CR Yamabe flow

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Yamabe problem

$M = \text{compact manifold of dimension } n \geq 3 \text{ without boundary}$

$g_0 = \text{Riemannian metric on } M$

$g$ is conformal to $g_0$ if $g = ug_0$ for some $0 < u \in C^\infty(M)$.

Yamabe problem: Find $g$ conformal to $g_0$ such that the scalar curvature of $g$, $R_g \equiv \text{constant}$. 
If we write \( g = u^{\frac{4}{n-2}} g_0 \), where \( u > 0 \), then

\[
\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u + R_g u^{\frac{n+2}{n-2}} = 0. \tag{1}
\]

Here, \( \Delta_{g_0} = \text{Laplacian of } g_0 \),

\( R_{g_0} = \text{scalar curvature of } g_0 \),

\( R_g = \text{scalar curvature of } g \).

Yamabe problem is to solve (1) with \( R_g \equiv \text{constant} \).
Define **Yamabe constant** $Y(M, g_0)$ by

$$Y(M, g_0) = \inf \left\{ E(u)|g = u^{\frac{4}{n-2}} g_0 \text{ conformal to } g_0 \right\}.$$  

Here,

$$E(u) = \frac{\int_M \frac{4(n-1)}{n-2} |\nabla_{g_0} u|^2 + R_{g_0} u^2 dV_{g_0}}{(\int_M u^{\frac{2n}{n-2}} dV_{g_0})^{\frac{n}{n-2}}}.$$

If $E(u) = Y(M, g_0)$, then $u$ satisfies (1).
Yamabe problem

- When $Y(M, g_0) \leq 0$, the Yamabe problem was solved by Trudinger.

- When $Y(M, g_0) > 0$, the Yamabe problem was solved by:
  - Aubin when $n \geq 6$ and $M$ is not locally conformally flat;
  - Schoen when $3 \leq n \leq 5$ or $M$ is locally conformally flat by using positive mass theorem.

- Bahri obtained the “same” result of Schoen by using “critical point at infinity”.

CR Yamabe flow
Hamilton introduced **Yamabe flow**:

\[ \frac{\partial}{\partial t} g(t) = -(R_g(t) - \bar{R}_g(t))g(t), \]

where

\[ \bar{R}_g(t) = \frac{\int_M R_g(t) dV_{g(t)}}{\int_M dV_{g(t)}}. \]
\[ 0 = \frac{\partial}{\partial t} g(t) = -(R_g(t) - \bar{R}_g(t))g(t) \iff R_g(t) = \bar{R}_g(t). \]

If we write \( g(t) = u(t)^{\frac{4}{n-2}} g_0 \), then
\[
\frac{\partial}{\partial t} \left( u(t)^{\frac{n+2}{n-2}} \right) = \frac{n+2}{4} \left( \frac{4(n-1)}{n-2} \Delta_{g_0} u(t) - R_{g_0} u(t) + \bar{R}_g(t) u(t)^{\frac{n+2}{n-2}} \right).
\]

\[ \frac{d}{dt} E(u(t)) \leq 0 \] along the Yamabe flow.
Yamabe flow

- Hamilton proved:
  - the long time existence;
  - convergence when $Y(M, g_0) \leq 0$.

- When $Y(M, g_0) > 0$, the convergence of Yamabe flow was studied by Chow, Ye, Schwetlick-Struwe.

- Brendle proved the convergence by using positive mass theorem.
$M = \text{strictly pseudoconvex compact CR manifold of real dimension } 2n + 1 \text{ with contact form } \theta_0$

**CR manifold:** $\exists$ subbundle $T^{1,0} \subset \mathbb{C} \otimes TM$ such that

- $\dim_{\mathbb{C}} T^{1,0} = n$;
- $T^{1,0} \cap \overline{T^{1,0}} = \{0\}$;
- $[T^{1,0}, T^{1,0}] \subset T^{1,0}$.

**strictly pseudoconvex:** Levi form $-\sqrt{-1}d\theta_0$ is positive definite on $T^{1,0} \times \overline{T^{1,0}}$. 
Example: \( M = S^{2n+1} \subset \mathbb{C}^{n+1} \), the standard contact form is given by \( \theta_{S^{2n+1}} = \sqrt{-1} \sum_{j=1}^{n+1} (z_j d\bar{z}_j - \bar{z}_j dz_j) \)

One can define the notion of scalar curvature, which is called Webster scalar curvature \( R_{\theta_0} \).

The Webster scalar curvature of \( S^{2n+1} \) is \( R_{\theta_{S^{2n+1}}} = n(n + 1)/2 \).
CR Yamabe problem: Find a contact form $\theta$ conformal to $\theta_0$ such that $R_{\theta} \equiv \text{constant}$. 

If we write $\theta = u^{\frac{2}{n}}\theta_0$, then

$$
\left(2 + \frac{2}{n}\right) \Delta_{\theta_0}u - R_{\theta_0}u + R_{\theta}u^{1+\frac{2}{n}} = 0.
$$

Here, $\Delta_{\theta_0} = \text{sub-Laplacian of } \theta_0$, $R_{\theta_0} = \text{Webster curvature of } \theta_0$, and $R_{\theta} = \text{Webster curvature of } \theta$. 

CR Yamabe problem was solved by

- Jerison and Lee when $n \geq 2$ and $M$ is not locally CR equivalent to $\mathbb{S}^{2n+1}$ ( = result of Trudinger and Aubin );

- Gamara and Yacoub when $n = 1$ or $M$ is locally CR equivalent to $\mathbb{S}^{2n+1}$ by using “critical point at infinity” ( = result of Schoen and Bahri ).
CR Yamabe flow is given by:

\[ \frac{\partial}{\partial t} \theta(t) = -(R_{\theta(t)} - \bar{R}_{\theta(t)})\theta(t), \]

where

\[ \bar{R}_{\theta(t)} = \frac{\int_M R_{\theta(t)} dV_{\theta(t)}}{\int_M dV_{\theta(t)}}. \]
CR Yamabe flow

\[ 0 = \frac{\partial}{\partial t} \theta(t) = -(R_{\theta(t)} - \overline{R}_{\theta(t)})\theta(t) \Leftrightarrow R_{\theta(t)} = \overline{R}_{\theta(t)}. \]

\[ \frac{d}{dt} E(u(t)) \leq 0 \text{ along the CR Yamabe flow, where} \]

\[ E(u) = \frac{\int_M (2 + \frac{2}{n})|\nabla_{\theta_0} u|^2 + R_{\theta_0} u^2 dV_{\theta_0}}{(\int_M u^{2+\frac{2}{n}} dV_{\theta_0})^{\frac{n}{n+1}}}. \]

If we write \( \theta(t) = u(t)^{\frac{2}{n}} \theta_0 \), then

\[ \frac{\partial}{\partial t} \left( u(t)^{\frac{2+n}{n}} \right) = \frac{n+2}{2} \left( \left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u(t) - R_{\theta_0} u(t) + \overline{R}_{\theta(t)} u(t)^{1 + \frac{2}{n}} \right). \]
CR Yamabe flow

- S. C. Chang and J. H. Cheng proved the short time existence.

- Y. B. Zhang proved the long time existence and convergence when $Y(M, \theta_0) < 0$.

- When $Y(M, \theta_0) > 0$, S. C. Chang, H. L. Chiu, and C. T. Wu proved the long time existence and convergence when $n = 1$ and torsion is zero.
Theorem (H.\(\underline{\_}\_\_\))

\textit{CR Yamabe flow exists for all time when } \(Y(M, \theta_0) > 0\).

Theorem (H.\(\underline{\_}\_\_\))

\textit{Suppose } \(M = S^{2n+1}\). \textit{If } \(\theta(t)|_{t=0}\) \textit{is conformal to } \(\theta_{S^{2n+1}}\), \textit{then CR Yamabe flow } \(\theta(t)\) \textit{converges to } \(\theta_{S^{2n+1}}\).
Recently, J. H. Cheng, H. L. Chiu, and P. Yang proved the CR positive mass theorem when $M$ is locally CR equivalent to $\mathbb{S}^{2n+1}$. J. H. Cheng, A. Malchiodi, and P. Yang proved the CR positive mass theorem when $n = 1$.

**Theorem (H.____)**

Suppose $n = 1$ or $M$ is locally CR equivalent to $\mathbb{S}^{2n+1}$. Then CR Yamabe flow $\theta(t)$ converges.
Theorem (X. Cao)

The first eigenvalue of \(-\Delta g(t) + \frac{1}{2} R_g(t)\) is nondecreasing along the Ricci flow

\[
\frac{\partial}{\partial t} g(t) = -2Ric_g(t)
\]

on a Riemannian manifold with nonnegative curvature operator.

Theorem (X. Cao et. al.)

For all \(a > 0\), the first eigenvalue of \(-\Delta g(t) + aR_g(t)\) is nondecreasing along the Ricci flow on a surface with \(R_g(t) \geq 0\).
Consider Yamabe flow, because:

- When dim = 2, Ricci flow becomes the unnormalized Yamabe flow:
  \[
  \frac{\partial}{\partial t} g(t) = -2 \text{Ric}_g(t) = -R_g(t)g(t).
  \]

- The condition $R_g(t) \geq 0$ is preserved along the unnormalized Yamabe flow.
Theorem (H.\_

Along the unnormalized Yamabe flow

$$\frac{\partial}{\partial t} g(t) = -R_g(t) g(t),$$

the first eigenvalue of $$-\Delta g(t) + aR_g(t)$$ is nondecreasing

(i) if $$0 \leq a < \frac{n-2}{4(n-1)}$$ and $$\min R_g(t) \geq \frac{n-2}{n} \min R_g(t) \geq 0,$$

(ii) if $$a \geq \frac{n-2}{4(n-1)}$$ and $$\min R_g(t) \geq 0.$$

Similar results hold for p-Laplacian and for manifolds with boundary.
Theorem (H.____)

Along the unnormalized CR Yamabe flow

\[
\frac{\partial}{\partial t} \theta(t) = -R_{\theta(t)} \theta(t),
\]

the first eigenvalue of \(-\Delta_{\theta(t)} + aR_{\theta(t)}\) is nondecreasing

(i) if \(0 \leq a < \frac{n}{2n+2}\) and \(\min R_{\theta(t)} \geq \frac{n}{n+1} \min R_{\theta(t)} \geq 0\),

(ii) if \(a \geq \frac{n}{2n+2}\) and \(\min R_{\theta(t)} \geq 0\).
Yamabe soliton

\[ M = \text{smooth manifold with boundary (possibly noncompact)} \]
\[ g_0 = \text{Riemannian metric on } M \]

\( g(t) \) is Yamabe soliton if

\[ g(t) = \sigma(t)\psi_t^*(g_0) \]

is solution for the unnormalized Yamabe flow

\[ \frac{\partial}{\partial t} g(t) = -R_{g(t)}g(t). \]

Here, \( \sigma(t) \) is a smooth function such that \( \sigma(0) = 1 \);
\( \psi_t \) is a family of diffeomorphism with \( \psi_0 = id_M \).
Yamabe soliton

Yamabe soliton has been studied by Daskalopoulos-Sesum, Hsu,...

**Theorem (Di Cerbo-Disconzi, Hsu)**

*On a compact manifold $M$, any Yamabe soliton has constant scalar curvature.*
CR Yamabe soliton

\( M = \) strictly pseudoconvex compact CR manifold of real dimension \( 2n + 1 \) with contact form \( \theta_0 \)

\( \theta(t) \) is CR Yamabe soliton if

\[
\theta(t) = \sigma(t) \psi_t^*(g_0)
\]

is solution for the unnormalized CR Yamabe flow

\[
\frac{\partial}{\partial t} \theta(t) = -R_{\theta(t)} \theta(t).
\]

Here, \( \sigma(t) \) is a smooth function such that \( \sigma(0) = 1 \);
\( \psi_t \) is a family of CR diffeomorphism with \( \psi_0 = id_M \).
Theorem (H.\___)

On a compact CR manifold $M$, any CR Yamabe soliton has constant Webster scalar curvature.
CR Yamabe soliton

Idea of proof:
Along the unnormalized CR Yamabe flow, we have

\[
\frac{d}{dt} \left( \frac{\int_M R_{\theta(t)} dV_{\theta(t)}}{\left( \int_M dV_{\theta(t)} \right)^{n+1}} \right) = -n \frac{\left( \int_M R_{\theta(t)}^2 dV_{\theta(t)} \right) \left( \int_M dV_{\theta(t)} \right) - \left( \int_M R_{\theta(t)} dV_{\theta(t)} \right)^2}{\left( \int_M dV_{\theta(t)} \right)^{n+1} + 1}.
\]

On the other hand, the quantity \( \frac{\int_M R_{\theta(t)} dV_{\theta(t)}}{\left( \int_M dV_{\theta(t)} \right)^{n+1}} \) is invariant if \( \theta(t) = \sigma(t) \psi^*_t(g_0) \).
As a generalization of Yamabe problem, we want to ask:
Given \( f \in C^\infty(M) \), \( \exists g \) conformal to \( g_0 \) such that its scalar curvature \( R_g = f \)?

If \((M, g_0) = (S^n, g_{S^n})\) the standard sphere, the problem is called \textbf{Nirenberg’s problem}.

Studied by Kazdan-Warner, Chang-Yang, Struwe, etc.
Kazdan-Warner obtained a necessary condition, the so-called Kazdan-Warner identity:

If there exists \( g = u^{\frac{4}{n-2}} g_{S^n} \) such that \( R_g = f \), we must have

\[
\int_{S^n} \left\langle \nabla f, \nabla x_i \right\rangle_{g_{S^n}} u^{\frac{2n}{n-2}} dV_{g_{S^n}} = 0 \quad \text{for } i = 1, 2, \ldots, n + 1
\]

where \( x_i \) is the coordinate function of \( \mathbb{R}^{n+1} \) restricted to \( S^n \).
Theorem (Chang-Yang)

If $f > 0$ is a Morse function on $S^n$ such that

$$\sum (-1)^{\text{ind}(f,x)} \neq -1$$

$$\nabla_{g_S} f(x) = 0, \Delta_{g_S} f(x) < 0$$

and $\|f - n(n-1)\|_{C^0}$ is sufficiently small, then $\exists g$ conformal to $g_{S^n}$ such that $R_g = f$. 
Nirenberg’s problem

Using prescribed scalar curvature flow

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \alpha(t)f) g(t),$$

X. Chen and X. Xu proved the following:

**Theorem (Chen-Xu)**

*If* $f > 0$ *is a Morse function on* $S^n$ *such that*

$$\sum \nabla_{g_{S^n}} f(x) = 0, \Delta_{g_{S^n}} f(x) < 0$$

$$\sum (-1)^{ind(f,x)} \neq -1$$

*and* $\|f - n(n-1)\|_{C^0} < \delta_n$ *where* $\delta_n = 2^{\frac{n}{n-2}}$, *then* $\exists$ *g conformal to* $g_{S^n}$ *such that* $R_g = f$.  

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Q: What about the CR case? That is, given $f$ on the CR sphere $S^{2n+1}$, find $\theta$ conformal to $\theta_{S^{2n+1}}$ such that $R_\theta = f$.

J. H. Cheng obtained the necessary condition corresponding to the Kazdan-Warner identity in the Riemannian case.
Theorem (Malchiodi-Uguzzoni)

If $f > 0$ is a Morse function on $S^{2n+1}$ such that

$$\sum (-1)^{\text{ind}(f,x)} \neq -1$$

$$\nabla_{g_{S^n}} f(x) = 0, \Delta_{\theta S^n} f(x) < 0$$

and $\|f - n(n + 1)/2\|_{C^0}$ is sufficiently small, then $\exists \theta$ conformal to $\theta_{S^{2n+1}}$ such that $R_\theta = f$. 

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CR Yamabe flow
Nirenberg’s problem

Using prescribed Webster scalar curvature flow

\[ \frac{\partial}{\partial t} \theta(t) = -(R_{\theta(t)} - \alpha(t)f)\theta(t), \]

one can obtain

**Theorem (H.____)**

*If* \( f > 0 \) *is a Morse function on* \( S^{2n+1} \) *such that*

\[ \sum_{\nabla f(x) = 0, \Delta_{\theta S^{2n+1}} f(x) < 0} (-1)^{ind(f,x)} \neq -1 \]

*and* \( \| f - n(n + 1)/2 \|_{C^0} < \delta_n \) *where* \( \delta_n = 2^{\frac{1}{n}} \), *then* \( \exists \theta \) *conformal to* \( \theta S^n \) *such that* \( R_{\theta} = f \).
Thank you very much for your attention!