Fourier coefficients of automorphic forms for higher rank groups

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1 Möbius Randomness Law in Classical Analytic Number Theory

- One of the basic general problems in analytic number theory is to understand as much as possible the fluctuation of the Möbius function $\mu(n)$. Obviously the trivial bound is

$$\left|\sum_{n\le x}\mu(n)\right| \le x.$$

- The Prime Number Theorem is equivalent to

$$\sum_{n\le x}\mu(n) = o(x).$$

- The Riemann Hypothesis is equivalent to

$$\sum_{n\le x}\mu(n) = O(x^{\frac{1}{2}+\varepsilon}).$$

- From ancient Greek to modern times the distribution of primes constitutes the core of number theory.

$$\sum_p f(p), \quad \text{or} \quad \sum_n \Lambda(n) f(n).$$

- In analytic number theory, one of the approach is the explicit formula, which connects these sums with zeros of L-functions (e.g. the Riemann zeta-function).
- However, even assuming the Riemann Hypothesis, this approach does not always work (or it is not always applicable). Find a substitute for the Riemann hypothesis or go beyond the capability of RH.

- Even from the simple expression in elementary number theory,

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d,$$

it is not difficult to predict the behavior of sums over primes

$$\sum_{n} \Lambda(n) f(n)$$

by appealing to a randomness of the Möbius function.

- **Möbius Randomness Law.** The Möbius function $\mu(m)$ changes sign randomly so that for any "reasonable" sequence of complex numbers $\mathcal{A} = (a_m)$ the twisted sum

$$\sum_{m \le x} \mu(m) a_m$$

is relatively small due to the cancellation of its terms.

- Classical analytic number theory:"reasonable".
- Sarnak: a_n arises from a zero-entropy dynamical system.
- Green, Bourgain: low-complexity $AC^0(d)$ functions, computed using a bounded depth circuit.

- Already the most trivial case $a_n \equiv 1$ is equivalent to the prime number theorem

$$\sum_{n \le N} \mu(n) \ll N \exp(-c(\log N)^{\frac{3}{5}}(\log\log N)^{-\frac{1}{5}}).$$

- A stronger version of the Sarnak conjecture can lead to some nontrivial bounds involving sums over primes. See e.g. Green and Tao [2008] in the case of nilsequences.
- H. Davenport [1937]: for any real number α, $X \ge 2$ and $A > 0$,

$$\sum_{n \le N} \mu(n) e(n\alpha) \ll_A N(\log N)^{-A}.$$

- Fouvry and Ganguly: two sequences (x_n) and (y_n) of complex numbers are strong asymptotically orthogonal if

$$\sum_{1\le n\le N} x_n y_n = O_A\left((\log N)^{-A}\sum_{n\le N}|x_n y_n|\right)$$

for every $A \ge 0$, uniformly for $N \ge 2$.

- Davenport's result means that $\{\mu(n)\}$ and $\{e(n\alpha)\}$ are strong orthogonal.

- Davenport's result, together with the techniques of the incomplete Λ-functions, implies Vinogradov's ternary Goldbach problems, i.e.

$$G_3(N) = \sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3).$$

- The heuristic Möbius Randomness Law justifies (of course not rigorously) the stronger form of the binary Goldbach conjecture, i.e.

$$G_2(N) = \sum_{n_1+n_2=N} \Lambda(n_1)\Lambda(n_2).$$

- Following the proof of Davenport, one can show

$$\sum_{n\le N} \mu(n)e(n^k\alpha) \ll_A N(\log N)^{-A}.$$

- Hajela and Smith [1987] found that provided that there are no Siegel zeros,

$$\sum_{n\le N} \mu(n)e(n\alpha) \ll N\exp(-c\sqrt{\log N}).$$

- Under the generalized Riemann Hypothesis, Baker and Harman [1989] proved that

$$\sum_{n\le N} \mu(n)e(n\alpha) \ll N^{\frac{3}{4}+\varepsilon}.$$

2 Automorphic forms and $\mathrm{GL}(2)$ analogue of Davenport's result

- Automorphic L-functions, introduced by Langlands in 1960s, are functions of a complex variable associated to an automorphic form, generalizing the Dirichlet L-function and the Mellin transform of a modular form. They are important tools in the Langlands functoriality conjecture.
- The Generalized Riemann Hypothesis, the Generalized Ramanujan Conjecture, and the Generalized Lindelöf Hypothesis are three main conjectures in the theory of automorphic L-functions.

- Fourier coefficients of cusp form are mysterious objects and an interesting question is how its Fourier coefficients are distributed.
- Let f be a primitive holomorphic or Maass cusp form for the group $SL(2,\mathbb{Z})$, and $a_f(n)$ its nth normalized Fourier coefficient

$$f(z) = \sum_{n\geq 1} a_f(n) n^{(k-1)/2} e^{2\pi i n z};$$

$$f(z) = \rho_f(1)\sqrt{y} \sum_{n\geq 1} a_f(n) K_{i\kappa_f p}(2\pi|n|y) \mathrm{e}_f(nx).$$

- Let $F(z)$ be a normalized Maass form of type $\nu = (\nu_1, \nu_2)$ for $SL(3,\mathbb{Z})$ and an eigenfunction of all Hecke operators T_n. Then it has a Fourier expansion

$$\sum_{\gamma\in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sum_{n_1=1}^{\infty} \sum_{n_2\neq 0} \frac{A_F(n_1,n_2)}{|n_1 n_2|} \times W_J\left(\begin{pmatrix} n_1|n_2| & & \\ & n_1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\frac{n_2}{|n_2|}} \right),$$

where $A_F(n_1,n_2) \in \mathbb{C}$, $A_F(1,1) = 1$, and W_J denotes the Jacquet Whittaker function.

- **Hecke multiplicative relation**

$$a_f(n)a_f(m) = \sum_{d|(m,n)} a_f\left(\frac{mn}{d^2}\right).$$

- **The Ramanujan conjecture** (Deligne's theorem)

$$|a_f(n)| \leq d(n).$$

- **The Sato-Tate conjecture** (Theorem of Barnet-Lamb, Geraghty, Harris, and Taylor)

  If $f(t)$ $(t \in [0, \pi])$ is a continuous function,

$$\sum_{p\leq x} f(\theta_p) \sim \left(\frac{2}{\pi}\int_0^{\pi} f(\theta)\sin^2\theta d\theta\right)\frac{x}{\log x},$$

  where $a_f(p) = 2\cos\theta_p$, p runs through the prime numbers.

Moments of Fourier Coefficients

- In 1927, Hecke proved that

$$\sum_{n\le x} a_f(n) \ll_f x^{\frac{1}{2}},$$

- Subsequent improvements are due to Wilton, Walfisz (Summation formula, Bessel functions)

$$\sum_{n\le x} a_f(n) \ll_f x^{\frac{1}{3}(1+\theta)}, \quad \text{if} \quad |a_f(n)| \ll n^{\theta}.$$

- Then progress towards θ [by Kloosterman, Davenport, Salié, Weil] implied better results. In particular, Deligne's result: $\sum_{n\le x} a_f(n) \ll_f x^{\frac{1}{3}}$.

- Rankin, Jie Wu etc

$$\sum_{n\leq x} a_f(n) \ll_f x^{\frac{1}{3}}(\log x)^{-0.0652}.$$

- Rankin, Selberg (1940):

$$\sum_{n\leq x} a_f(n)^2 = c_2 x + O_f(x^{\frac{3}{5}}),$$

$$\sum_{n\leq x} a_f(n)a_g(n) \ll x^{\frac{3}{5}}.$$

- Remark: $\frac{1+\theta}{3}$ and $\frac{3}{5}$ have never been improved.
- Moreno and Shahidi (1983):

$$\sum_{n\leq x} a_f(n)^4 \sim c_4 x \log x.$$

Fourier coefficients over arithmetic progressions

- R.A. Smith [1993]: for $q \le x^{\frac{2}{3}-\varepsilon}$

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q, (a,q)=1}} a_f(n) \ll x/q.$$

- This problem is similar to the distribution of divisor function over arithmetic progressions studied by Selberg, Hooley, ...
- Here the main point is NOT how sharp the error term, but rather the range of uniformity in the modulus q.
- **Shifted convolution sums, Sign Changes, Estimates over sparse sequences....**

Strong orthogonality between $a_f(n)$ and $e(n\alpha)$

- **Square Root Cancellation:** For any real α

$$\sum_{n\leq x} a_f(n)e(n\alpha) \ll_f x^{\frac{1}{2}} \log x,$$

where the implied constant depends only on f (not on α).

- N.J.E. Pitt [2001]

$$\sum_{n\leq x} a_f(n)e(\alpha n^2 + \beta n) \ll_f x^{1-\delta},$$

where $\delta < 1/8$, and the implied constant depends only on f (not on α and β).

Strong orthogonality between $A_F(n,1)$ and $e(n\alpha)$

- S. Miller [2006]: for any $\alpha \in \mathbb{R}$ and any $\epsilon > 0$ one has

$$\sum_{n \le N} A_F(n,1)e(n\alpha) \ll N^{\frac{3}{4}+\epsilon},$$

  where the implied constant depends only on the form F and ϵ.

- These sums over integers signify an enormous number of cancellations.

- We show that a better life exists in the world of automorphic forms than in the zoo of degree one L-functions by applying these cancellations.

- Fouvry and Ganguly [2014] ask whether strong orthogonality for

$$\sum_{n\le X} a_f(n)e(n\alpha)$$

is manifested if, instead of sums over integers, when we consider the corresponding sum over primes, or $\mathrm{GL}(2)$ analogue of Davenport's theorem

$$\sum_{n\le X} \Lambda(n)a_f(n)e(n\alpha),$$

$$\sum_{n\le X} \mu(n)a_f(n)e(n\alpha).$$

- A. Perelli [1984] studied the exponential sums connected with Ramanujan's τ-function over primes,

$$\sum_{n\le N}\Lambda(n)\tau(n)e(n\alpha),$$

$$\ll N^{\frac{11}{2}}(N\rho^{-\frac{1}{2}}+N^{\frac{1}{2}}\rho^{\frac{1}{2}}+N^{\frac{5}{6}})\log^c N,$$ where

$$\left|\alpha-\frac{a}{\rho}\right|\le\frac{1}{\rho^2},\quad (a,\rho)=1,\quad \rho\ge 1$$

for some integers a, ρ.

- As a corollary, he derived a ”Vinogradov three primes theorem” for the coefficients $\tau(n)$

$$\sum_{n_1+n_2+n_3=N} \Lambda(n_1)\tau(n_1)\Lambda(n_2)\tau(n_2)\Lambda(n_3)\tau(n_3) \ll \frac{N^{\frac{37}{2}}}{(\log N)^A}.$$

- A. Perelli also remarked that similar results hold in general for the Fourier coefficients of the cusp forms for the full modular group which are eigenfunctions for the Hecke operators.

- **Theorem 0.** Fouvry and Ganguly [2014]: There exists an effective absolute $c_0 > 0$ such that for cusp form f for the group $SL(2,\mathbb{Z})$, there exists an effective constant $C_0(f) > 0$ such that

$$\Big|\sum_{n\le X}\Lambda(n)a_f(n)e(n\alpha)\Big| \le C_0(f)X\exp(-c_0\sqrt{\log x}),$$

and

$$\Big|\sum_{n\le X}\mu(n)a_f(n)e(n\alpha)\Big| \le C_0(f)X\exp(-c_0\sqrt{\log x}),$$

for every $\alpha\in\mathbb{R}$ and $X\ge 2$.

- The significant improvement is the removal of the dependence on α.
- It can be interpreted as the PNT for Fourier coefficients of cusp forms with additive twists.
- It is closely related to the general programme advanced by Sarnak, *Three lectures on the Möbius function randomness and dynamics*.
- The sequences $\{a_f(n)e(n\alpha)\}$ does not fit immediately in Sarnak's context, because they are unbounded (the unknown Ramanjuan Conjecture). A suitable reformulation would apply to these sequences is expected.

- Davenport's (weaker) bound is a reflection of the exceptional zero, which is not yet ruled out in the $\mathrm{GL}(1)$ case.
- Hoffstein and Ramakrishnan [1995] have shown that there are no exceptional zeros for L-functions on $\mathrm{GL}(2)$ that are not associated to grossen-characters of quadratic fields.

- Could we establish strong orthogonality for sums

$$\sum_{n\le N} \Lambda(n)a_f(n)e(\alpha n^2+\beta n),$$

$$\sum_{n\le N} \mu(n)a_f(n)e(\alpha n^2+\beta n),$$

$$\sum_{n\le N} \Lambda(n)A_F(n,1)e(n\alpha),$$

and

$$\sum_{n\le N} \mu(n)A_F(n,1)e(n\alpha)?$$

3 Our Results

Theorem 1. (JNT, 2015)
Let $N \geq 2$ and f be a primitive holomorphic or Maass cusp form for the group $SL(2,\mathbb{Z})$. Let $a_f(n)$ denote the nth normalized Fourier coefficient of the form f. Then there exists an effective absolute $c > 0$ such that, for any $\alpha, \beta \in \mathbb{R}$, there exists an effective constant $C(f) > 0$ such that

$$\Big|\sum_{n\leq N} \Lambda(n)a_f(n)e(\alpha n^2+\beta n)\Big|$$

$$\leq C(f)N\exp(-c\sqrt{\log N}).$$

Theorem 2. (Quart. J. Math., 2016)
Let $N \geq 2$ and $L(s,F)$ be the L-function associated to a Hecke-Maass form F for $SL(3,\mathbb{Z})$. Let $A_F(n,1)$ denote the nth coefficient of the Dirichlet series for $L(s,F)$. Then for any $\alpha \in \mathbb{R}$ there exists an effective constant $c > 0$, such that

$$\sum_{n\leq N} \Lambda(n)A_F(n,1)e(n\alpha) \ll N\exp(-c\sqrt{\log N}),$$

where the implied constant depends only on the form F.

Corollary. ”Vinogradov three primes theorem” for coefficients of the Rankin-Selberg L-functions

$$\sum_{N=p_1+p_2+p_3} a_f(p_1)^2 a_f(p_2)^2 a_f(p_3)^2$$

$$= \sum_{N=p_1+p_2+p_3} 1 + O(N^2 \exp(-c\sqrt{\log N})).$$

Idea.

- *Major arcs:*

  Suitable version of PNT for automorphic L-functions. (No poles and exceptional zero)

- *Minor arcs:*
  - Vinogradov's method for exponential sum via Vaughan's identity;
  - Strong orthogonality between $a_f(n)$ and $e(\alpha n^2 + \beta n)$ (or $A_F(n,1)$ and $e(n\alpha)$). (to deal with Type I' sums)

4 Proof of Theorem 1.

By Dirichlet's theorem on Diophantine Approximation, there exist two integers l and q such that

$$\left|\alpha - \frac{l}{q}\right| \le \frac{1}{qQ}, \quad 1 \le q \le Q, \quad (l, q) = 1.$$

- *Major arcs:*

$$1 \le q \le \exp(C_0\sqrt{\log N}).$$

- *Minor arcs:*

$$\exp(C_0\sqrt{\log N}) < q \le N^2 \exp(-C_0\sqrt{\log N}).$$

Major arcs:

- By partial summation, it suffices to consider

$$\sum_{n\le x}\Lambda(n)a_f(n)e\left(\frac{ln^2}{q}+\beta n\right)$$

- It equals

$$\frac{1}{q}\sum_{b \bmod q}\sum_{a \bmod q} e\left(\frac{la^2-ba}{q}\right)$$
$$\times\sum_{n\le x}\Lambda(n)a_f(n)e\left(n\left(\frac{b}{q}+\beta\right)\right).$$

- The estimate for major arcs follows from Theorem 0 proved by Fouvry and Ganguly.

Minor arcs: We introduce some notations.

- The Hecke L-function associated to the form f, for $\Re s > 1$,

$$L(f,s) = \sum_{n=1}^{\infty} a_f(n) n^{-s} = \prod_p (1 - a_f(p) p^{-s} + p^{-2s})^{-1}.$$

-

$$L(f,s)^{-1} = \sum_{n=1}^{\infty} \mu(n,f)\, n^{-s},$$

-

$$-\frac{L'(f,s)}{L(f,s)} = \sum_{n=1}^{\infty} \Lambda(n,f)\, n^{-s},$$

where

$$\mu(n,f)=\sum_{\substack{s,t\geq 1\\ st^2=n}}\mu(s)\mu^2(st)a_f(s),$$

$$\Lambda(n,f)=\begin{cases} a_f(p)\log p, & \text{if } n=p,\\ (a_f(p^k)-a_f(p^{k-2}))\log p, & \text{if } n=p^k,\\ 0, & \text{otherwise.}\end{cases}$$

Minor arcs:

- Analogue of Vaughan's identity

$$
\begin{aligned}
&-\frac{L'(f,s)}{L(f,s)} \\
&= F(s) - L'(f,s)G(s) - L(f,s)F(s)G(s) \\
&\quad + \Big(-\frac{L'(f,s)}{L(f,s)} - F(s) \Big) \times (1 - L(f,s)G(s)).
\end{aligned}
$$

- Here

$$F(s) = \sum_{n \le X} \Lambda(n,f) n^{-s},$$

and

$$G(s) = \sum_{n \le Y} \mu(n,f) n^{-s}.$$

Minor arcs:

- By Kim-Sarnak's bound, it suffices to consider

$$\sum_{n\leq N} \Lambda(n,f)e(\alpha n^2+\beta n).$$

- After applying the analogue of Vaughan's identity, we have

$$\sum_{n\leq N} \Lambda(n,f)e(\alpha n^2+\beta n)) = S_1+S_2-S_3+S_4+O((XY)^{1+\varepsilon}).$$

Minor arcs: Here

$$S_1 = \sum_{n\le X} \Lambda(n,f)e(\alpha n^2+\beta n),$$

$$S_2 = \sum_{m\le Y} \mu(m,f) \sum_{mn\le N} a_f(n)(\log n) \times e(\alpha(mn)^2+\beta mn),$$

$$S_3 = \sum_{m\le XY} a(m,f) \sum_{mn\le N} a_f(n)e(\alpha(mn)^2+\beta mn),$$

$$S_4 = \sum_{X<m<N/Y} \sum_{Y<n\le N/m} b(n,f)\Lambda(m,f) \times e(\alpha(mn)^2+\beta mn),$$

Minor arcs, Type I' Sums:

We call S_2 and S_3 Type I' sums (which means that the n-indexed coefficient is no longer 1, but we have some control over the inner exponential sum)

$$\sum_{m \le Y} \mu(m,f) \sum_{mn \le N} a_f(n)(\log n) e(\alpha(mn)^2 + \beta mn),$$

$$\sum_{m \le XY} a(m,f) \sum_{mn \le N} a_f(n) e(\alpha(mn)^2 + \beta mn),$$

where

$$a(m,f) = \sum_{\substack{m_1 m_2 = m \\ m_1 \le X, m_2 \le Y}} \Lambda(m_1,f)\mu(m_2,f).$$

Minor arcs, Type I' Sums:

- Our observation is that Pitt's result (or Liu-Ren's), which states that uniformly for α and β,

$$\sum_{n\leq x} a_f(n)e(\alpha n^2+\beta n) \ll_f x^{\frac{7}{8}+\varepsilon},$$

can be directly applied to treat the inner sum of S_2 and S_3.

- Hence after choosing

$$X = Y^4 = \exp(C_0\sqrt{\log N}),$$

we have

$$S_2,\ S_3 \ll N^{\frac{7}{8}+\varepsilon}.$$

Minor arcs, Type II Sums:

Recall that

$$S_4 = \sum_{X<m<N/Y} \sum_{Y<n\le N/m} b(n,f)\Lambda(m,f) \times e(\alpha(mn)^2 + \beta mn),$$

where

$$b(n,f) = \sum_{\substack{n_1 n_2 = n \\ n_2 \le Y}} a_f(n_1)\mu(n_2,f).$$

We shall estimate it by applying classical method in analytic number theory.

One Lemma:

Let $M, N, x \geq 2$. Let $\{a(m) : 1 \leq m \leq M\}$ and $\{b(n) : 1 \leq n \leq N\}$ be any two complex-valued sequences. Suppose that the bounds

$$a) \sum_{m \sim M} |a(m)|^2 \ll M \log^{c_0} M,$$

$$b) \sum_{n \sim N} |b(n)|^2 \ll N \log^{c_1} N$$

hold for some $c_0, c_1 > 0$. Then there exists a positive constant c, depending on c_0, c_1, such that, *One Lemma, Continued:*

for any $\alpha \in \mathbb{R}$ satisfying

$$\left|\alpha - \frac{l}{q}\right| \le \frac{1}{qQ}, \quad 1 \le q \le Q, \quad (l, q) = 1.$$

and any $\beta \in \mathbb{R}$, one has

$$\sum_{\substack{m\sim M, n\sim N \\ mn\sim x}}\sum a(m)b(n)e(\alpha(mn)^2 + \beta mn)$$

$$\ll x\Big(\frac{1}{\rho} + \frac{1}{M} + \frac{1}{N^4} + \frac{1}{\rho}x^2\Big)^{\frac{1}{8}} \log^c x.$$

Minor arcs, Type II Sums:

- In the absence of the Ramanujan conjecture, it remains to show

$$\sum_{n\sim N} |\Lambda(n,f)|^2 \ll N \log^{c_0} N,$$

$$\sum_{n\sim N} |b(n,f)|^2 \ll N \log^{c_1} N.$$

- These can be handled by higher power moments of $a_f(n)$ and the divisor function

$$\sum_{n\le N} a_f(n)^8 \ll N(\log N)^{13},$$

$$\sum_{n\le N} d(n)^k \ll N(\log N)^{2^A-1}.$$

Minor arcs, Type II Sums:

- Now it is standard that S_4 can be written as a linear combination of $O(\log^2 N)$, each of which is of the form

$$\sum\sum_{\substack{m\sim M, n\sim N' \\ mn\sim N''}} b(n,f)\Lambda(m,f)e(\alpha(mn)^2+\beta mn).$$

- After applying Lemma, we have

$$S_4 \ll N\exp(-c\sqrt{\log N}).$$

5 Proof of Theorem 2.

Additive twists of Fourier coefficients of $\mathrm{GL}(3)$ Maass forms have appealed to a number of researchers.

- S. Miller [2006], Cancellation in additively twisted sums on $\mathrm{GL}(n)$.
- X. Li and M. Young [2012], Additive twists of Fourier coefficients of symmetric-square lifts.
- D. Godber [2013], Additive twists of Fourier coefficients of modular forms.
- X. Li [2014], Additive twists of Fourier coefficients of $\mathrm{GL}(3)$ Maass forms.

Our Theorem 2 is one result on additive twists of Fourier coefficients of $\mathrm{GL}(3)$ Maass formst at prime arguments.

Fourier coefficients of GL(3) Maass forms:

- $A_F(n_1, n_2)$ satisfies the following multiplicativity relations: if $(n_1n_2, n_1'n_2') = 1$,

$$A_F(n_1n_1', n_2n_2') = A_F(n_1, n_2)A_F(n_1', n_2'),$$

$$A_F(n,1)A_F(n_1,n_2) = \sum_{\substack{d_0d_1d_2=n \\ d_1|n_1, d_2|n_2}} A_F\left(\frac{n_1d_0}{d_1}, \frac{n_2d_1}{d_2}\right),$$

$$A_F(1,n)A_F(n_1,n_2) = \sum_{\substack{d_0d_1d_2=n \\ d_1|n_1, d_2|n_2}} A_F\left(\frac{n_1d_2}{d_1}, \frac{n_2d_0}{d_2}\right),$$

$$A_F(n_1,1)A_F(1,n_2) = \sum_{d|(n_1,n_2)} A_F\left(\frac{n_1}{d}, \frac{n_2}{d}\right),$$

$$A_F(n_1,n_2) = \overline{A_F(n_2,n_1)}.$$

The Generalized Ramanujan Conjecture:

- The generalized Ramanujan conjecture asserts that

$$|A_F(n,1)| \ll n^{\varepsilon}.$$

- The current best estimate is due to Kim and Sarnak

$$|A_F(n,1)| \le n^{\frac{5}{14}+\varepsilon}.$$

Major arcs, $1 \le q \le \exp(C_0\sqrt{\log N})$*:*

- Let $N \ge 2$ and F be a Hecke-Maass form for $SL(3,\mathbb{Z})$. Let χ be any Dirichlet character modulo q. Suppose that α belongs to the major arcs. Then there exists a constant $c > 0$ such that

$$\sum_{n \le N} \Lambda(n) A_F(n,1)\chi(n) \ll q^{\frac{3}{2}} N \exp(-c\sqrt{\log N}),$$

  where the implied constant only depends on the form F.

- The proof is based on analytic properties (zero-free region, etc.) of $L(s, F \times \chi)$.

Notations:

- Recall that the L-function $L(s, F)$ is defined by

$$L(s, F) = \sum_{n=1}^{\infty} \frac{A_F(n, 1)}{n^s}.$$

- We also denote

$$L(s, F)^{-1} = \sum_{n=1}^{\infty} \mu(n, F)\, n^{-s},$$

$$-\frac{L'(s, F)}{L(s, F)} = \sum_{n=1}^{\infty} \Lambda(n, F)\, n^{-s}.$$

$$\mu_F(n) = \begin{cases} 0, & \text{if } p^4 | n, \\ \prod_{l=1}^{3} \prod_{p^\ell \| n} (-1)^\ell \sum_{1 \le j_1 < \cdots < j_\ell \le 3} \\ \times \alpha_F(p, j_1) \cdots \alpha_F(p, j_\ell), & \text{otherwise.} \end{cases}$$

$$\Lambda_F(n) = \begin{cases} \log p \sum_{j=1}^{3} \alpha_F(p, j)^k, & \text{if } n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

Minor arcs: Namely,

$$\exp(C_0\sqrt{\log N}) < q \le N\exp(-C_0\sqrt{\log N}).$$

- Analogue of Vaughan's identity on $\mathrm{GL}(3)$ Maass forms gives

$$S_2 = \sum_{m\le Y} \mu(m,F) \sum_{mn\le N} A_F(n,1)(\log n)e(mn\alpha)$$

$$S_3 = \sum_{m\le XY} a_F(m) \sum_{mn\le N} A_F(n,1)e(mn\alpha),$$

$$S_4 = \sum_{X<m<N/Y} \sum_{Y<n\le N/m} b_F(n)\Lambda(m,F)e(mn\alpha).$$

Minor arcs:

- To go further, we need evaluate sums involving arithmetic functions $d(n)|\mu(n,F)|^2$, $d(n)|A_F(n,1)|^2$, and $d(n)|\Lambda(n,F)|^2$. We have

$$\sum_{n\leq x} d(n)|\mu(n,F)|^2 \ll x\log x,$$

$$\sum_{n\leq x} d(n)|A_F(n,1)|^2 \ll x\log x,$$

$$\sum_{n\leq x} d(n)|\Lambda(n,F)|^2 \ll x\log^3 x.$$

Minor arcs:

- $L(s, F)$ can be written as an Euler product

$$L(s,F) = \prod_p \prod_{1 \le j \le 3} \left(1 - \frac{\alpha_F(p,j)}{p^s}\right)^{-1}.$$

- Here the local parameters satisfy

$$\alpha_F(p,1)\alpha_F(p,2)\alpha_F(p,3) = 1,$$
$$A_F(p,1) = \alpha_F(p,1) + \alpha_F(p,2) + \alpha_F(p,3),$$
$$A_F(1,p) = \alpha_F(p,1)\alpha_F(p,2) + \alpha_F(p,1)\alpha_F(p,3)$$
$$+\alpha_F(p,2)\alpha_F(p,3),$$

which are implied by the multiplicative properties of $A_F(n_1, n_2)$.

Minor arcs:

Then the average behavior of $|\mu(n,F)|^2$, $|A_F(n,1)|^2$ and $|\Lambda(n,F)|^2$ can be controlled by applying

- the Selberg-Delange method,
- the multiplicative properties of $A_F(n_1,n_2)$,
- the Rankin-Selberg L-function $L(s,F\times F)$.

We shall omit the detailed proof of Theorem 2. As for the corollary, the proof follows directly from Theorem 2, the basic identity of the circle method, and the identity

$$a_f(p)^2 = 1 + a_f(p^2).$$

Thank You !