Applications of the Deligne-Kazhdan philosophy to the Langlands correspondence for split classical groups

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(joint work with Sandeep Varma)

July 12, 2016

Outline of the talk

1. The local Langlands correspondence
2. An overview of the Deligne-Kazhdan philosophy
3. Some results on classical groups

Local fields

F: a non-archimedean local field

$\mathcal{O}_F$: ring of integers

$\mathfrak{p}_F$: its maximal ideal

$\mathfrak{f} = \mathcal{O}_F/\mathfrak{p}_F$ denote the residue field of F.

A non-archimedean local field is:

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A non-archimedean local field is:

is a finite extension of $\mathbb{Q}_p$ (these are the non-archimedean local fields of characteristic 0) or

is isomorphic to $\mathbb{F}_q((t))$, (where $q = p^n$), the field of Laurent series in the indeterminate t. (these are the non-archimedean local fields of characteristic p).

The Galois group and the Weil group

Let $\bar{F}$ be a separable closure of F. Let F^{un} denote the maximal unramified extension of F. Recall that $\mathrm{Gal}(F^{\mathsf{un}}/F) \cong \hat{\mathbb{Z}}$. Note that we have an exact sequence

$$1 \longrightarrow \mathrm{Gal}(\bar{F}/F^{\mathsf{un}}) \longrightarrow \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Gal}(F^{\mathsf{un}}/F) \longrightarrow 1$$

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W_F is called the Weil group of F. The group $\mathrm{Gal}(\bar{F}/F^{\mathsf{un}})$ is denoted by I_F and is called the inertia group of F.

Local class field theory

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The main theorem gives a topological isomorphism

$$\phi_F : F^\times \to W_F^{\mathsf{ab}}$$

that induces an isomorphism

$$\hat{F}^\times \overset{\cong}{\to} \mathrm{Gal}(\bar{F}/F)^{\mathsf{ab}}.$$

Here $\hat{F}^\times \cong \mathcal{O}_F^\times \times \hat{\mathbb{Z}}$ is the profinite completion of $F^\times$ (Note that $F^\times \cong \mathcal{O}_F^\times \times \mathbb{Z}$).

Local class field theory

The inertia group I_F admits a nice descending filtration of ramification subgroups with *upper numbering* $\{I_F^m\}$ and the isomorphism

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in fact maps

$$\mathcal{O}_F^\times \to I_F^{\mathsf{ab}} \text{ and } 1 + \mathfrak{p}_F^m \xrightarrow{\phi_F} (I_F^m)^{\mathsf{ab}}.$$

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Hence local class field theory gives an isomorphism

$$\mathrm{Hom}(F^\times, \mathbb{C}^\times) \cong \mathrm{Hom}(W_F^{\mathsf{ab}}, \mathbb{C}^\times) \cong \mathrm{Hom}(W_F, \mathbb{C}^\times).$$

The local Langlands correspondence

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Let $\mathbf{G} = \mathrm{GL}_n$. To describe the correspondence, we replace

$$\mathrm{Hom}(F^\times, \mathbb{C}^\times) \rightsquigarrow \{ \text{ Irreducible smooth representations of } \mathrm{GL}_n(F)\}$$

and

$$\mathrm{Hom}(W_F, \mathbb{C}^\times) \rightsquigarrow \{ \text{ semi-simple } n\text{-dim. representations of } \mathrm{WD}_F\}.$$

Here $\mathrm{WD}_F := W_F \times \mathrm{SL}_2(\mathbb{C})$ is the Weil-Deligne group of F.

The local Langlands correspondence for GL_n

The local Langlands correspondence for GL_n can be described a follows: There is a bijection between

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The LLC has the property that for each (τ, W) of $\mathrm{GL}_r(F)$, $1 \leq r \leq n-1$

$$L(s, \sigma \times \tau) = L(s, \phi_\sigma \otimes \phi_\tau)$$
$$\epsilon(s, \sigma \times \tau, \psi) = \epsilon(s, \phi_\sigma \otimes \phi_\tau, \psi)$$

and furthermore, there is a unique map (1) that satisfies this property.

Proofs

There is a unique bijection between $\pi \to \phi_\pi$,

$$\{ \text{ Irreducible smooth representations of } \mathrm{GL}_n(F)\}$$

$$\stackrel{\text{LLC}}{\longleftrightarrow} \{ \text{ semi-simple } n\text{-dim. representations of } \mathrm{WD}_F\}.$$

such that

$$L(s, \pi \times \sigma) = L(s, \phi_\pi \otimes \phi_\sigma)$$

$$\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \phi_\pi \times \phi_\sigma, \psi)$$

- Proof over local function fields was done in 1993 (Laumon-Rapoport-Stuhler).
- Proof in characteristic 0 was completed in 2000 (Harris-Taylor, Henniart, Scholze (2013)).

Beyond GL_n

For other split reductive groups $\mathbf{G}(F)$ (like $\mathrm{GSp}_4(F), \mathrm{SO}_{2n+1}(F)$), the local Langlands correspondence will no longer be a bijection and will only be a surjective finite-to-one map:

$$\{\text{Irr. smooth reps. of } \mathbf{G}(F)\} \twoheadrightarrow \{\text{Homs } \phi : \mathrm{WD}_F \to \mathbf{G}^\vee\}/\{\mathbf{G}^\vee - \text{conj}\}$$

where $\mathbf{G}^\vee$ is the "Langlands dual group" of $\mathbf{G}$ (the complex group associated to the dual root datum).

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where $\mathbf{G}^\vee$ is the "Langlands dual group" of $\mathbf{G}$ (the complex group associated to the dual root datum). Given a ϕ, its fiber under the map above is called the L-packet attached to ϕ, denoted by Π_ϕ. The Langlands correspondence is also expected to parametrize the elements of Π_ϕ in terms of the set of irr. representations of the component group $\mathcal{S}_\phi$. Here $\mathcal{S}_\phi = S_\phi/S_\phi^0 Z(\hat{\mathbf{G}})$, where $S_\phi = \mathrm{Cent}_{\mathbf{G}^\vee}(\mathrm{Im}(\phi))$.

Langlands Correspondence beyond GL_n

The LLC has been established for

- For GSp_4 (Gan-Takeda in char 0, (-) in characteristic $p > 2$)
- For classical groups (Arthur/Mok/KMSW in char 0, G-Varma for split classical groups in sufficiently large characteristic)

Close local fields

Definition

Let $m \geq 1$. Two non-archimedean local fields F and F' are m-*close* if the quotient rings $\mathcal{O}_F/\mathfrak{p}_F^m$ and $\mathcal{O}_{F'}/\mathfrak{p}_{F'}^m$ are isomorphic.

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Example

The fields $\mathbb{F}_p((t))$ and $\mathbb{Q}_p(p^{1/m})$ are m-close.

In fact,

$$\mathbb{Z}_p[p^{1/m}]/(p) \cong \mathbb{Z}_p[X]/(X^m - p, p) \cong \mathbb{F}_p[X]/(X^m) \cong \mathbb{F}_p[[X]]/(X^m).$$

Note

Given a local field F' of characteristic p and an integer $m \geq 1$, there is a non-archimedean local field F of characteristic 0 such that F' is m-close to F.

Deligne's theory

For an object X defined over a field F, we use the notation X' to denote the corresponding object over F'.

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- I_F^m - m-th higher ramification subgroup with upper numbering.

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Theorem (Deligne)

If the fields F and F' are m-close, then

$$\mathrm{Gal}(\bar{F}/F)/I_F^m \xrightarrow{\cong} \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m.$$

Properties of Del_m: Local class field theory

The Deligne isomorphism is compatible with local class field theory.

Deligne proved that if the fields F and F' are m-close, then the following diagram is commutative:

$$\begin{array}{ccc} (\mathrm{Gal}\,\bar{F}/F)/I_F^m)^{ab} & \xrightarrow{\mathrm{Del}_m} & (\mathrm{Gal}(\bar{F}'/F')/I_{F'}^m)^{ab} \\ \Big\downarrow \text{LCFT} & & \Big\downarrow \text{LCFT} \\ (F^\times/(1+\mathfrak{p}_F^m))\widehat{} & \xrightarrow[\mathrm{cl}_m]{} & (F'^\times/(1+\mathfrak{p}_{F'}^m))\widehat{} \end{array}$$

In the above, we have used that if F and F' are m-close, then

$$F^\times/1+\mathfrak{p}_F^m \cong F'^\times/1+\mathfrak{p}_{F'}^m$$

.

Properties of Del_m: Representations of the Galois group

Now let $\phi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(V)$ be an irreducible n-dimensional representation such that $\phi|_{I_F^m} = 1$. Then ϕ factors through $\mathrm{Gal}(\bar{F}/F)/I_F^m$. If the fields F and F' are m-close, then

$$\mathrm{Gal}(\bar{F}/F)/I_F^m \overset{\mathrm{Del}_m}{\cong} \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m.$$

Hence

$$\phi' = \phi \circ \mathrm{Del}_m^{-1} : \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m \to \mathrm{GL}(V).$$

The isomorphism Del_m induces a bijection

$\{$Isomorphism classes of representations of $\mathrm{Gal}(\bar{F}/F)$ trivial on $I_F^m\}$

$\leftrightarrow$

$\{$Isomorphism classes of representations of $\mathrm{Gal}(\bar{F}'/F')$ trivial on $I_{F'}^m\}$.

Artin factors

Let (ϕ, V) be a representation of $\mathrm{Gal}(\bar{F}/F)$ trivial on I_F^m and let (ϕ', V') be the representation of $\mathrm{Gal}(\bar{F}'/F')$ obtained using Del_m. Then their Artin L- and ϵ- factors remain the same:

$$L(s, \phi) = L(s, \phi')$$

$$\epsilon(s, \phi, \psi) = \epsilon(s, \phi', \psi')$$

Here ψ is a non-trivial additive character of F, $k = \mathrm{cond}(\psi)$, and ψ' is a character of F' that satisfies the following conditions:
$\mathrm{cond}(\psi') = k$ and $\psi'|_{\mathfrak{p}'^{k-m}/\mathfrak{p}'^{k}} = \psi|_{\mathfrak{p}^{k-m}/\mathfrak{p}^{k}}$.

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Remark: These properties remain true if the $\mathrm{Gal}(\bar{F}/F)$ is replaced by Weil group W_F or the Weil-Deligne group $\mathrm{WD}_F := W_F \times \mathrm{SL}_2(\mathbb{C})$.

Kazhdan isomorphism

Let $\mathbf{G}$ be split, connected reductive group defined over $\mathbb{Z}$.

Let $G = \mathbf{G}(F)$ and

$$K_m = Ker(\mathbf{G}(\mathcal{O}) \to \mathbf{G}(\mathcal{O}/\mathfrak{p}^m))$$

be the m-th usual congruence subgroup of G. Consider the Hecke algebra $\mathcal{H}(G, K_m)$. This is spanned as a $\mathbb{C}$-vector space by $\{\mathbb{1}_{K_m g K_m} \mid g \in G\}$.

Theorem (Kazhdan)

Fix a non-archimedean local field F and $m \geq 1$. There exists $l \geq m$ such that for any field F' which is l-close to F, the Hecke algebras $\mathcal{H}(G, K_m)$ and $\mathcal{H}(G', K'_m)$ are isomorphic.

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Since irreducible representations (π, V) of G with $V^{K_m} \neq 0$ correspond to a simple $\mathcal{H}(G, K_m)$-modules, we get a natural bijection between

$$\{\text{Irr. ad. representations } (\pi, V) \text{ of } G \text{ such that } V^{K_m} \neq 0\}$$
$$\xrightarrow{\mathrm{Kaz}_m} \{\text{Irr. ad. representations } (\pi', V') \text{ of } G' \text{ such that } V'^{K'_m} \neq 0\}.$$

A variant of the Kazhdan isomorphism for $\mathrm{GL}_n(F)$

Let I_m be the m-th filtration subgroup of the standard Iwahori subgroup I of $GL_n(F)$.

- Howe wrote down a presentation of the Hecke algebra $\mathcal{H}(GL_n(F), I_m)$.

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$$\updownarrow \zeta_m$$

$$\{\text{Irr. ad. representations } (\sigma', V') \text{ of } \mathrm{GL}_n(F') \text{ such that } V'^{I'_m} \neq 0\}.$$

Assume F and F' are $(m+1)$-close and let $\sigma \overset{\zeta_{m+1}}{\longleftrightarrow} \sigma'$. Lemaire proved that if σ is ψ-generic then σ' is ψ'-generic, where $\psi \leftrightarrow \psi'$ as before.

Generalizing Howe's and Lemaire's work

For a split reductive group $\mathbf{G}$, let I be the standard Iwahori subgroup of $G := \mathbf{G}(F)$ and I_m be its m-th filtration subgroup.

- A presentation for the Hecke Algebra $\mathcal{H}(G, I_m)$.
- If F and F' are m-close, we have that $\mathcal{H}(G, I_m) \cong \mathcal{H}(G', I'_m)$.
- Hence we have a bijection

  $\{$Iso. classes of irr., ad. representations (σ, V) of G with $\sigma^{I_m} \neq 0\}$

  $\overset{\zeta_m}{\longleftrightarrow}$

  $\{$Iso. classes of irr., ad. representations (σ', V') of G' with $\sigma'^{I'_m} \neq 0\}$.

- Let F and F' be $(m+1)$-close and let $\sigma \overset{\zeta_{m+1}}{\longleftrightarrow} \sigma'$. If σ is generic, then so is σ'.

The Deligne-Kazhdan theory and Local Langlands Correspondence

Question: Assume that F and F' are two sufficiently close local fields, and consider the following diagram:

$$\begin{array}{ccc} \{(\sigma, V) \text{ of } G \mid \text{depth}(\sigma) < m\} & \xrightarrow{\text{LLC}} & \{\phi : \mathrm{WD}_F \to {}^L G \mid \text{depth}(\phi) < l\} \\ \text{Kazhdan} \downarrow & & \downarrow \text{Deligne} \\ \{(\sigma', V') \text{ of } G' \mid \text{depth}(\sigma') < m\} & \xrightarrow{\text{LLC}} & \{\phi' : \mathrm{WD}_{F'} \to {}^L G \mid \text{depth}(\phi') < l\} \end{array}$$

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Is this diagram commutative?

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This diagram is

known to be commutative when $\mathbf{G} = \mathrm{GL}_n$ (- (2012), ABPS (2013))

was used to prove the LLC for $\mathrm{GSp}_4(F'), \mathrm{char}(F') > 2$ using the LLC for $\mathrm{GSp}_4(F)$ of Gan-Takeda in characteristic 0 (- (2013)).

Work for classical groups

For the rest of the talk, $\mathbf{G} = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}$ (split), N = Rank of $\mathbf{G}^\vee$.

Arthur proved the LLC for classical groups over local fields of characteristic 0. We want to use the Deligne-Kazhdan theory to establish the correspondence over local function fields.

Proposition

Let $\mathrm{char}(F) = 0$ *and let* $G = \mathbf{G}(F)$ *with* $\mathbf{G}$ *as above. Let* (σ, V) *be an irreducible, admissible, tempered representation of* G *with Langlands parameter* ϕ_σ *(and functorial lift* σ^{GL}*). If* $\mathrm{depth}(\sigma) \leq m$*, then* $\mathrm{depth}(\phi_\sigma) \leq m+1$ *provided the residue characteristic of* F *is* > 2*.*

The Deligne-Kazhdan theory and Local Langlands Correspondence

Assume $\mathrm{char}(F') = p > 2$. Let $m \geq 1$ and F be any non-archimedean local field of characteristic 0 that is $(m+1)$-close to F'. Consider the diagram

$$\begin{array}{ccc}
\{(\sigma', V') \text{ of } G' \mid \text{depth}(\sigma) < m'\} & \xrightarrow{\text{LLC?}} & \{\phi' : \mathrm{WD}_{F'} \to {}^L G \mid \text{depth}(\phi') < m+1\} \\
\text{Kazhdan} \downarrow & & \downarrow \text{Deligne} \\
\{(\sigma, V) \text{ of } G \mid \text{depth}(\sigma) < m\} & \xrightarrow{\text{Arthur}} & \{\phi : \mathrm{WD}_{F} \to {}^L G \mid \text{depth}(\phi) < m+1\}
\end{array}$$

We want to assign the Langlands parameter of σ' using this diagram. To do this, we have to study a set of characterizing properties of the Langlands correspondence over close local fields.

Recharacterization of the Langlands correspondence

Arthur defined and characterized his correspondence using character identities. We obtain the following recharacterization (using the works of Arthur, CKPSS, and a strengthened version of the local converse theorem for GL_n).

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Proposition

Assume $\mathrm{char}(F) = 0$. *Let* $m \geq 1$ *and let* $l = Nm + 2N$. *Let* (σ, V) *be an irreducible, admissible, discrete series representation of* G *with depth* $\leq m$ *and with Langlands parameter* ϕ_σ. *It satisfies:*

(A) ϕ_σ *does not factor through any proper Levi subgroup of* $G^\vee$.

(B) If σ *is generic, then for each irr. ad. supercuspidal* τ *of* $\mathrm{GL}_r(F)$, $r \leq N - 1$, *and* $\mathrm{depth}(\tau) \leq 2l$, *we have*

$L(s, \sigma \times \tau) = L(s, \phi_\sigma \otimes \phi_\tau); \quad \epsilon(s, \sigma \times \tau, \psi) = \epsilon(s, \phi_\sigma \otimes \phi_\tau, \psi)$

Recharacterization of the Langlands correspondence

Proposition (contd.)

(C) If σ is non-generic, then for each irr. ad. discrete series τ of $\mathrm{GL}_r(F)$, $r \leq N-1$, and $\mathrm{depth}(\tau) \leq m+1$, we have

$\mu(s, \sigma \times \tau, \psi) =$

$\gamma(s, \phi_\sigma \otimes \phi_\tau, \psi)\gamma(-s, \phi_\sigma \otimes \phi_\tau^\vee, \bar{\psi})\gamma(2s, r_2 \circ \phi_\tau, \psi)\gamma(-2s, r_2 \circ \phi_\tau^\vee, \bar{\psi})$.

Here

$$r_2 = \begin{cases} Sym^2 & \textit{if } G = \mathrm{SO}_{2n+1}(F) \\ \wedge^2 & \textit{if } G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n}(F) \end{cases}.$$

Furthermore, ϕ_σ is uniquely characterized by these properties if $G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)$ and is determined up to $\mathrm{O}_{2n}(\mathbb{C})$ conjugacy if $G = \mathrm{SO}_{2n}(F)$.

Comparing local factors over close local fields

Deligne proved that Artin factors are compatible over close local fields.

We have

Proposition

Let σ be an irreducible representation of $G = \mathbf{G}(F)$ of depth $\leq m$ and let τ be an irreducible representation of $\mathrm{GL}_r(F)$ of depth $\leq m$. There exists $l \geq m$ such that for any field F' that is l-close to F we have

1. *If σ, τ are generic, then*

$$L(s, \sigma \times \tau) = L(s, \sigma' \times \tau'), \;\; \epsilon(s, \sigma \times \tau, \psi) = \epsilon(s, \sigma' \times \tau', \psi')$$

where $\sigma, \tau \leftrightarrow \sigma', \tau'$ via the Kazhdan isomorphism and $\psi \leftrightarrow \psi'$ as before.

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2. *$\mu(s, \sigma \times \tau, \psi) = \mu(s, \sigma' \times \tau', \psi')$ where $\sigma, \tau \leftrightarrow \sigma', \tau'$ and $\psi \leftrightarrow \psi'$.*

Attaching Langlands parameter in positive characteristic

Assume $\mathrm{char}(F') = p > 2$. We use this diagram

$$\begin{array}{ccc} \{(\sigma', V') \text{ of } G' \mid \text{depth}(\sigma') < m\} & \xrightarrow{\text{LLC}} & \{\phi' : \mathrm{WD}_{F'} \to {}^L G \mid \text{depth}(\phi') < m+1\} \\ \text{Kazhdan} \downarrow & & \downarrow \text{Deligne} \\ \{(\sigma, V) \text{ of } G \mid \text{depth}(\sigma) < m\} & \xrightarrow{\text{Arthur}} & \{\phi : \mathrm{WD}_{F} \to {}^L G \mid \text{depth}(\phi) < m+1\} \end{array}$$

to attach the Langlands parameter to σ', prove that it has the required properties (i.e. preserves L- and ϵ-factors of pairs and Plancherel measures), and is uniquely characterized by these properties.

Internal structure of the packet

Let $\mathbf{G} = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$. Let $\mathrm{char}(F) = 0$ and let $\phi : \mathrm{WD}_F \to \mathbf{G}^\vee$ be a discrete Langlands parameter. Arthur also contructed a bijection between $\Pi_\phi \to \hat{\mathcal{S}}_\phi$ (depending on the choice of a Whitaker datum).

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Let F' be a local field of sufficiently large characteristic and let $\phi' : \mathrm{WD}_{F'} \xrightarrow{\epsilon'} \mathbf{G}^\vee$ be a discrete Langlands parameter. We show that there is a bijection $\Pi_{\phi'} \to \hat{\mathcal{S}}_{\phi'}$ (depending on the choice of a Whittaker datum). Furthermore, for sufficiently close local fields F and F', if $\phi \leftrightarrow \phi'$ via the Deligne isomorphism, the following diagram is commutative.

$$\begin{array}{ccc} \Pi_{\phi'} & \xrightarrow{\epsilon'} & \hat{\mathcal{S}}_{\phi'} \\ \text{Kazhdan} \downarrow & & \downarrow \\ \Pi_\phi & \xrightarrow{\epsilon} & \hat{\mathcal{S}}_\phi \end{array}$$

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Ingredients in proof:

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(a) $\mathrm{depth}(\sigma) \leq \mathrm{depth}(\phi) \; \forall \, \sigma \in \Pi_\phi$.

(b) It follows from Arthur/Moeglin that the character $\epsilon(\sigma)$ can be described in terms of certain normalized intertwining operators, and these are compatible over sufficiently close local fields.

Thank you for your attention!