An Introduction to SubRiemannian Geometry

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Abstract

A few important discoveries in the field of Thermodynamics in 1800s made the first steps towards sub-Riemannian geometry. Carnot discovered the principle of an engine in 1824 involving two isotherms and two adiabatic processes, Jule studied adiabatic processes and Clausius formulated the existence of the entropy in the second law of Thermodynamics in 1854. In 1909 Carathéodory made the point regarding the relationship between the connectivity of two states by adiabatic processes and non-integrability of a distribution, which is defined by the one-form of work. Chow proved the general global connectivity in 1934, and the same hypothesis was used by Hörmander in 1967 to prove the hypoellipticity of a sum of squares of vector fields operator. However, the study of the invariants of a horizontal distribution, known as non-holonomic geometry, was initiated by the Romanian mathematician George Vranceanu in 1936.

A position of a ship on a sea is determined by three parameters: two coordinates \( x \) and \( y \) for the location and an angle to describe the orientation. Therefore, the position of ship can be described by a point in a manifold. One can ask what is the shortest distance one should navigate to get from one position to another; this defines a Carnot-Carathéodory metric on the manifold \( \mathbb{R}^2 \times S^1 \). In a similar way, a Carnot-Carathéodory metric can be defined on a general sub-Riemannian manifold. The study of sub-Riemannian geodesics is useful in determining the Carnot-Carathéodory distance between two points.

The study of the geometry of the Heisenberg group, which is the prototype of the sub-Riemannian geometry was started by Gaveau in 1975. The understanding of the geometry of this group led Beals, Gaveau and Greiner to characterize the fundamental solutions for heat-type sub-elliptic operators and Heisenberg sub-Laplacian operator in 1990’s. Meantime many examples have been considered. Some of them have a behavior similar to the Heisenberg operator but others don’t. However, a unitary and general theory of these sub-Riemannian manifolds is still missing at the moment. This mini-course series is based on the aforementioned ideas and contains the following three parts:

• Part I. The Origins of SubRiemannian Geometry
• Part II. Hamiltonian Formalism in SubRiemannian Geometry
• Part II. Hamiltonian-Jacobi Theory in SubRiemannian Geometry
Part I. The Origins of SubRiemannian Geometry

SubRiemannian manifold: \((\mathbb{R}^n, \mathcal{H}, g)\)

Horizontal distribution: \(\mathcal{H} : x \rightarrow \mathcal{H}_x \subset T_x\mathbb{R}^n \) \(\dim \mathcal{H}_x = k, \ k \leq n - 1.\)

Particular case: \(k = n - 1, \ \mathcal{H} = \ker \omega, \ \omega\) one-form on \(\mathbb{R}^n\)

Horizontal curve: \(\gamma : [0, 1] \rightarrow \mathbb{R}^3, \ \omega(\dot{\gamma}(s)) = 0\) i.e. \(\dot{\gamma}(s) \in \mathcal{H}_{\gamma(s)}\)

Connectivity problem: Can any two points \(A\) and \(B\) be connected by a piece-wise horizontal curve?

Two aspects: local, global.
SubRiemannian metric: $\mathbb{R}^n \ni p \to g_p$, $g_p : \mathcal{H}_p \times \mathcal{H}_p \to \mathbb{R}$, Riemannian metric.

Length of a horizontal curve:

$$\ell(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(s), \dot{\gamma}(s))} \, ds$$

Example: Heisenberg case

$\mathbb{R}^3$, $\mathcal{H} = \ker \omega, \omega = dx_3 - 2x_2dx_1 + 2x_1dx_2$

$\mathcal{H} = \text{span}\{X_1, X_2\}, \omega(X_1) = \omega(X_2) = 0$

$X_1 = \partial x_1 + 2x_2 \partial x_3, \quad X_2 = \partial x_2 - 2x_1 \partial x_3$

SubRiemannain metric: $g(X_i, X_j) = \delta_{ij}$

Horizontal curves: $\gamma(s) = (x_1(s), x_2(s), x_3(s))$

$$\dot{x}_2(s) - 2x_2(s)\dot{x}_1(s) + 2x_1(s)\dot{x}_2(s) = 0$$

Connectivity: global
**SubRiemannian space** = **Carnot-Carathéodory space**

1824 Carnot: 4 cycle engine (2 isothermal and 2 adiabatic curves)

- **adiabatic curve** = no heat exchange along the curve
- **isothermal curve** = the same temperature along the curve

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**Diagram:**

- Red and blue arrows labeled as **isothermal** and **adiabatic**
- Points labeled **A**, **B**, **C**, and **D**
- **V (volume)** and **P (pressure)** axes
The 2\textsuperscript{nd} Law of Thermodynamics (Clau-sius formulation, 1854)
\[ Q = \text{heat transfer}, \quad T = \text{temperature} \]

\[ Q \text{ depends on the curve: } \int_{\gamma_1} dQ \neq \int_{\gamma_2} dQ. \]
**Theorem** (Clausius 1854): $\int_{\gamma} \frac{dQ}{T}$ is independent on the curve $\gamma$; it depends only on the endpoints $\gamma(A), \gamma(B)$.

$$
\int_{\gamma_1} \frac{dQ}{T} = \int_{\gamma_2} \frac{dQ}{T}
$$

**The 2nd Law of Thermodynamics:** There is a function $S$, called *entropy*, such that

$$dQ = T dS.$$ 

$$
\int_{\gamma_i} \frac{dQ}{T} = \int_{\gamma_i} dS = S_B - S_A, \quad \text{for } i = 1, 2.
$$
The **heat exchanged** along curve $\gamma$:

$$Q = \int_{\gamma} TdS$$

$\gamma$ **adiabatic curve**: no heat exchanged, i.e.

$$Q = \text{constant along } \gamma \iff dQ(\dot{\gamma}) = 0 \iff \dot{\gamma} \in \ker dQ = \mathcal{H} \text{ (horizontal distribution)}$$

Thus, $\textbf{adiabatic curve} = \textbf{horizontal curve}$

going back to subRiemannian geometry...
\textbf{2\textsuperscript{nd} Law of Thermodynamics for subRiemannian manifolds}

Replace $dQ$ by the one-form $\omega$.

$\gamma$ \textbf{adiabatic curve} if horizontal curve, i.e.

$$\omega(\dot{\gamma}) = 0$$

$\mathcal{H} = \ker \omega$, horizontal (adiabatic) distribution

$\mathcal{H}$ integrable $\iff$ $\omega$ has an integral factor

$\iff \exists f \neq 0$ such that $d(f\omega) = 0$

$\iff f\omega$ closed

$\iff f\omega$ exact

$\iff \exists S$ such that $f\omega = dS$

$\iff \omega = TdS$, where $T = \frac{1}{f}$.

$\mathcal{H}$ integrable $\iff \omega = TdS$. 
Connectedness and Non-integrability of $\mathcal{H}$

Theorem (Carathéodory, 1909): Given a Pfaff equation
\[ \omega = dx_0 + X_1 dx_1 + X_2 dx_2 + \cdots + X_n dx_n = 0, \]
with $X_i$ finite, continuous, differentiable of $x_i$, such that know that in the neighborhood of any point $A$ there are points which are not path connected by curves that satisfies the above equation, then the one-form $\omega$ has an integral factor.

$\iff$

Theorem: If \textbf{locally}, the space is not connected by piece-wise horizontal curves, then $\omega$ has an integral factor (i.e. $\mathcal{H} = \ker \omega$ is integrable).
Consequence:

\[
\text{locally non-connectedness} \quad \quad \longrightarrow \\
\mathcal{H} \text{ integrable} \quad \quad \longleftrightarrow \quad \omega = TdS
\]

(2-nd law of thermodynamics)

In classical thermodynamics we have non-connectedness by adiabatic curves...
Non-connectedness in thermodynamics

adiabatic curve: \( P = \frac{c}{V^k}, \ k, c > 0 \) (Jule)

Given point \( A \), for any neighborhood \( U \) of \( A \), there are points \( B \in U \) such that \( A \) and \( B \) cannot be connected by an adiabatic curve.
Carathéodory’s Theorem (contrapositive):

**Theorem:** If $\mathcal{H}$ is non-integrable, then $\forall A$, $\exists \mathcal{U}$ neighborhood such that $\forall B \in \mathcal{U}$ the points $A$ and $B$ can be connected by a piece-wise horizontal curve.
Teleman's Theorem

Theorem (Teleman, 1957)

Let $\mathcal{H}$ be a non-integrable distribution of rank $k$ on $\mathbb{R}^n$. Then any domain $U \subset \mathbb{R}^n$ contains a subdomain $U' \subset U$ such that $\forall p, q \in U'$, there is a piece-wise horizontal curve which joins $p$ and $q$.

When $k = n - 1$ we get Carathéodory's theorem.

$$C_p = \{q \in \mathbb{R}^n; p \text{ and } q \text{ can be joined by a piece-wise horizontal curve}\}$$
$p$ is called free if $p \in Int C_p$

$$Free(U) = \{p \in U; p \text{ free}\}$$

$$Bd(U) = U \setminus Free(U)$$

**Corollary:** If $\mathcal{H}$ is non-integrable we have

(i) $Int Bd(U) = \emptyset$

(ii) $\forall U_1 \subset U, U_1 \cap Free(U) \neq \emptyset$.

Need an example...
**Example:** $\omega = dx - xydz$ one-form on $\mathbb{R}^3_{(xyz)}$

**horizontal distribution:** $\mathcal{H} = \ker \omega = \text{span}\{X_1, X_2\}$

$$X_1 = xy\partial_x + \partial_z, \quad X_2 = \partial_y$$

No global connectivity...

$Bd(R^3) = \{x = 0\}$

$Free(R^3) = \{x > 0\} \cup \{x < 0\}$.

$$x(s) = x(0)e^{\int_0^s y\dot{z}}$$

There are no horizontal curves connecting points from upper half space to points in the lower half space.
Question: What condition $\mathcal{H}$ should satisfy to have **GLOBAL CONNECTIVITY**?
Bracket generating condition

Let $p$ be a point. Define

$$\mathcal{H}^1_p = \mathcal{H}_p$$
$$\mathcal{H}^2_p = \mathcal{H}^1_p + [\mathcal{H}_p, \mathcal{H}^1_p]$$
$$\mathcal{H}^3_p = \mathcal{H}^2_p + [\mathcal{H}_p, \mathcal{H}^2_p]$$

$$\ldots \ldots \ldots$$

$$\mathcal{H}^{n+1}_p = \mathcal{H}^n_p + [\mathcal{H}_p, \mathcal{H}^n_p],$$

where $[\mathcal{H}_p, \mathcal{H}^j_p] = \{[X, Y]; X \in \mathcal{H}_p, Y \in \mathcal{H}^j_p\}$.

**Definition:** $\mathcal{H}$ is said to be bracket generating at $p$ if

$$\exists r > 1 \text{ integer, with } \mathcal{H}^r_p = T_p \mathbb{R}^n.$$  

$r =$ the step at point $p$. 


Example: $X_1 = \partial_x + 2y\partial_t$, $X_2 = \partial_y - 2x\partial_t$

$[X_1, X_2] = -4\partial_t$,

$X_1, X_2, [X_1, X_2]$ linear independent in $\mathbb{R}^3$

$\mathcal{H}^2_p = T_p\mathbb{R}^3$

Heisenberg distribution is step 2 at each point.

Example: $X_1 = xy\partial_x + \partial_z$, $X_2 = \partial_y$

$[X_1, X_2] = x\partial_x$

$[X_1, [X_1, X_2]] = 0$

$[X_2, [X_1, X_2]] = 0$,

$\mathcal{H}$ has

step 2 on $\{x \neq 0\} = \text{Free}(\mathbb{R}^3)$

infinite step on $\{x = 0\} = \text{Bd}(\mathbb{R}^3)$
**Theorem (Chow, 1939)** If $\mathcal{H}$ is a bracket generating distribution on $\mathbb{R}^n$, then any two points can be joined by a horizontal piece-wise curve.

bracket generating $\implies$ global connectedness.

**Theorem (Hörmander, 1967)**

If $\mathcal{H} = \text{span}\{X_1, \ldots, X_m\}$ is a bracket generating distribution on $\mathbb{R}^n$, $m < n$, then

$$\Delta_X = X_1^2 + \cdots + X_m^2$$

is hypoelliptic ($\Delta_X u = f \in C^\infty \implies u \in C^\infty$.)

bracket generating $\implies$ hypoellipticity.
Explicit formulas for horizontal curves

In the Heisenberg case the points $(0, 0, 0)$ (the origin) and $(x_1, x_2, x_3)$ are connected by a family of horizontal curves with components

\[
\begin{align*}
x_1(s) &= as + bs^2 \\
x_2(s) &= \alpha s^2 + \beta s^3 \\
x_3(s) &= -\frac{2}{3} \alpha as^3 - a\beta s^4 - \frac{2}{5} b\beta s^5,
\end{align*}
\]

where $\beta = \frac{-15x_3-10x_1x_210bx_2}{5x_1+b}$,

$\alpha = x_2 - \beta$, $a = x_1 - b$ and $b$ satisfies $b \neq -5x_1$. 
Part II. Hamiltonian Formalism in SubRiemannian Geometry

SubRiemannian manifold: \((\mathbb{R}^n, \mathcal{D}, g)\)

Horizontal distribution: \(\mathcal{D} : x \to \mathcal{D}_x \subset T_x \mathbb{R}^n \) \(\dim \mathcal{D}_x = k, k \leq n - 1\).

SubRiemannian metric: \(g\)

\[\mathcal{D} = \text{span}\{X_1, X_2, \ldots, X_k\}, \quad g(X_i, X_j) = \delta_{ij}.\]

\(X\)-Laplacian: \(\Delta_X = \frac{1}{2}(X_1^2 + \cdots + X_k^2)\)

Hamiltonian: principal symbol of \(\Delta_X\)

\[H(x, p) = \frac{1}{2} \sum_{j=1}^{k} \langle X_j(x), p \rangle^2 = \frac{1}{2} \sum_{j=1}^{k} h^{ij}(x) p_i p_j\]

\[= \frac{1}{2} h(p, p).\]

Riemannian geometry: \(h^{-1} = g, gh = I\)

SubRiemannian geometry: \(\det h(x) = 0, \forall x, (gh)^T h = h.\)
Example: Heisenberg case:

\[
X_1 = \partial_{x_1} + 2x_2 \partial_t = (1, 0, 2x_2) \\
X_2 = \partial_{x_2} - 2x_1 \partial_t = (0, 1, -2x_1),
\]

Hamiltonian function \( H(x, p) = \)

\[
\frac{1}{2} \langle X_1(x), p \rangle^2 + \frac{1}{2} \langle X_2(x), p \rangle^2 \\
= \frac{1}{2} (p_1 + 2x_2p_3)^2 + \frac{1}{2} (p_2 - 2x_1p_3)^2 \\
= \frac{1}{2} \left( p_1^2 + 4x_2p_1p_3 + 4x_2^2p_3^2 \right) \\
+ \frac{1}{2} \left( p_2^2 - 4x_1p_2p_3 + 4x_1^2p_3^2 \right) \\
= \frac{1}{2} \left( \begin{pmatrix} p_1 + 2x_2p_3 \\ p_2 - 2x_1p_3 \\ 2x_2p_1 - 2x_1p_2 + 4|x|^2p_3 \end{pmatrix} \right) \left( \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \right) \\
= \frac{1}{2} \left( \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4|x|^2 \end{pmatrix} \right) \left( \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \right) = \frac{1}{2} h(p, p) = \frac{1}{2} h^{ij}(x) p_i p_j.
\]
Hence

\[ h^{ij}(x) = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4|x|^2 \end{pmatrix}. \]

\[ \det h^{ij}(x) = 4|x|^2 - 4x_1^2 + 2x_2(-2x_2) = 0, \]

the matrix \( h^{ij}(x) \) is degenerate everywhere.

**Example:**

Consider on \( \mathbb{R}^3 \) the following vector fields

\[ X_1 = \partial x_1 + A_1(x)\partial x_3, \quad X_2 = \partial x_2 - A_2(x)\partial x_3 \]

then a similar computation as the one above yields the Hamiltonian

\[ \frac{1}{2}(p_1 + A_1(x)p_3)^2 + \frac{1}{2}(p_1 - A_2(x)p_3)^2 = \frac{1}{2} h^{ij}(x)p_ip_j, \]

where

\[ h^{ij}(x) = \begin{pmatrix} 1 & 0 & A_1(x) \\ 0 & 1 & -A_2(x) \\ A_1(x) & -A_2(x) & A_1(x)^2 + A_2(x)^2 \end{pmatrix} \]

is a degenerate matrix everywhere.
**Example:**

If \( X_1 = \cos x_3 \partial x_1 + \sin x_3 \partial x_2, \ X_2 = \partial x_3 \), we have

\[
2H(x, p) = (\cos x_3 p_1 + \sin x_3 p_2)^2 + p_3^2
\]

\[
= p_1(\cos^2 x_3 p_1 + \sin x_3 \cos x_3 p_2)
+ p_2(p_2 \sin^2 x_3 + \sin x_3 \cos x_3 p_1) + p_3^2
\]

\[
= \begin{pmatrix}
\cos^2 x_3 p_1 + \sin x_3 \cos x_3 p_2 \\
\sin^2 x_3 p_2 + \sin x_3 \cos x_3 p_1 \\
p_3
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}
\]

\[
= \langle \begin{pmatrix}
\cos^2 x_3 & \sin x_3 \cos x_3 & 0 \\
\sin x_3 \cos x_3 & \sin^2 x_3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}, \begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix} \rangle.
\]

Hence the coefficients are

\[
h^{ij}(x) = \begin{pmatrix}
\cos^2 x_3 & \sin x_3 \cos x_3 & 0 \\
\sin x_3 \cos x_3 & \sin^2 x_3 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with \( \det h^{ij}(x) = 0 \).
A more effective way to construct the contravariant metric coefficients $h^{ij}$:

Let

$$X_j = \sum_{i=1}^{n} a^i_j(x) \partial x_i, \quad j = 1, \ldots, k,$$

with $\text{rank}(a^i_j(x)) = k, \forall x \in M$.

i.e., the horizontal distribution $\text{rank} \mathcal{D} = k$

There is any way we can express $h^{ij}(x)$ in terms of $a^i_j(x)$?
**Proposition:** The coefficients of the quadratic form which define the Hamiltonian $H$ are given by

$$h^{\alpha\beta} = \sum_{j=1}^{k} a^{\alpha}_j a^{\beta}_j. \quad (1)$$

**Example:** Consider the distribution defined by the vector fields

$$\partial x_1 = (1, 0, 0) = (a_1^1, a_1^2, a_1^3)$$
$$\partial x_2 - 2x_1 \partial x_3 = (0, 1, -2x_1) = (a_2^1, a_2^2, a_2^2).$$

The coefficients $h^{ij}$ are given by

- $h^{11} = a_1^1 a_1^1 + a_2^1 a_2^1 = 1^2 + 0 = 1$
- $h^{12} = a_1^1 a_1^2 + a_2^1 a_2^2 = 0 = h^{21}$
- $h^{22} = a_1^2 a_1^2 + a_2^2 a_2^2 = 1$
- $h^{13} = a_1^1 a_1^3 + a_2^1 a_2^3 = 2x_2 = h^{31}$
- $h^{23} = a_1^2 a_1^3 + a_2^2 a_2^3 = -2x_1 = h^{32}$
- $h^{33} = a_1^3 a_1^3 + a_2^3 a_2^3 = 4x_1^2$. 

26
**Proposition:** The matrix $h^{\alpha\beta}(x)$ is degenerate for any $x \in M$.

The $h^{i\bar{j}}(x)$ is the analogue of the raised index metric from Riemannian geometry. Since $h^{i\bar{j}}(x)$ is nowhere invertible, there is no analogue of the lowered index metric. One may say, as a general rule, that any formula of Riemannian geometry that can be extended in terms of raised indices alone is still valid in subRiemannian geometry.
**Definition:** A normal geodesic between the points $A$ and $B$ is a solution $x(s)$ of the Hamiltonian system

\[
\begin{align*}
\dot{x}^i(s) &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i(s) &= -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, n,
\end{align*}
\]

with the boundary conditions $x(0) = A$ and $x(\tau) = B$.

**Remark:** A similar definition for geodesics works also in the case of Riemannian manifolds.
The equation of normal geodesics as a 2\textsuperscript{nd} order ODE

Define the raised Christoffel symbols

\[ \Gamma^{ia b}(x) = \frac{1}{2} \left( \frac{\partial h^{ia}(x)}{\partial x^r} h^{rb}(x) + \frac{\partial h^{ib}(x)}{\partial x^r} h^{ra}(x) - \frac{\partial h^{ab}(x)}{\partial x^r} h^{ri}(x) \right). \]  \hspace{1cm} (2)

In subRiemannian geometry the raised Christoffel symbols were used for the first time by N. C. Günter, see *Hamiltonian mechanics and optimal control*, Thesis, Harvard University, 1982. The following result can be found also in Strichartz (J. Diff. Geom. 1986).
**Proposition:** The equation of normal geodesics are given by

\[ \ddot{x}^i(s) = \Gamma^{ijk}(x(s))p_j(s)p_k(s), \]  

where \((x(s), p(s))\) is a solution of the Hamiltonian system.

**Riemannian geometry:**
\[ \dot{x}^i = h^{ij}p_j \implies p_j = g_{ij}\dot{x}^i \]
\[ \ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0. \]

**subRiemannian geometry:** No obvious way to solve \(p_j\) in terms of \(\dot{x}^i\) since \(\det h^{ij} = 0\). Difficulty of solving (\(\ast\)) depends from model to model.
**Heisenberg case:** Substituting the formulas of $h^{ij}$ found before into the formula for the raised Christoffel symbols yields

\[
\begin{align*}
\Gamma^{111} &= 0 & \Gamma^{211} &= 0 & \Gamma^{311} &= 0 \\
\Gamma^{112} &= 0 & \Gamma^{212} &= 0 & \Gamma^{312} &= 0 \\
\Gamma^{113} &= 0 & \Gamma^{213} &= -2 & \Gamma^{313} &= 4x_1 \\
\Gamma^{123} &= 2 & \Gamma^{223} &= 0 & \Gamma^{323} &= 4x_2 \\
\Gamma^{133} &= -8x_1 & \Gamma^{222} &= 0 & \Gamma^{333} &= 0 \\
\Gamma^{122} &= 0 & \Gamma^{233} &= -8x_2 & \Gamma^{322} &= 0. \\
\end{align*}
\]

Using the symmetry in the last two indices, $\Gamma^{ijk} = \Gamma^{ikj}$, yields $3^3 = 27$ functions. The equation (*) become

\[
\begin{align*}
\ddot{x}^1(s) &= 4p_3(p_2 - 2x_1p_3) \\
\ddot{x}^2(s) &= -4p_3(p_1 + 2x_2p_3) \\
\ddot{x}^3(s) &= 8p_3(x_1p_1 + x_2p_2). \\
\end{align*}
\]
In this case we can solve for $p$ in terms of $\dot{x}$.

The right hand side of the first two above equations can be written in terms of $\dot{x}(s)$ as follows

$$\dot{x}^1(s) = \frac{\partial H}{\partial p_1} = p_1 + 2x_2p_3$$
$$\dot{x}^2(s) = \frac{\partial H}{\partial p_2} = p_2 - 2x_1p_3,$$

so we get

$$\ddot{x}^1(s) = 4p_3\dot{x}^2(s)$$
$$\ddot{x}^2(s) = -4p_3\dot{x}^1(s).$$

$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = 0 \implies p_3$$ is constant.

The constant $p_3$ depends on the boundary conditions of the above system. In general, for given boundary conditions $x^i(0) = x^i_0$ and $x^i(\tau) = x^i_f$, the constant $p_3$ is not unique.
**Proposition:** Along the normal geodesics the Hamiltonian is preserved.

**Proof:** The proof is a consequence of Hamiltonian equations. Let \((x(s), p(s))\) be a solution of the Hamiltonian system. Since the Hamiltonian \(H\) does not depend explicitly on \(s\) we have

\[
\frac{d}{ds} H(x(s), p(s)) = \frac{\partial H}{\partial x^i} \dot{x}^i + \frac{\partial H}{\partial p_j} \dot{p}_j
\]

\[
= \frac{\partial H \partial H}{\partial x^i \partial p_j} - \frac{\partial H \partial H}{\partial p_j \partial x_j}
\]

\[
= 0,
\]

and hence \(H(x(s), p(s))\) is constant along the solutions.
**Proposition:** Any normal geodesic is a horizontal curve (the reciprocal is false).

**Example:** Heisenberg case:

\[
H(x, p) = \frac{1}{2}(p_1 + 2x_2p_3)^2 + \frac{1}{2}(p_2 - 2x_1p_3)^2
\]

\[
\dot{x}_1 = H_{p_1} = p_1 + 2x_2p_3
\]
\[
\dot{x}_2 = H_{p_2} = p_2 - 2x_1p_3
\]
\[
\dot{x}_3 = H_{p_3} = 2x_2(p_1 + 2x_2p_3) - 2x_1(p_2 - 2x_1p_3)
\]
\[
= 2x_2\dot{x}_1 - 2x_1\dot{x}_2 \implies \text{horizontal constraint.}
\]
The covariant subRiemannian metric:
Let $\mathcal{D} = \text{span}\{X_1, \ldots, X_k\}$ be the horizontal distribution.

Covariant subRiemannian metric: a symmetric degenerate tensor 2-covariant $g$ on $M$ such that $g(X_l, X_r) = \delta_{lr}$.

The restriction $g|_{\mathcal{D} \times \mathcal{D}}$ is a positive definite metric.

Example:
In the case of the Heisenberg group we have

$$a^i_1 = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix}, \quad a^i_2 = \begin{pmatrix} 0 \\ 1 \\ -2x_1 \end{pmatrix}, \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$
**Proposition:** Let $c(s) = (x_1(s), \ldots, x_n(s))$ be a horizontal curve and let

$$\dot{c}(s) = \sum_{j=1}^{k} \dot{c}^j(s) X_j$$

be its velocity vector. Then

$$g(\dot{c}(s), \dot{c}(s)) = (\dot{c}^1)^2 + \ldots (\dot{c}^k)^2.$$
**Definition:** If $c : [0, \tau] \to \mathbb{R}^n$ is a horizontal curve, its **energy** is defined by

$$E(c) = \frac{1}{2} \int_0^\tau g(\dot{c}(s), \dot{c}(s)) \, ds$$

$$= \frac{1}{2} \int_0^\tau ((\dot{c}^1(s))^2 + \cdots + (\dot{c}^k(s))^2) \, ds.$$ 

Consequently, we shall define the **length** of the horizontal curve $c$ as

$$\ell(c) = \int_0^\tau \sqrt{g(\dot{c}(s), \dot{c}(s))} \, ds$$

$$= \int_0^\tau \sqrt{(\dot{c}^1(s))^2 + \cdots + (\dot{c}^k(s))^2} \, ds.$$
Covariant versus contravariant:

**Riemanian case:** $gh = I$ (g is the inverse of h).

**subRiemanian case:** neither g nor h is invertible.

**Proposition:** The following relation holds at any point

$$g_{ij} h^{i\alpha} h^{j\beta} = h^{\alpha\beta},$$

or, in matrix form $(gh)^T h = h$, where $A^T$ denotes the transpose of the matrix A.
**Example:** In the case of the Heisenberg group we have

\[ g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4|x|^2 \end{pmatrix}, \]

\[ gh = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ (gh)^T h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4|x|^2 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4|x|^2 \end{pmatrix} = h, \]

with \( \text{rank } g = \text{rank } h = 2 \) everywhere.
**Definition:** Let $S_p : T_p M \to T_p M$ be a linear map defined by

$$S_p(v^i \partial_{x_i}) = \langle (gh)^T v, \partial_x \rangle_p,$$

where $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$. In the matrix notation we have $Sv = (gh)^T v$.

**Proposition:**

The operator $S$ has the following properties:

1) $S(\partial_x) = (gh) \partial_x$

2) $S_p(U) = U$, $\forall U \in \mathcal{D}_p$

3) $S|_{\mathcal{D}} = I_\mathcal{D}$ (identity)

4) If $x(s)$ is a solution of the Hamiltonian system, then $S\dot{x}(s) = \dot{x}(s)$. 


Conservation of energy theorem:

**Theorem:** Let \((x(s), p(s))\) be a solution of the Hamiltonian system. Then

1) \(\langle hp, p \rangle = \langle g \dot{x}, \dot{x} \rangle\), i.e.,

\[ h^\alpha\beta(x)p_\alpha p_\beta = g_{ij}(x)\dot{x}^i\dot{x}^j. \]

2) \(\langle g \dot{x}, \dot{x} \rangle\) is constant along the solution.
**Example:**

In the Heisenberg group case the Hamiltonian is

\[ H(p, x) = \frac{1}{2} h^{ij} p_i p_j = \frac{1}{2} (p_1 + 2x_2 p_3)^2 + \frac{1}{2} (p_2 - 2x_1 p_3)^2. \]  

(3)

Using the Hamiltonian equations

\[ \dot{x}_1 = H_{p_1} = p_1 + 2x_2 p_3 \]
\[ \dot{x}_2 = H_{p_2} = p_2 - 2x_1 p_3 \]

the Hamiltonian (3) can be written as

\[ H = \frac{1}{2} h^{ij} p_i p_j = \frac{1}{2} \dot{x}_1^2 + \frac{1}{2} \dot{x}_2^2 = \frac{1}{2} g_{ij} \dot{x}_i \dot{x}_j \]

with the coefficients \( g_{ij} \) given by

\[ g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
The eigenvectors of $h^{ij}$

Can we recover the horizontal distribution from the contravariant matrix $h^{ij}$?

The matrix $h^{ij}$ is degenerate at each point.

$$Eig(h) = \{ \lambda \in \mathbb{R}; \det(h^{ij} - \lambda \delta_{ij}) = 0 \}$$

$$= \{ \lambda_1 = \cdots = \lambda_r = 0, \lambda_{r+1}, \ldots, \lambda_n \}$$

is the set of eigenvalues of $h$.

For each $\lambda \in Eig(h)$ consider the space of eigenvectors

$$V_{\lambda} = \{ v \in \mathbb{R}^n; hv = \lambda v \}. $$
**Notation:** If \( \omega = \omega_i dx_i \) is a one-form on \( \mathbb{R}^n \), then \( \omega^# = \omega_i \partial x_i \) is the associated vector field on \( \mathbb{R}^n \).

**Proposition:** Let \( \lambda \in Eig(h) \) and \( \omega^# \in V_\lambda \). Then

\[
\omega(X_\beta) = \lambda g(X_\beta, \omega^#),
\]

where \( X_\beta \in \mathcal{D} \).

**Corollary:** The one-forms \( \omega \) with \( \omega^# \) eigenvector corresponding to the eigenvalue \( \lambda = 0 \), are vanishing on the distribution \( \mathcal{D} \).

\( \omega^# = \text{missing direction} \)
**Proposition:** Let $\lambda \in Eig(h)$, $\lambda \neq 0$. Then $V_\lambda \subset \mathcal{D}$, i.e., for any vector $\eta^\# \in V_\lambda$ we have $\eta^\#$ horizontal. The $h$-components of $\eta^\#$ are $\eta(X_\alpha)/\lambda$.

**Conclusion:** If $\lambda \in Eig(h), \lambda \neq 0 \implies V_\lambda \subset \mathcal{D}$

If $\lambda \in Eig(h), \lambda = 0 \implies V_\lambda$ transversal to $\mathcal{D}$
**Example:** Heisenberg case:

\[ X_1 = \partial x_1 + 2x_2 \partial x_3, \quad X_2 = \partial x_2 - 2x_1 \partial x_3 \]

\[ h^{ij}(x) = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4(x_1^2 + x_2^2) \end{pmatrix} \]

\[
\det(h^{ij} - \lambda \delta_{ij}) = \lambda(1 - \lambda)(\lambda - 4(x_1^2 + x_2^2) - 1)
\]

\[ \lambda_1 = 0 \implies V_{\lambda_1} = \{(-2x_2, 2x_1, 1)\} \perp \mathcal{D} \]

\[ \lambda_2 = 1 \implies V_{\lambda_2} = \{(x_1, x_2, 0)\} \subset \mathcal{D} \]

\[ \lambda_3 = 1 + 4(x_1^2 + x_2^2) \implies V_{\lambda_3} = \{(x_2, -x_1, 2(x_1^2 + x_2^2))\} \subset \mathcal{D} \]
SubRiemannian geodesics:
A (normal) geodesic between points $A$ and $B$ is a solution $x(s)$ of the Hamiltonian system

$$\dot{x}^i(s) = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i(s) = -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, n,$$

with the b.c. $x(0) = A, x(\tau) = B$.

Properties:
- Hamiltonian is preserved along the geodesics.
- Any (normal) geodesic is a horizontal curve.
- (normal) geodesics are locally length (energy) minimizing curves.
Caution: There are length minimizing horizontal curves which are not solutions of Hamiltonian system (abnormal geodesics).

Example (Liu & Sussman, 1995):

\[ X = \partial_x, \quad Y = (1 - x)\partial_y + x^2\partial_t \]

\( \gamma : [a, b] \to \mathbb{R}^3, \gamma(s) = (0, s, 0). \) If \( b - a \leq 2/3 \) then \( \gamma \) is an abnormal geodesic.
A more general result: $D = \text{span}\{X_1, X_2\}$,

$X_1 = \partial x_1 + A_1(x)\partial t, \quad X_2 = \partial x_2 - A_2(x)\partial t,$

$x = (x_1, x_2), \quad t = x_3.$

Curvature 2-form: $\Omega = \left(\frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1}\right)dx_1 \wedge dx_2$

Proposition: The abnormal geodesics are among the horizontal curves $\gamma(s)$ with

$$\Omega(\gamma(s)) = 0.$$ 

No abnormal geodesics in the following cases:

- $X_1 = \partial x_1 + 2x_2\partial t, \quad X_2 = \partial x_2 - 2x_1\partial t,$
  
  (step 2)

- $X_1 = \partial x_1 + 2kx_2|x|^{2(k-1)}\partial t,$

  $X_2 = \partial x_2 - 2kx_1|x|^{2(k-1)}\partial t,$

  (step $k + 1$).

- $X_1 = \partial x_1, \quad X_2 = \partial x_2 - e^{x_1}dt$

  (step 2).
Solving Hamiltonian system

1. Heisenberg case

\[ X_1 = \partial_{x_1} + 2x_2 \partial_t, \quad X_2 = \partial_{x_2} - 2x_1 \partial_t \]

Hamiltonian:

\[ H(x, p) = \frac{1}{2} (p_1 + 2x_2 p_3)^2 + \frac{1}{2} (p_2 - 2x_1 p_3)^2 \]

Hamiltonian system of equations:

\[ \dot{x}_1 = \frac{\partial H}{\partial p_1} = p_1 + 2x_2 p_3 \]
\[ \dot{x}_2 = \frac{\partial H}{\partial p_2} = p_2 - 2x_1 p_3 \]
\[ \dot{t} = \frac{\partial H}{\partial p_3} = 2x_2 \dot{x}_1 - 2x_1 \dot{x}_2 \]
\[ \dot{p}_1 = 2\theta \dot{x}_2 \]
\[ \dot{p}_2 = -2\theta \dot{x}_1 \]
\[ \dot{p}_3 = 0 \implies p_3 = \theta (constant). \]
Geodesics equations:

\[ \begin{align*}
\ddot{x}_1 &= 4\theta \dot{x}_2 \\
\ddot{x}_2 &= -4\theta \dot{x}_1 \\
\dot{t} &= 2x_2 \dot{x}_1 - 2x_1 \dot{x}_2 \Longleftrightarrow \omega(\dot{\gamma}) = 0 \\
\theta &= \text{constant}.
\end{align*} \]

If \( \theta \neq 0 \) the solutions are

\[ \begin{align*}
x(s) &= \frac{R}{4\theta} \left( \sin \phi(s), \cos \phi(s) \right), \\
t(s) &= t_0 + \frac{\sin 2(\phi - \phi_0) - 2(\phi - \phi_0)}{8\theta^2}
\end{align*} \]

where \( \phi(s) = 4\theta s + \alpha \)
Consider $\mathcal{D} = \text{span}\{X_1, X_2\}$, with

$$X_1 = \partial x_1 + A_1(x)\partial_t, \quad X_2 = \partial x_2 - A_2(x)\partial_t. \quad (4)$$

$$\varphi(x) := \frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} \neq 0.$$  

Since $[X_1, X_2] = -\varphi(x)\partial_t \notin \mathcal{D}$, the distribution $\mathcal{D}$ is not involutive and hence non-integrable. Hamiltonian system:

$$\ddot{x}_1(s) = \lambda \varphi(x) \dot{x}_2(s) \quad (5)$$

$$\ddot{x}_2(s) = -\lambda \varphi(x) \dot{x}_1(s) \quad (6)$$

$$\dot{t}(s) = A_1(x) \dot{x}_1(s) + A_2(x) \dot{x}_2(s). \quad (7)$$
Assume
\[
\frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} = ax_1 + bx_2 + c.
\]
In particular, when \( c = 0 \), this covers two important families of vector fields
\[
X_1 = \partial_{x_1} + (ax_1 x_2 + f(x_1)) \partial_t, \\
X_2 = \partial_{x_2} - (bx_1 x_2 + g(x_2)) \partial_t
\]
and
\[
X_1 = \partial_{x_1} + \left(\frac{1}{2} bx_2^2 + f(x_1)\right) \partial_t, \\
X_2 = \partial_{x_2} - \left(\frac{1}{2} ax_1^2 + g(x_2)\right) \partial_t,
\]
with \( f, g \) arbitrary smooth real functions. When \( a = b = 0 \) and \( c \neq 0 \), choosing \( A_1 = \frac{c}{2}x_2 \) and \( A_2 = \frac{c}{2}x_1 \) leads to the Heisenberg group case.
**Proposition:** Let $\varphi(x) = ax_1 + bx_2 + c$. Then the solution of the Hamiltonian system is

$$
\begin{align*}
x_1(s) &= x_1(0) - \frac{1}{\sqrt{a^2 + b^2}} \left( as - \frac{2}{\omega} (aE(\omega s, k) \\
&+ kb (\text{cn}(\omega s, k) - 1)) \right), \\
x_2(s) &= x_2(0) + \frac{1}{\sqrt{a^2 + b^2}} \left( bs - \frac{2}{\omega} (bE(\omega s) \\
&+ ka (\text{cn}(\omega s, k) - 1)) \right),
\end{align*}
$$

where $E(u, k) = \int_0^u \text{dn}^2(s) \, du$ is the Jacobi’s epsilon function, $\omega^2 = |\lambda| \sqrt{a^2 + b^2}$ and $k = \sin \frac{\alpha}{2}$, with $\alpha = \max \theta(s)$.
Elliptic functions

\[ z = \text{sn}^{-1} w = \int_{0}^{w} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}. \]

The function \( w = \text{sn} z \) is called a Jacobian elliptic function.

By analogy with the trigonometric functions, it is convenient to define other elliptic functions

\[ \text{cn} z = \sqrt{1 - \text{sn}^2 z}, \quad \text{dn} z = \sqrt{1 - k^2 \text{sn}^2 z}. \]

A few properties of this functions are

\[ \text{sn}(0) = 0, \quad \text{cn}(0) = 1, \quad \text{dn}(0) = 1, \]

\[ \text{sn}(-z) = \text{sn}(z), \quad \text{cn}(-z) = \text{cn}(z), \]

\[ \frac{d}{dz} \text{sn} z = \text{cn} z \, \text{dn} z, \quad \frac{d}{dz} \text{cn} z = -\text{sn} z \, \text{dn} z, \]

\[ -1 \leq \text{cn} z \leq 1, \quad -1 \leq \text{sn} z \leq 1, \quad 0 \leq \text{dn} z \leq 1 \]
Invariants of a distribution

A mathematical object which depends only on the horizontal distribution $\mathcal{D}$ and the sub-Riemannian metric defined on $\mathcal{D}$, is called an invariant. In other words, an invariant does not depend on the vector fields which generate $\mathcal{D}$. In the following we shall discuss a few invariants.
1. The Hamiltonian. Given the orthonormal vector fields $X_1, \ldots, X_k$ which span the distribution $\mathcal{D}$, we define the associate Hamiltonian

$$H_X(x, p) = \frac{1}{2} a^i_\alpha a^j_\alpha p_i p_j.$$ 

Let $Y_1, \ldots, Y_k$ be another orthonormal basis of $\mathcal{D}$, with $Y_\beta = A^\alpha_\beta X_\alpha$, where $A \in \mathcal{O}(k)$, i.e., $AA^T = I_k$. Identifying the components of $Y_\beta$ in the expressions

$$Y_\beta = b^i_\beta \partial x_i$$

$$Y_\beta = A^\alpha_\beta X_\alpha = A^\alpha_\beta a^i_\alpha \partial x_i$$

yields $b^i_\beta = A^\alpha_\beta a^i_\alpha$. Then the Hamiltonian associated with the basis $\{Y_j\}$ can be written as

$$H_Y(x, p) = \frac{1}{2} b^i_\beta b^j_\beta p_i p_j = \frac{1}{2} A^\alpha_\beta a^i_\alpha A^{\alpha'}_\beta a^{j'}_\alpha p_i p_j$$

$$= \frac{1}{2} (A^\alpha_\beta A^{\alpha'}_\beta) a^i_\alpha a^{j'}_\alpha p_i p_j$$

$$= \frac{1}{2} \delta_{\alpha \alpha'} a^i_\alpha a^{j'}_\alpha p_i p_j = \frac{1}{2} a^i_\alpha a^{j'}_\alpha p_i p_j$$

$$= H_X(x, p).$$
Hence the Hamiltonian is the same for all orthonormal bases of $\mathcal{D}$. 
2. The curvature tensor. Recall the curvature tensor along the horizontal curve $c(s)$ associated with the one-form $\omega \in \mathcal{I}$ is given by

$$K_X(\dot{c}) = \sum_{j=1}^{k} \Omega(\dot{c}, X_j)X_j,$$

where $\Omega = d\omega$. Let $A \in \mathcal{O}(k)$ and $Y_\alpha = A_\alpha^\beta X_\beta$. The curvature tensor associated with the basis $\{Y_j\}$ is given by

$$K_Y(\dot{c}) = \sum \Omega(\dot{c}, Y_\alpha)Y_\alpha = \sum \Omega(\dot{c}, A_\alpha^\beta X_\beta)A_\alpha^\beta X_\beta' = \sum A_\alpha^\beta A_\alpha^{\beta'} \Omega(\dot{c}, X_\beta)X_\beta' = \delta_\beta^{\beta'} \Omega(\dot{c}, X_\beta)X_\beta = \sum \Omega(\dot{c}, X_\beta)X_\beta = K_X(\dot{c}),$$

i.e., the curvature does not depend on the orthonormal basis.
4. Geodesics. Since the Hamiltonian does not depend on the choice of the orthonormal basis \( \{X_j\} \), the solution of the Hamiltonian system will do the same. Hence the geodesics depend on the distribution \( \mathcal{D} \) and the metric defined on it. In particular, the conjugate points and the CC-distance do not depend on the basis.
5. The length of a horizontal curve. Let $c : [0, 1] \to \mathbb{R}^n$ be a horizontal curve, with

$$\dot{c} = \dot{c}^\alpha X_\alpha = \dot{c}^\alpha A_{\alpha \beta} Y_\beta = \dot{\gamma}^\beta Y_\beta.$$

The length does not depend on the orthonormal basis chosen

$$\ell(c) = \int_0^1 \sqrt{(\dot{c}^1(s))^2 + \cdots + (\dot{c}^k(s))^2}$$

$$= \int_0^1 \sqrt{(\dot{\gamma}^1(s))^2 + \cdots + (\dot{\gamma}^k(s))^2}.$$
6. **The step of the connection.** The step of the distribution at a point does not depend on the vector fields, neither the subRiemannian metric. The same for the missing directions.
7. **The horizontal distribution.** The horizontal connection $\mathcal{D} : \mathcal{D} \times \mathcal{D} \to \mathcal{D} f$,

$$D_{UV} = \sum U(V^k)X_k$$

does not depend on the orthonormal basis $X_i$. 
Part III. Hamiltonian-Jacobi Theory on SubRiemannian Geometry

SubRiemannian manifold: \((M, \mathcal{D}, g)\)

Horizontal distribution: \(\mathcal{D}: x \mapsto \mathcal{D}_x \subset T_x \mathbb{R}^n\) \(\dim \mathcal{D}_x = k, k \leq n - 1\).

\(\mathcal{D} = \text{span}\{X_1, X_2, \ldots, X_k\}\), not integrable

\(X\)-Laplacian: \(\Delta_X = \frac{1}{2} (X_1^2 + \cdots + X_k^2)\)

SubRiemannian metric: \(g_x: \mathcal{D}_p \times \mathcal{D}_p \to \mathbb{R}\)

\[ g(X_i, X_j) = \delta_{i,j}. \]
A curve $c : [0, \tau] \to M$ is called horizontal curve if $\dot{c}(s) \in D_{c(s)}$, $\forall s \in [0, \tau]$, i.e., the velocity vector of the curve belongs to the horizontal distribution.

We are allowed to measure the length of horizontal vectors only. The metric $g$ can be used also for defining the length of horizontal curves $c : [0, \tau] \to M$ as

$$\ell(c) = \int_0^\tau g(\dot{c}(s), \dot{c}(s))^{1/2} \, ds.$$
The following concept plays a similar role to the Euclidean distance in Euclidean geometry or Riemannian distance in Riemannian geometry.

**Definition** Let $P, Q$ be two points on the manifold $M$. If there is a horizontal curve joining them, $c : [0, \tau] \to M$, $c(0) = P$, $c(\tau) = Q$, then the subRiemannian distance, $d_C(P, Q)$, between $P$ and $Q$ is defined by

$$\inf \{ \ell(\gamma); \gamma \text{ horizontal curve joinning } P \text{ and } Q \}.$$

$d_C$ is also called the *Carnot-Carathéodory* distance of the subRiemannian manifold.
Question: How can we find the length minimizing curves between two given points?
The horizontal gradient

The set of horizontal vector fields (sections in the horizontal bundle):

$$\Gamma(\mathcal{D}) = \{ X; X_x \in \mathcal{D}_x \}$$

The variation of a function $f$ along the horizontal distribution is measured by a horizontal vector field which is defined below.

**Definition:** For any function $f \in \mathcal{F}(M)$ the horizontal gradient of $f$ is the horizontal vector field $\nabla_h f \in \Gamma(\mathcal{D})$ defined by

$$g(\nabla_h f, X) = X(f) = df(X), \quad \forall X \in \Gamma(\mathcal{D}).$$
\textbf{Proposition:} We have
\[ (\nabla_h f)_p = \sum_{i=1}^{k} X_i(f) X_i|_p. \]

\textbf{Recall:} \( \mathcal{D} = \text{span}\{X_1, \ldots, X_k\} \), with \( X_i \) orthonormal horizontal vector fields

\textbf{Definition:} The energy of the function \( f \) is defined by
\[ H(\nabla f) = \frac{1}{2} \left| \nabla_h f \right|_h^2 = \frac{1}{2} h (\nabla_h f, \nabla_h f) \]
\[ = \frac{1}{2} \sum_{j=1}^{k} (X_j(f))^2. \]
**Proposition:** Let $f$ be a smooth function defined on the connected manifold $M$ with $\mathcal{D}$ not integrable. Then the energy $H(\nabla f) = 0$ if and only if $f$ is a constant.

Proof: $f$ is constant along horizontal curves. Since $\mathcal{D}$ is not integrable, by Caratheodory’s theorem $M$ is locally connected by horizontal curves. So $f$ is locally constant.

**Corollary:** $f$ is a constant function if and only if $\nabla_h f = 0$. 
**Corollary:** Given a horizontal vector field $U \in \Gamma(D)$, the following equation in $f$

$$\nabla_h f = U$$

(8)

has the uniqueness property, up to an additive constant.

Suppose there are two solutions $\phi$ and $\psi$ for the equation (8)

$$\nabla_h \phi = U, \quad \nabla_h \psi = U.$$ 

Subtracting, yields

$$\nabla_h (\psi - \phi) = 0.$$ 

There is a constant $C$ such that $\psi = \phi + C$. 
**Existence:** Given a horizontal vector field $U \in \Gamma(D)$, does the equation

$$\nabla_h f = U$$

have a solution $f$?

**Reformulation:** Given the functions $u_i$, can we find a function $f$ such that

$$X_j f = u_j, \quad \forall j = 1 \ldots k?$$

- Uniqueness $\sqrt{\text{ }}$
- Existence $?$(see the conference talk)
**Example:** The Heisenberg distribution on $\mathbb{R}^3$ is generated by the vector fields

$$X_1 = \partial_x - 2y\partial_t, \quad X_2 = \partial_y + 2x\partial_t.$$ 

If $X_1 f$ and $X_2 f$ are given, we can recover the function $f$ up to an additive constant if certain integrability conditions are satisfied.

Let $u_1, u_2$ be two functions on $\mathbb{R}^3$ such that

$$X_1 f = u_1(x, y, t), \quad X_2(f) = u_2(x, y, t).$$

(10)

The system (10) has solution iff

$$X_1^2 u_2 = (X_1 X_2 + [X_1, X_2]) u_1$$
$$X_2^2 u_1 = (X_2 X_1 + [X_2, X_1]) u_2.$$
**Hamilton-Jacobi equation**

**Definition:** Let $S$ be a real valued function defined on $\mathbb{R} \times M$. The Hamilton-Jacobi equation for subRiemannian manifolds is

$$\frac{\partial S}{\partial \tau} + H(\nabla S) = 0,$$

with the initial condition $S|_{\tau=0} = 0$, where

$$H(\nabla S) = \frac{1}{2} h(\nabla h S, \nabla h S) = \frac{1}{2} |\nabla h S|^2_h.$$

The Hamilton-Jacobi equation (11) does not have a unique solution as can be inferred from the following example. This is expected since the equation is non-linear.
**Example:** Solving the Hamilton-Jacobi equation in the case of the distribution in $\mathbb{R}^3$ spanned by $X_1 = \partial_x + 2y\partial_t$ and $X_2 = \partial_y - 2x\partial_t$ (Heisenberg distribution).

Since $X_1$ and $X_2$ are orthonormal, the Hamilton-Jacobi equation in this case is

$$\frac{\partial S}{\partial \tau} + \frac{1}{2}X_1(S)^2 + \frac{1}{2}X_2(S)^2 = 0. \quad (12)$$

We shall look for a solution of the form

$$S(x, y, t; \tau) = A(x, y)B(\tau) + C(t).$$
Substituting in the equation, yields
\[ B'(\tau) + \frac{(\partial_x A)^2 + (\partial_y A)^2}{2A(x, y)} B^2(\tau) + \frac{2(x^2 + y^2)}{A(x, y)} C'(t)^2 = 0. \]

We would like the coefficients of \( B^2(\tau) \) and \( C'(t)^2 \) to be constants, so we choose
\[ A(x, y) = 2(x^2 + y^2). \]

The equation becomes
\[ B'(\tau) + 4B^2(\tau) + C'(t)^2 = 0, \]
which after the separation of the variables \( \tau \) and \( t \) becomes
\[ B'(\tau) + 4B^2(\tau) = -C'(t)^2. \]

There is a separation constant \( \theta \) such that
\[ C'(t) = \theta \]
\[ B'(\tau) + 4B^2(\tau) = -\theta^2. \]
Integrating we get

\[ C(t) = \theta t \]

\[ B(\tau) = -\frac{\theta}{2} \tan(2\theta\tau). \]

Hence the action is

\[ S(x, y, t; \tau) = A(x, y)B(\tau) + C(t) \]

\[ = -\theta(x^2 + y^2) \tan(2\theta\tau) + \theta t. \]

Another solution of the Hamilton-Jacobi equation is

\[ S = A(x, y)B(\tau) = \frac{K(x^2 + y^2)}{2(C_0 + K\tau)}, \]

with \( C_0 \) and \( K \) constants. We note that if \( C_0 = 0 \) the action becomes the Euclidean action

\[ S = \frac{x^2 + y^2}{2\tau}. \]
Example: Consider
\[ X_1 = \partial x_1 + 2x_2 |x|^{2k} \partial_t, \quad X_2 = \partial x_2 - 2x_1 |x|^{2k} \partial_t. \]

Looking for an action of the form
\[ S = W(|x|) + U(\tau) + V(t) \]
yields
\[
S(x, t, \tau) = \theta t + \frac{1}{k + 1} \left( k E \tau + |x| \sqrt{\frac{E}{2} + |x|^{4k+2}} \right. \\
- \left. |x_0| \sqrt{\frac{E}{2} + |x_0|^{4k+2}} \right).
\]
Solving Hamilton-Jacobi using the eiconal equation

Try a solution of the form

\[ S(x, \tau) = A(\tau) + B(x) \]

for the Hamilton-Jacobi equation

\[ \partial_\tau S + H(\nabla h S) = 0. \]

Separation of variables yields

\[ \underbrace{A'(\tau)}_{= -E} + \underbrace{H(\nabla h B)}_{= E} = 0, \]

where \( E \) is the energy constant.

Hence

\[ S(x, \tau) = -E \tau + B(x), \]

\( B(x) \) solution of the eiconal equation:

\[ H(\nabla h B) = E, \]

or \( \frac{1}{2} |\nabla h B|^2_h = E. \)
Eiconal equation on a Riemannian manifold $(M, g)$

\[ f(x) = \text{dist}(x_0, x) \] is a solution for the eiconal equation

\[
|\nabla_h f|^2_g = 1 \\
\]
\[ f(x_0) = 0. \]

In general, the eiconal equation does not have an unique solution. For instance, the eiconal equation on $\mathbb{R}^2$

\[
\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1
\]

has infinitely many solutions of the type

\[
f_\lambda(x, y) = (x - x_0) \cos \lambda + (y - y_0) \sin \lambda
\]

with $f_\lambda(x_0, y_0) = 0$. Also

\[
f(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}
\]

is a solution with $f(x_0, y_0) = 0$. 

Lagrange-Charpit method
Solves the non-linear PDE
\[ F(x, z, \nabla z) = 0, \]
where \( x = (x_1, \ldots, x_n) \), \( \nabla z = (\partial x_1 z, \ldots, \partial x_n z) \)
and \( F(x, z, p) \) is a differentiable function defined on an open subset of \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \).

Replacing \( \partial x_j z \) by \( p_j \) the Lagrange-Charpit system of characteristics
\[
\dot{x}_i = \frac{\partial F}{\partial p_i}, \\
\dot{p}_i = - \left( \frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial z} \right), \\
\dot{z} = \frac{\partial F}{\partial p_i}, \quad i \in \{1, \ldots, n\}. 
\]
The solution of the above system
\[
x_i = x_i(s), \quad p_i = p_i(s), \quad z = z(s)
\]
verify the equation
\[ F(x, p, z) = 0. \]

The main difficulty is to eliminate the parameter \( t \) and write the solution explicitly as \( z = z(x) \).
Eiconal equation on $\mathbb{R}^2$

Consider the equation

$$\left( \frac{\partial z}{\partial x_1} \right)^2 + \left( \frac{\partial z}{\partial x_2} \right)^2 = 1, \quad (13)$$

Apply Lagrange-Charpit method of characteristics. Consider

$$F(x_1, x_2, z, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}.$$

Since $\frac{\partial F}{\partial p_i} = p_i$, $\frac{\partial F}{\partial x_i} = 0$, $\frac{\partial F}{\partial z} = 0$, the Lagrange-Charpit system becomes

$$\dot{x}_1 = p_1$$
$$\dot{x}_2 = p_2$$
$$\dot{p}_1 = 0 \implies p_1 = c_1 \ (constant)$$
$$\dot{p}_2 = 0 \implies p_2 = c_2 \ (constant)$$
$$\dot{z} = p_1^2 + p_2^2 = c_1^2 + c_2^2.$$
Solving the system yields
\[ x_i(t) = c_i t + x_i(0) \quad \Rightarrow \quad c_i = \frac{x_i(t) - x_i(0)}{t}, \]
and hence
\[ z(t) = \left( c_1^2 + c_2^2 \right) t + z(0) \]
\[ = \frac{\left( x_1(t) - x_1(0) \right)^2 + \left( x_2(t) - x_2(0) \right)^2}{t} + z(0) \]
\[ = \frac{d_{Eu}^2(x(0), x(t))}{t} + z(0). \]

We shall eliminate the parameter \( t \) from the above formula. We can do this by choosing \( t \) to be the arc length parameter along the curve \( x(t) \). In this case the length of the curve is the same as the Euclidean distance between the end points of the curve, \( i.e., \)
\[ t = d_{Eu}(x(0), x(t)). \]
Hence we get a solution which depends only on $x$

$$z(x) = d_{Eu}(x^0, x) + z_0 = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + z_0},$$

where $x^0$ is an arbitrary point and $z_0$ is an arbitrary constant.

**SubRiemannian eiconal equation**

$$H(\nabla f) = \frac{1}{2}|\nabla h f|^2_h = \frac{1}{2} \sum_{i=1}^{k} (X_i f)^2.$$  

We are interested in solving the eiconal subRiemannian equation

$$H(\nabla f) = c,$$  \hspace{1cm} (14)

where $c > 0$ is a constant.
We shall write the energy in a slightly different way. If the vector fields in local coordinates are

\[ X_j = \sum_{i=1}^{m} X_j^i \partial x_i, \quad j = 1, \ldots, k, \]

then

\[ X_j f = \sum_{i=1}^{k} X_j^i \partial x_i f = \langle X_j(x), \nabla f \rangle, \]

where \( \langle \ , \ \rangle \) denotes the Euclidean inner product. Then the energy of the function \( f \) can be written in the following inner product form

\[ H(\nabla f) = \frac{1}{2} \sum_{j=1}^{k} \langle X_j(x), \nabla f \rangle^2. \]

Replacing \( \nabla f \) by \( p \) we obtain the principal symbol of \( \Delta_X = \frac{1}{2}(X_1^2 + \cdots + X_n^2) \).
\[ H(x, p) = \frac{1}{2} \sum_{j=1}^{k} \langle X_j(x), p \rangle^2. \]

This is called the Hamiltonian.

We shall employ the Lagrange-Charpit method by choosing

\[ F(x, f, p) = \frac{1}{2} \sum_{j=1}^{k} \langle X_j(x), p \rangle^2 - c = H(x, p) - c. \]

The Lagrange-Charpit system of characteristics can be written as

\[ \dot{x}_i = \frac{\partial F}{\partial p_i} = \frac{\partial H}{\partial p_i}, \]
\[ \dot{p}_i = -\frac{\partial F}{\partial x_i} = -\frac{\partial H}{\partial x_i}, \]
\[ \dot{f} = \sum_{i=1}^{k} p_i \frac{\partial F}{\partial p_i} = \sum_{i=1}^{k} p_i \frac{\partial H}{\partial p_i}, \]

where \( i = 1 \dotsc k \).
The first $2k$ equations of the above Lagrange-Charpit system is called the Hamiltonian system associated with the Hamiltonian $H(x, p)$. Its solutions $(x(t), p(t))$ are called bicharacteristics. The $x$-component of the bicharacteristics is called a (regular) subRiemannian geodesic.

**Formal solution:** By Euler’s formula for homogeneous functions, we have

$$\sum_{i=1}^{k} p_i \frac{\partial H}{\partial p_i} = 2H,$$

and hence the third Lagrange-Charpit equation becomes

$$\dot{f}(t) = \sum_{i=1}^{k} p_i \frac{\partial H}{\partial p_i} = 2H(x(t), p(t)) = E,$$

with the solution

$$f(t) = Et + f(0).$$
**Length minimizing curves**

If the distribution $\mathcal{D}$ is bracket generating ($X_i$ and their iterated brackets span the tangent space $TM$ at every point), then any two points can be connected by a horizontal curve (Chow’s theorem).
Let \( p, q \in M \) be two distinct points. We are interested in characterizing the horizontal curves \( \phi : [0, \tau] \rightarrow M \) with endpoints \( \phi(0) = p \), \( \phi(\tau) = q \) for which the length of the curve defined by

\[
\ell(\phi) = \int_0^\tau \frac{1}{2} |\dot{\phi}(t)|_h \, dt
\]

is minimum.
A horizontal curve joining $P$ and $Q$

Minimizing length $\int_0^\tau \frac{1}{2}|\dot{\phi}(t)|_h \, dt \iff$
Minimizing energy $\int_0^\tau \frac{1}{2}|\dot{\phi}(t)|^2_h \, dt$.

for any function $S$ the integrals

$$I(\phi) = \int_0^\tau \frac{1}{2}|\dot{\phi}(t)|^2_h \, dt \quad \text{and}$$
$$J(\phi) = \int_0^\tau \left(\frac{1}{2}|\dot{\phi}(t)|^2_h - dS\right)$$

are related by the relation

$$J(\phi) = I(\phi) - S(\tau, \phi(\tau)) + S(0, \phi(0)).$$

Since $\phi(0) = p$ and $\phi(\tau) = q$ are given, then $I(\phi)$ and $J(\phi)$ reach the minimum value for the same curve $\phi(t)$ and the relationship between the minima is

$$\min J(\phi) = \min I(\phi) - S(\tau, q) + S(0, p). \quad (15)$$
By computation:

\[ J(\phi) = \int_0^\tau \left( \frac{1}{2}|\dot{\phi} - \nabla_h S|^2_h - \left( \frac{1}{2}|\nabla_h S|^2_h + \frac{\partial S}{\partial t} \right) \right) dt. \]

Since this works for any function \( S \), we shall choose it such that the integral \( J(\phi) \) has a simplified form:

\[ \frac{1}{2}|\nabla_h S|^2_h + \frac{\partial S}{\partial t} = 0 \quad (\text{Hamilton Jacobi eq.}) \]

For this \( S \)

\[ J(\phi) = \int_0^\tau \frac{1}{2}|\dot{\phi} - \nabla_h S|^2_h dt, \]

and \( J(\phi) \) is minimum if and only if \( |\dot{\phi} - \nabla_h S|^2_h = 0 \), i.e., when \( \dot{\phi}(t) = \nabla_h S|_{\phi(t)} \). Since in this case \( \min J(\phi) = 0 \), relation (15) yields

\[ \min I(\phi) = S(\tau, q) - S(0, p). \]
**Theorem** Given two distinct points $p, q \in M$, consider the energy functional

$$\phi \rightarrow I(\phi) = \int_0^\tau \frac{1}{2} |\dot{\phi}(t)|_h^2 \, dt$$

where $\phi : [0, \tau] \rightarrow M$ is a horizontal curve with fixed end points $\phi(0) = p$, $\phi(\tau) = q$.

Let $S : [0, \tau] \times M \rightarrow \mathbb{R}$ be a solution of the Hamilton-Jacobi equation (11). Then $\phi$ is a minimizer if and only if

$$\dot{\phi}(t) = \nabla_h S|_{\phi(t)}.$$ 

In this case the minimal value of $I(\phi)$ is $S(\tau, q) - S(0, p)$. 
Integrating the equation $\dot{\phi} = \nabla_h S$ in the Heisenberg case

One possible action on the subRiemannian manifold defined by the Heisenberg distribution is

$$S(x, y, t; s) = \theta t - \theta(x^2 + y^2)\tan(2\theta s).$$

Let $\phi(s) = (x(s), y(s), t(s))$ be a length-minimizing curve. The equation $\dot{\phi} = \nabla_h S$ written on components becomes

\begin{align*}
\dot{x}(s) &= 2\theta(y(s) - x(s)\tan(2\theta s)) \\
\dot{y}(s) &= -2\theta(x(s) + y(s)\tan(2\theta s)) \\
\dot{t}(s) &= 2x(s)y(s) - 2y(s)x(s).
\end{align*}
Differentiating in the expression of $\dot{x}(s)$ yields

$$
\ddot{x} = 2\theta \left( \dot{y} - \dot{x} \tan(2\theta s) - 2\theta x \left( 1 + \tan^2(2\theta s) \right) \right)
$$

$$
= 2\theta \left( \dot{y} - 2\theta \left( x + y \tan(2\theta s) \right) \right)
$$

$$
= 2\theta (\dot{y} + \dot{y}) = 4\theta \dot{y}.
$$

Hence $\ddot{x} = 4\theta \dot{y}$. In a similar way we can show that $\ddot{y} = -4\theta \dot{x}$, which leads to the system of equations:

$$
\begin{align*}
\ddot{x} &= 4\theta \dot{y} \\
\ddot{y} &= -4\theta \dot{x} \\
\dot{t} &= 2yx - 2xy \\
\theta &= \text{constant}.
\end{align*}
$$

If $\theta \neq 0$ the solutions are

$$
(x(s), y(s)) = \frac{R}{4\theta} \left( \sin \alpha(s), \cos \alpha(s) \right),
$$

$$
t(s) = t_0 + \frac{\sin 2(\alpha - \alpha_0) - 2(\alpha - \alpha_0)}{8\theta^2}
$$

where $\alpha(s) = 4\theta s + c$
Horizontal connection

Definition Let \((M, \mathcal{D}, h)\) be a subRiemannian manifold and \(\{X_1, \ldots, X_k\}\) be an orthonormal frame on an open set \(U \subset M\). Define the horizontal connection \(D : \Gamma(\mathcal{D}, U) \times \Gamma(\mathcal{D}, U) \to \Gamma(\mathcal{D}, U)\) by

\[
D(V, W) = D_V W = \sum_{j=1}^{k} V g(W, X_j) X_j.
\]

(16)

The above definition is correct, in the sense that it does not depend on the horizontal frame. Let \(X' = \{X'_1, \ldots, X'_k\}\) be another horizontal frame on an open set \(U'\). If \(U \cap U' \neq \emptyset\) we shall show that

\[
\sum_{j=1}^{k} V g(W, X'_j) X'_j = \sum_{j=1}^{k} V g(W, X_j) X_j.
\]

(17)
**Proposition** \( D \) is a linear connection, i.e., \( D \) is \( \mathbb{R} \)-linear in both arguments, \( \mathcal{F}(M) \)-linear in the first argument and satisfies Leibniz rule of differentiation in the second argument

\[
DV(fW) = V(f)W + fDVW, \quad \forall f \in \mathcal{F}(M).
\]  

(18)

**Proposition** The linear connection \( D \) is a metric connection, i.e.,

\[
DUg(V, W) = g(DUV, W) + g(V, DUW).
\]  

(19)
**Horizontal divergence**

**Definition** Let $Z$ be a horizontal vector field. The horizontal divergence of $Z$ is defined as the trace of the horizontal connection

$$div_h Z = \text{Trace}(V \rightarrow D_V Z), \quad (20)$$

where the Trace is taken over $\mathcal{D}$.

**Proposition:** Let $\{X_1, \ldots, X_k\}$ be a horizontal orthonormal frame and $Z = \sum_j Z^j X_j$ be a horizontal vector field. Then

$$div_h Z = \sum_{j=1}^k X_j(Z^j).$$

**Proposition:** For any $f \in \mathcal{F}(M)$ we have

$$\Delta_X f = \frac{1}{2} div_h \nabla_h f.$$
The action and conjugate points  We shall assume that the subRiemannian manifold \((M, \mathcal{D})\) has dimension \(2n + 1\) and it is a contact manifold. This means that there is a one form \(\omega\) on \(M\) such that Frobenius non-integrability condition \(\omega \wedge (d\omega)^n \neq 0\) is satisfied at each point. The horizontal distribution is defined by
\[
\mathcal{D} = \{X \in TM; \omega(X) = 0\} = \ker \omega.
\]
Frobenius’ condition is equivalent with the non-integrability of the one-form \(\omega\) and hence with the non-integrability of the horizontal distribution. By Carathédory’s theorem it follows that the subRiemannian manifold \(M\) is locally connected by horizontal curves.

The \(2n + 1\) form \(dv = \omega \wedge (d\omega)^n\) plays the role of the volume element. There is a function \(f\), which depends on \(X\), such that
\[
L_X dv = f(X) dv.
\]
Assume that \(M\) has dimension 3.
**Example:** Let $M = \mathbb{R}^3$, $X_1 = \partial_{x_1} + 2x_2\partial_t$, $X_2 = \partial_{x_2} - 2x_1\partial_t$, and $Y = [X_1, X_2] = -4\partial_t$. Choosing the contact form

$$\omega = \frac{1}{4}dt + \frac{1}{4}(x_2dx_1 + x_1dx_2),$$

yields $\omega(X_i) = 0$, $d\omega = dx_2 \wedge dx_1$, $\omega(Y) = 1$, $d\omega(Y, \cdot) = 0$, and the volume element is

$$dv = \omega \wedge d\omega = \frac{1}{4}dx_1 \wedge dx_2 \wedge dt.$$

**Theorem:** If $X = a^1X_1 + a^2X_2$ is a horizontal vector field, then

$$L_X dv = (\text{div}_h X) dv,$$

where $\text{div}_h X = X_1(a^1) + X_2(a^2)$ is the horizontal divergence of $X$. 

99
**Corollary:** (i) The volume element $dv$ is preserved along the horizontal vector field $X$ if and only if $div_h X = 0$.

(ii) The volume element expands (contracts) along the horizontal vector field $X$ if and only if $div_h X > 0$ ($div_h X > 0$).

(iii) The vector field $X$ is singular at the points where $div_h X = \pm \infty$.

In the following we shall consider a special flow. It is known that if $c(s)$ is a length minimizing curve then $\dot{c}(s) = \nabla_h S|_{c(s)}$, where $S$ denotes the action. Denote by $X$ the vector field in the direction of a length minimizing flow

$$X_p = \dot{c}(s), \quad p = c(s).$$

Taking $h$-divergence in $X = \nabla_h S$ yields

$$div_h X = div_h \nabla_h S = X_1^2 S + X_2^2 S = 2\Delta_X S.$$

Using above Theorem yields:
**Proposition:** If $X$ is the vector field considered above

$$L_X dv = 2(\Delta_X S) dv.$$ 

Two points $p$ and $q$ are called conjugate along the geodesics flow $X$ if $p$ and $q$ are singularities for $X$ with

$$(\text{div}_h X)_p = +\infty, \quad (\text{div}_h X)_q = -\infty.$$ 

It follows that at the conjugate points $\Delta_X S$ is singular.
**Example:** Consider a geodesic flow starting at the origin. The action function from the origin is

\[ S = \theta t - (x^2 + y^2) \tan(2\theta s), \quad (22) \]

so \( \Delta_X S = -2\theta \tan(2\theta s) \) with the singularities at \( 2\theta s = \frac{n\pi}{2}, \ z \in \mathbb{Z} \).

If \( c(s) \) is a geodesic starting at \( c(0) \), the conjugate points to the origin will occur at \( c(s_n) \), where \( 2\theta s_n = \frac{n\pi}{2} \). The constant \( \theta \) satisfies the relation

\[ \frac{t(s)}{x^2(s) + y^2(s)} = \mu(2\theta s) \]

and then at \( 2\theta s = \frac{n\pi}{2} \) the function \( \mu \) is singular. This corresponds to points for which \( x^2 + y^2 = 0 \), which are points on the \( t \)-axis. It follows that the points conjugate to the origin belong to the \( t \)-axis.
References


