The conformal curvature flow

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Abstract: In this talk, one will survey some recent progress on the conformal curvature flow for various prescribed curvature problem, including the perturbation theory for prescribing scalar curvature problem on the unit sphere and similar question for the prescribing mean curvature on the $n$-dimensional unit ball as well as the prescribing Q-curvature problem on the unit spheres.
Conformal curvature flow
1 Conformal curvature flow

2 Prescribed scalar curvature problem
1. Conformal curvature flow
2. Prescribed scalar curvature problem
3. Prescribed Q-curvature problem
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3. Prescribed Q-curvature problem
4. Prescribed mean curvature problem
Flow method

Several important results by Flow Method:

- Harmonic maps: 1964, Eells-Simpson
- Hamilton’s Ricci flow 1982
- Yamabe flow 1990’s?
- Conformal curvature flow, S. Brendle, 2002
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S. Brendle: large energy
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\[
\frac{\partial g}{\partial t} = -(Q - \bar{Q}f)g,
\]

Here \(\bar{Q}, \bar{f}\) mean the average values with respect to the time metric and \(f > 0\) is a smooth function.

Brendle proved the following statement: If \(\int_M Q_0 d\mu_g < (n - 1)!\omega_n\) and its corresponding Paneitz operator is non-negative with constant kernel, then the flow exists globally and the metrics converge to a metric with Q curvature \(\bar{Q}f\).
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In this talk, I will focus on the first special case.
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We may restate this as the following question:

**Question** Given a smooth function \( f \) on \( S^n \) with standard metric \( g \), find a metric which is point-wise conformal to \( g \) and with the scalar curvature \( f \).
Such problem is equivalent to solving the following equation:

$$\Delta u + \frac{n-2}{4(n-1)} f u^{\frac{n+2}{n-2}} = \frac{n(n-2)}{4} u$$

on $S^n$ with $u > 0$. 
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**Necessary conditions:**
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1. $f > 0$ somewhere; (easy to see)

2. (Kazdan- Warner) $\int_{S^n} < \nabla f, \nabla x > u^{2n/(n-2)} = 0$. 
**Classical approach:** Variational method: Above differential equation is the critical point of the functional:

\[
J(u) = \int_{S^n} |\nabla u|^2 + \frac{n(n-2)}{4} u^2 \frac{(\int_{S^n} fu^{2n/(n-2)})^{(n-2)/n}}{\left(\int_{S^n} fu^{2n/(n-2)}\right)^{(n-2)/n}},
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over the set

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H = \{u \in H^1, u \not\equiv 0\}.
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Notice that if \( f > 0 \), then \( J(u) \) is non-negative. Hence it is natural to find the minimizer of \( J(u) \) over the set \( H \).
However, due to second necessary condition, the minimizer exists if and only if \( f \) is a constant. In other words, if \( f \) is not a constant, the minimizer never achieves.
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T. Aubin realized that such variational problem is closely related to best Sobolev embedding $H^1 \rightarrow L^{2n/(n-2)}$.

T. Aubin was able to show that for any given smooth function $f$, there exists a vector $a$ such that $f - \langle a, x \rangle$ is the scalar function of some conformal metric. In particular, this is true for $f$ positive.
Hence the problem reduces to question: for which $f$, we can ensure $a = 0$. 
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Naturally for each point $p \in S^n$ and $t \geq 1$, we can construct a conformal transformation $\phi_{p,t}$ on $S^n$. For each fixed $p$, $t$, we can consider the following map from $S^n \times [1, \infty) \to \mathbb{R}^{n+1}$:

$$G(p, t) = \int_{S^n} f \circ \phi_{p,t} x.$$ 

Since $f$ is non-degenerate, for $t$ sufficiently large, $G(p, t)$ is never zero, $\text{Deg}(G)$ is well defined. Here non-degenerate means $|\nabla f|^2 + (-\Delta f)^2 \neq 0$. 

**Theorem (Chang and Yang)** If $f$ is a positive smooth function with non-degenerate critical points and $\text{Deg}(G)$ is not equal to zero and $f$ is sufficiently close to $n(n-1)$, then the equation has a solution.
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Let $u(x, t)$ be a smooth positive function such that $u^{\frac{4}{n-2}}g_0$ is a smooth conformal metric on $S^n$. 

$$u_t = \frac{n-2}{4} (\alpha(t) f - R) u$$

where $R$ is the scalar curvature of the metric $g = u^{\frac{4}{n-2}}g_0$, i.e. in term of $u$,

$$R = u^{-n+2} \left[ -c_n \Delta u + n(n-1)u \right].$$
Recently we adopt the flow method to recheck above perturbation theory. Here is the set up.

Let $u(x,t)$ be a smooth positive function such that $u^{\frac{4}{n-2}}g_0$ is a smooth conformal metric on $S^n$.

We consider the following scalar curvature flow:

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where $R$ is the scalar curvature of the metric $g = u^{4/(n-2)}g_0$, i.e, in term of $u$, $R = u^{-\frac{n+2}{n-2}}[-c_n\Delta u + n(n-1)u]$. 
\[ \alpha(t) \] is chosen such that the flow preserves the volume, which means:

\[
\frac{d}{dt} \int_{S^n} u^{\frac{2n}{n-2}} d\mu_{g_0} = \frac{n}{2} \int_{S^n} (\alpha f - R) d\mu_g = 0.
\]

So

\[
\alpha(t) = \frac{\int_{S^n} Rd\mu_g}{\int_{S^n} f d\mu_g}.
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Thus it turns the problem to investigate the convergence of the flow. The first observation is that for any $p \geq 1$

$$\int_{S^n} |\alpha f - R|^p \, d\mu_g \to 0 \text{ as } t \to \infty.$$
Moreover we also have the following fact:

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Nevertheless, with those preliminary results, we can conclude that the scalar curvature flow will converge to \( \alpha f \) in \( H^1 \) norm. However, except the measure is time dependent, \( \alpha \) also depends on time \( t \). It is still far away from point-wise convergence which is what we want.
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The simple bubble condition is given by

$$\max f \leq \delta_n \min f$$

where $\delta_n = 2^{2/n}$ if $n \leq 4$ and $= 2^{2/n-2}$ if $n \geq 5$. 
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The argument is based on argument by contradiction.
Assume $f$ cannot be realized as a scalar curvature for some conformal metric. Then we can pick a constant

$$\beta > n(n - 1)(\min f)^{\frac{n}{n-2}}$$

such that for every initial date $u_0$ with $E_f(u_0) \leq \beta$, the flow will blow-up at some critical point of $f$. Let me explain this in step by steps.
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First recall the energy functional is defined by

$$E_f(u) = \frac{\int_{S^n} Rdv_g}{\left(\int_{S^n} fdv_g\right)^{n/(n-2)}}$$

as a functional on $H^1$. 
If the flow does not converge, as $t$ tends infinity,

$$E_f(u) \to \frac{n(n-1)}{f(p)^{n-2}},$$

for some point $p \in S^n$ such that $\nabla f(p) = 0$ and $\Delta f(p) \leq 0$. 
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for some point $p \in S^n$ such that $\nabla f(p) = 0$ and $\Delta f(p) \leq 0$.

Notice that the choice of the constant $\beta$ is strictly greater than the possible blow-up energy level.
If \( \frac{\max f}{\min f} \leq \delta_n \) with the initial energy under the control, then either \( u \) is bounded in \( H^{2,p} \) for some \( p > n/2 \) or its normalized flow (with the center of mass at origin) will converge.
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For latter, we need blow-up analysis, under the assumption of simple bubble condition, the flow can only concentrate at at most one point. In fact, I should say that it will concentrate at exactly one point.
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One bubble case implies the eigen-values of laplace for conformal metric $g$ converge to the ones of Laplace for standard metric. This, together with free-center of mass and constant volume, implies that the normalized conformal factors bounded above while simple bubble condition implies they also bounded from below.
We also need to analyze the conformal vector field induced by the normalization for free center of mass in terms of the $L^2$ norm of $(\alpha f - R)$. 
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Then we completely study the spectral decomposition in order to study the speed of convergence of the center of mass of conformal metric $g$ to a blow-up point. Here a large amount of computation is needed.
Hence $E_f(u)$ defines a contraction on the domain
$\Omega = \{ u \in H^1 : E_f(u_0) \leq \beta \}$ to a single point for some suitable
constant $\beta$ which is related to what we have chosen before.
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Let $p_1, p_2, \cdots, p_N$ be all of the critical points of $f$ such that
$f(p_i) \leq f(p_j)$ if $i < j$. Without loss of generality, we may assume
$f(p_i) < f(p_j)$ if $i < j$. 
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$f(p_i) < f(p_j)$ if $i < j$.

Define $\beta_j = \frac{n(n-1)}{f(p_j)^{n-2}}$. Then clearly $\beta_i > \beta_j$ if $i < j$. So we can
choose a constant $\nu > \sigma$ such that $\beta_i - \nu > \beta_{i+1}$. 
First we shall show that for each $1 \leq i \leq N$, the sub-level set $L_{\beta_i - \nu}$ is homotopic equivalent to $L_{\beta_i + 1 + \nu}$.
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Morse identity can be obtained by standard Morse theory for infinity dimensional case for domain $\Omega$ which voids the assumption on degree by notice that if $f$ is non-degeneracy, then above cases include all critical points of $f$. 

Xingwang Xu

Curvature Flow
Thus we can state the result we have obtained as

Theorem (Chen and Xu, 2012) Suppose $f$ is a positive smooth non-degeneracy function. If $\frac{\max f}{\min f} \leq \delta_n$ and $\deg(G) \neq 0$, then $f$ can be realized as a scalar curvature in its standard conformal class.
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Theorem (Chen and Xu, 2012) Suppose $f$ is a positive smooth non-degeneracy function. If $\frac{\max f}{\min f} \leq \delta_n$ and $\text{deg}(G) \neq 0$, then $f$ can be realized as a scalar curvature in its standard conformal class.

Notice that if $|f - n(n - 1)| \leq \gamma_n$ with $\gamma_n < (2^{2/n} - 1)n(n - 1)/(2^{2/n} + 1)$, then we can show that $\frac{\max f}{\min f} \leq 2^{2/n} \leq \delta_n$. It seems that our assumption on $f$ is very precise contrast to original one for existence. Or in other words, we get some precise estimate on their smallness condition. Observe that when $n$ is large, $\gamma_n$ here is not small at all.
As we have pointed out that the flow methods for prescribing curvature problem was first introduced by Brendle for Guassian curvature type (Q-curvature) problem.
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First let us discuss the Gaussian curvature equation on $S^2$. The equation is given by

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In terms of blow-up behaviors, it seems that the solution set of this two dimensional equation has at most simple blow-up. Therefore with degree and non-degeneracy assumptions, the solution always exists. There is no extra assumption ”the close to constant 1” needed.
The best result in Gaussian curvature case can be stated as follows:

**Theorem** (Chang-Gursky-Yang) If $f$ is a positive non-degenerate smooth function with $\deg(G) \neq 0$, then the equation has a solution.
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Of course, there are many other kind results, for example, relax the non-degeneracy condition, symmetric property on prescribed functions. I will not mention them here since it is not what we will focus on.
Other type of such equations is so-called higher order $Q$—curvature equation:

$$Pw + fe^{nw} = Q,$$

where $P$ is generalized Paneitz operator. Wei and I have been able to generalize above Chang-Gursky-Yang’s statement to this case with exactly the same assumption on $f$. 
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One should point out that S. Brendle has done the same thing.
As long as conformal invariant equations concern, we may also consider

\[ (-\Delta)^p u = fu^{\frac{n+2p}{n-2p}} \]

on \( \mathbb{R}^n \) after stereo-graphic projection from \( S^n \). Similar behavior occurs. However it has its own difficulty which mainly comes from the lack of maximum principle for this equation.
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On generic manifolds, one can purpose similar problems, in particular for $p = 2$. The basic problem there is the analogy of Yamabe problem which in general is still open. This is other motivation for us to study the flow method for scalar curvature equation. One hopes such method can be applied to this type of problem.
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Then Struwe and Malchiodi used the flow method to study the fourth order $Q$–curvature problem on $S^4$ (JDG, 2006). It further demonstrated that if the curvature problem only allows the simple blow-up, then the flow method will be successful to produce the solution. In particular, Morse theory argument in their paper made the evidence for flow to convergence. That means that if the flow does not converge, then Morse theory for infinite dimensional manifolds gives arise some identity for critical points of $f$ while the degree of the map defined above provides some information to conclude that such identity could not be true. That forces the flow to converge.
Recently we adopt the same scheme as above to get the following statement:
Recently we adopt the same scheme as above to get the following statement:

**Theorem:** (Chen & Xu, 2011) Let $n \geq 2$ be an even integer. Let $f$ be a positive smooth Morse function with only non-degenerate critical points. Let

$$\gamma_i = \# \{ q \in S^n : \nabla f(q) = 0; \Delta u(q) < 0; \text{ind}(f, q) = n - i \}$$

and the system

$$\gamma_0 = 1 + k_0, \quad \gamma_i = k_i + k_{i-1}, \text{ for all } 1 \leq i \leq n - 1; \quad k_n = 0$$

has no non-negative integer solutions. Then the flow method generates a solution to the prescribed Q-curvature equation.
We observe that in previous statement $f > 0$ should not be necessary. We can remove this condition rather define $\gamma_i$ for those critical points with $f(q) > 0$. The rest of conditions keeps unchange, then the conclusion still holds true. This has been realized in a joint work with X. Chen and L. Ma. I do not have time to discuss this in the detail.
Now we discuss the following problem:
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Find a positive harmonic function on the unit ball $B^{n+1}$ such that

$$\frac{2}{n-1} \frac{\partial u}{\partial \eta} + u = fu^{\frac{n+1}{n-1}},$$

where $f$ is pre-given function.
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It is well known that such problem is similar to the prescribed scalar curvature problem.
Again, by divergence theorem, we have

\[ \int_{S^n} fu^{n+1} d\mu_g = \int_{S^n} ud\mu_g. \]

Thus if \( u > 0 \), then \( f \) must be positive somewhere.
Again, by divergence theorem, we have
\[
\int_{S^n} f u^{n+1} d\mu_g = \int_{S^n} u d\mu_g.
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It is not hard to see that
\[
\int_{S^n} (\nabla f \cdot \nabla x) u^{2n} d\mu_g = 0.
\]
Here \( x \) is the position vector of \( S^n \) in \( R^{n+1} \).
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on the sphere \( S^n \). Here

\[ h = u^{-\frac{n+1}{n-1}}\left(\frac{2}{n-1} \frac{\partial u}{\partial \eta} + u\right). \]

And harmonically extend \( u(x, t) \) to the ball \( B^{n+1} \).
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And harmonically extend $u(x, t)$ to the ball $B^{n+1}$.

Follow’s Brendle’s work, we can show that for any given smooth function $f$ on $S^n$ and initial data, there exists a global solution.
Again, follow the same scheme as for prescribed scalar curvature problem, to conclude the following statement:
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Let $f$ be a smooth Morse function on $S^n$. Define

$$m_i = \#\{\theta \in S^n; \nabla f(\theta) = 0; \Delta(\theta) < 0; \text{ind}(f, \theta) = n - i\},$$

for $0 \leq i \leq n$. 

Xingwang Xu  
Curvature Flow
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for $0 \leq i \leq n$.

Simple bubble condition:

$$\frac{\max f}{\min f} < 2^{\frac{1}{n-1}}.$$
Theorem: (Xu & Zhang, 2012, work in progress) If the smooth non-degenerate Morse function $f$ satisfies the simple bubble condition and the system

$$m_0 = 1 + k_0, \quad m_i = k_{i-1} + k_i; \quad 1 \leq i \leq n - 1; \quad k_n = 0$$

has no non-negative integer solutions, then above prescribed mean curvature problem has a positive smooth solution.
THANK YOU VERY MUCH!