u(x) : \{|x| \leq R\} \rightarrow \mathbb{R}^N$, 

$$\Delta u(x) = a(x, u(x), \nabla u(x)) \quad (1)$$
$$u(0) = u_0 \in \mathbb{R}^N \quad (2)$$
$$\nabla u(0) = u_1 \in \mathbb{R}^n \otimes \mathbb{R}^N \quad (3)$$

has solutions of $C^{k+2+\alpha}$ for $R \approx 0$. 
Theorem 0 Let \(a(p, q, r) = (a_1(p, q, r), \ldots, a_N(p, q, r))\) be of class \(C^{k}_{loc} (2 \leq k \leq \infty, 0 < \alpha < 1)\), where \(p \in \mathbb{R}^N, q \in \mathbb{R}^n \otimes \mathbb{R}^N\), and \(r \in \text{Sym}(n) \otimes \mathbb{R}^N\). Assume that \(a\) contain no linear term, i.e.,

\[
\begin{align*}
    a(0) &= 0, \quad (4) \\
    \nabla a(0) &= 0. \quad (5)
\end{align*}
\]

Then the following system: \(u(x) : \{|x| \leq R\} \rightarrow \mathbb{R}^N\),

\[
\Delta u(x) = a(u(x), \nabla u(x), \nabla^2 u(x))
\]

has solutions in \(\{|x| \leq R\} \, x \in \mathbb{R}^n\) of \(C^{k+2+\alpha}\) with vanishing order 2 at the origin for any given value of \(R\).
Higher Dimension: Local solvability of Poisson Type in $\mathbb{R}^n$

A simple corollary is that there are infinite many solutions to the well-studied equation in PDE

$$\Delta u = K(x)|u|^{p-1}u,$$

where $p > 1$ in $|x| \leq R$. For systems it seems new. On the other hand, by Pohozaev (famous) identity, non-trivial positive solutions with zero boundary value exist possibly only if $p \leq \frac{n+2}{n-2}$ (here one needs $K(x) = 1$). More generally, Gidas-Ni-Nirenberg famous theorem states any positive solution of

$$\Delta u = f(u), \quad u|_{\{|x|=R\}} = 0$$

must be radial. We prove there are many solutions to

$$\Delta u = f(u)$$

provided $f$ contains no linear term ($f(0) = f'(0) = 0$).
Thank You!