Waldspurger formula over function fields

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(Joint work with Chih-Yun Chuang)

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Outline

1. Statement of the main theorem.
2. Central critical values of Rankin-Selberg L-functions.
3. Gross points on definite Shimura curves and the theta element.
4. Proof of the main theorem (sketch).

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Basic settings

- k: a global function field with odd characteristic.
- $\mathbb{A}$: the adele ring of k.
- $\mathcal{D}$: a quaternion algebra over k.
- $\mathcal{D}_{\mathbb{A}} := \mathcal{D} \otimes_k \mathbb{A}$.
- K: a quadratic field extension over k, together with an embedding $\iota : K \hookrightarrow \mathcal{D}$.
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Toric period integrals

Let Π be an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, with a unitary central character ω_Π. Assume that Π corresponds to an automorphic representation $\Pi^{\mathcal{D}}$ of $\mathcal{D}_{\mathbb{A}}^\times$ via the Jacquet-Langlands correspondence. Given a unitary Hecke character $\chi : K^\times \backslash K_{\mathbb{A}}^\times \to \mathbb{C}^\times$, suppose $\omega_\Pi \cdot \chi\big|_{\mathbb{A}^\times} \equiv 1$.

For each $f \in \Pi^{\mathcal{D}}$, define

$$\mathcal{P}_\chi(f) := \int_{K^\times \mathbb{A}^\times \backslash K_{\mathbb{A}}^\times} f(\iota(a)) \cdot \chi(a)\, d^\times a.$$

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Toric period integrals

Here the chosen measure $d^\times a$ satisfies

$$\mathrm{vol}(K^\times \mathbb{A}^\times \backslash K_{\mathbb{A}}^\times, d^\times a) = 2L(1, \xi_{K/k}),$$

where $\xi_{K/k}$ is the quadratic character associated to K/k. Let $\widetilde{\Pi}^{\mathcal{D}}$ be the contragredient representation of $\Pi^{\mathcal{D}}$. Define $\mathbb{P}_\chi : \Pi^{\mathcal{D}} \otimes \widetilde{\Pi}^{\mathcal{D}} \to \mathbb{C}$ by

$$\mathbb{P}_\chi(f \otimes \tilde{f}) := \mathcal{P}_\chi(f) \cdot \mathcal{P}_{\chi^{-1}}(\tilde{f}).$$

Toric period integrals

Writing $\Pi^{\mathcal{D}}$ (resp. $\widetilde{\Pi}^{\mathcal{D}}$) as $\otimes_v \Pi_v^{\mathcal{D}}$ (resp. $\otimes_v \widetilde{\Pi}_v^{\mathcal{D}}$), we choose a local pairing $\langle \cdot, \cdot \rangle_v : \Pi_v^{\mathcal{D}} \times \widetilde{\Pi}_v^{\mathcal{D}} \to \mathbb{C}$ for each place v of k so that

$$\langle \cdot, \cdot \rangle_{\mathrm{Pet}} = \frac{2L(1, \Pi, \mathrm{Ad})}{\zeta_k(2)} \cdot \prod_v \langle \cdot, \cdot \rangle_v .$$

Here $\langle \cdot, \cdot \rangle_{\mathrm{Pet}} : \Pi^{\mathcal{D}} \times \widetilde{\Pi}^{\mathcal{D}} \to \mathbb{C}$ is the pairing induced from the Petersson inner product (with respect to the Tamagawa measure on $\mathcal{D}_{\mathbb{A}}^{\times} / \mathbb{A}^{\times}$).

Toric period integrals

For each place v of k, define $\mathscr{P}_{\chi,v} : \Pi_v^{\mathcal{D}} \otimes \widetilde{\Pi}_v^{\mathcal{D}} \to \mathbb{C}$ by

$$\mathscr{P}_{\chi,v}(f_v \otimes \tilde{f}_v) \quad := \quad \frac{L_v(1, \xi_{K/k}) \cdot L_v(1, \Pi, \mathrm{Ad})}{L_v(\frac{1}{2}, \Pi \times \chi) \cdot \zeta_{k,v}(2)} \cdot \int_{K_v^\times / k_v^\times} \langle \Pi^{\mathcal{D}}(\iota(a_v)) f_v, \tilde{f}_v \rangle_v \cdot \chi_v(a_v)\, d^\times a_v.$$

It is observed that when v is "good," one has $\mathscr{P}_v(f_v \otimes \tilde{f}_v) = 1$. We may define $\mathscr{P}_\chi : \Pi^{\mathcal{D}} \otimes \widetilde{\Pi}^{\mathcal{D}} \to \mathbb{C}$ by

$$\mathscr{P}_\chi := \otimes_v \mathscr{P}_{\chi,v}.$$

Waldspurger formula over function fields

Theorem 1 (Chuang-W.)

Let Π be an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, with a unitary central character ω_Π. Given a unitary Hecke character $\chi : K^\times \backslash K_\mathbb{A}^\times \to \mathbb{C}^\times$, suppose $\omega_\Pi \cdot \chi\big|_{\mathbb{A}^\times} \equiv 1$. Then

$$\mathbb{P}_\chi = L(\frac{1}{2}, \Pi \times \chi) \cdot \mathscr{P}_\chi.$$

It is known ([Tunnell] and [Waldspurger], also [Gross-Prasad]) that for each place v of k, $\mathscr{P}_{\chi,v} \neq 0$ if and only if

$$\epsilon_v(\Pi_v \times \chi_v) = \chi_v(-1)\xi_{K/k,v}(-1)\epsilon_v(\mathcal{D}). \qquad (*)$$

Here $\epsilon_v(\Pi_v \times \chi_v)$ is the local root number of $L_v(s, \Pi_v \times \chi_v)$, and $\epsilon_v(\mathcal{D})$ is the Hasse invariant of $\mathcal{D}$.

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Non-vanishing criterion of $L(1/2, \Pi \times \chi)$

Corollary

Suppose $\prod_v \epsilon_v(\Pi_v \times \chi_v) = 1$. Let $\mathcal{D}$ be the quaternion algebra over k satisfying $(*)$ for every place v of k. Take an embedding $\iota : K \hookrightarrow \mathcal{D}$. Then $L(1/2, \Pi \times \chi)$ is non-vanishing if and only if there exists $f \in \Pi^{\mathcal{D}}$ such that

$$\mathcal{P}_\chi(f) = \int_{K^\times \mathbb{A}^\times \backslash K_{\mathbb{A}}^\times} f(\iota(a))\chi(a)\, d^\times a \neq 0.$$

Gross-type formula of $L(1/2, \Pi \times \chi)$

From now on, for simplicity we assume that $k = \mathbb{F}_q(T)$ with q odd, the central character of Π is **trivial**, and the conductor of Π is $\mathfrak{n}\infty$, where $\mathfrak{n}$ is a square-free ideal of $A = \mathbb{F}_q[T]$.

Let $K = k(\sqrt{D})$, where $D \in A$ is square-free with non-zero even degree and the leading coefficients of D is not a square in $\mathbb{F}_q$ (then ∞ is inert in K). Let $O_K := A[\sqrt{D}]$ and $O_\mathfrak{c} := A + \mathfrak{c} \cdot A[\sqrt{D}]$ for each ideal $\mathfrak{c}$ of A. Every character χ of $\mathrm{Pic}(O_\mathfrak{c})$ can be viewed as a Hecke character on $K^\times \backslash K_\mathbb{A}^\times$ via the isomorphism

$$\mathrm{Pic}(O_\mathfrak{c}) \cong K^\times \backslash K_\mathbb{A}^\times / \widehat{O}_\mathfrak{c}^\times \cdot K_\infty^\times.$$

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$$\mathrm{Pic}(O_{\mathfrak{c}}) \cong K^\times \backslash K_{\mathbb{A}}^\times / \widehat{O}_{\mathfrak{c}}^\times \cdot K_\infty^\times.$$

For simplicity, assume $(\mathfrak{n}, \mathfrak{c}D) = 1$. Write $\mathfrak{n} = \mathfrak{n}^+ \cdot \mathfrak{n}^-$, where

$$\mathfrak{n}^- = \prod_{\substack{\mathfrak{p} \mid \mathfrak{n}, \\ \mathfrak{p} \text{ inert in } K}} \mathfrak{p} \quad \text{and} \quad \mathfrak{n}^+ := \frac{\mathfrak{n}}{\mathfrak{n}^-}.$$

Assume that the number of prime factors of $\mathfrak{n}^-$ is odd. Let $\mathcal{D}$ be the quaternion algebra over k which is ramified precisely at ∞ and primes $\mathfrak{p}$ dividing $\mathfrak{n}^-$. Choose an Eichler A-order $R_{\mathfrak{n}^+,\mathfrak{n}^-} \subset \mathcal{D}$ of type $(\mathfrak{n}^+, \mathfrak{n}^-)$, together with an optimal embedding $\iota : O_{\mathfrak{c}} \hookrightarrow R_{\mathfrak{n}^+,\mathfrak{n}^-}$.

Gross-type formula of $L(1/2, \Pi \times \chi)$

Let

$$\phi_\Pi : \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}_0(\mathfrak{n}\infty) \to \mathbb{C}$$

and

$$\phi_{\Pi^\mathcal{D}} : \mathcal{D}^\times \backslash \mathcal{D}_\mathbb{A}^\times/(\widehat{R}^\times_{\mathfrak{n}^+,\mathfrak{n}^-} O^\times_{\mathcal{D}_\infty}) \to \mathbb{C}$$

be newforms associated to Π and $\Pi^\mathcal{D}$, respectively. Assume that ϕ_Π is **normalized**. Then the central critical value $L(1/2, \Pi \times \chi)$ can be expressed as follows:

Gross-type formula of $L(1/2, \Pi \times \chi)$

Theorem 2 (Chuang-W.)

Suppose χ is primitive of conductor $\mathfrak{c}$. Then

$$L(\frac{1}{2}, \Pi \times \chi) = \frac{\|\phi_\Pi\|_{\text{Pet}}}{\|\mathfrak{c}\| \cdot |D|^{1/2}} \cdot \frac{\left|\sum_{[\mathfrak{A}]\in \text{Pic}(O_\mathfrak{c})} \phi_{\Pi^{\mathcal{D}}}\left(\iota(\mathfrak{A})\right)\chi([\mathfrak{A}])\right|^2}{\|\phi_{\Pi^{\mathcal{D}}}\|_{\text{Pet}}}.$$

Here $\|\mathfrak{c}\| := \#(A/\mathfrak{c})$ and $|D| := q^{\deg D}$.

Remark:
1. In fact, we are able to derive such a formula for $L(\frac{1}{2}, \Pi \times \chi)$ only under the assumptions that:
(i) the central character of Π is unramified everywhere;
(ii) $\mathfrak{n}$ is square-free.

2. This formula was known in the following special cases:
(i) k is rational, $\mathfrak{n}$ is prime, D is irreducible, and $\mathfrak{c} = 1$ (Papikian 2005);
(ii) k is rational, $\mathfrak{n}$ is square-free, D is irreducible, and $\mathfrak{c} = 1$ (W.-Yu 2011);
(iii) $\mathfrak{n}$ is square-free and $\mathfrak{c} = 1$ (Chuang-W.-Yu).

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Theta element

Let $G_{\mathfrak{c}} := \mathrm{Pic}(O_{\mathfrak{c}})$. There exists a unique element

$$\mathcal{L} = \mathcal{L}^{\mathfrak{c}}_{\Pi,K} = \sum_{\sigma \in G_{\mathfrak{c}}} c_\sigma \cdot \sigma \in \mathbb{C}[G_{\mathfrak{c}}]$$

so that for each character $\chi : G_{\mathfrak{c}} \to \mathbb{C}^\times$, we have

$$\chi(\mathcal{L}^{\mathfrak{c}}_{\Pi,K}) = \frac{L^{\mathfrak{c}}(1/2, \Pi \times \chi)}{\|\phi_\Pi\|_{\mathrm{Pet}}}.$$

From the above Gross-type formula, we may describe $\mathcal{L}^{\mathfrak{c}}_{\Pi,K}$ explicitly by using the “Gross points” on definite Shimura curves.

Definite Shimura curve

Let Y be the genus 0 curve over k so that the points of Y over any k-algebra M are

$$Y(M) = \{x \in \mathcal{D} \otimes_k M : \mathrm{Tr}(x) = \mathrm{Nr}(x) = 0\}/M^\times,$$

where Tr and Nr are respectively the reduce trace and the reduced norm on $\mathcal{D}$. The group $\mathcal{D}^\times$ acts on Y (from the left) by conjugation.

Definition

The definite Shimura curve $X = X_{\mathfrak{n}^+,\mathfrak{n}^-}$ of type $(\mathfrak{n}^+, \mathfrak{n}^-)$ is

$$X := \mathcal{D}^\times \backslash \left(Y \times \mathcal{D}_{\mathbb{A}}^{\infty,\times} / \widehat{R}_{\mathfrak{n}^+,\mathfrak{n}^-} \right).$$

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Let $I_1, ..., I_n$ be representatives of right ideal classes of $R = R_{\mathfrak{n}^+,\mathfrak{n}^-}$. Then

$$X = \coprod_{i=1}^{n} R_i^\times \backslash Y$$

where for each i, R_i is the left order of I_i. Hence X is a finite disjoint union of genus 0 curves, and the components correspond canonically to left ideal classes of R. Therefore we may identify Pic(X) with the free abelian group generated by the double cosets in

$$\mathcal{D}^\times \backslash \mathcal{D}_{\mathbb{A}}^{\infty,\times} / \widehat{R}_{\mathfrak{n}^+,\mathfrak{n}^-}.$$

Moreover, $\phi_{\Pi^\mathcal{D}}$ can be viewed as an element in Hom(Pic(X), $\mathbb{C}$) by extending additively.

Gross points

There is a canonical identification of $Y(K)$ with $\mathrm{Hom}(K, \mathcal{D})$. We call a point $x = [y, g] \in X$ a Gross point of conductor $\mathfrak{c}$ over K if

$$x \in \mathrm{Image}\left[Y(K) \times \mathcal{D}_{\mathbb{A}}^{\infty,\times} / \widehat{R}^{\times} \to X(K) \right]$$

satisfying that

$$\iota_y(K) \cap g^{-1} \widehat{R} g = \iota_y(O_{\mathfrak{c}}).$$

Here ι_y is the embedding of K into $\mathcal{D}$ corresponding to y. We have a natural free action of $G_{\mathfrak{c}} = \mathrm{Pic}(O_{\mathfrak{c}})$ on the set of Gross points of conductor $\mathfrak{c}$ over K.

Gross points

We assumed that $(\mathfrak{n}, \mathfrak{c}D) = 1$. Fix a Gross point $x = x_{(\mathfrak{c})} \in X$ of conductor $\mathfrak{c}$ over K. For each divisor $\mathfrak{c}'$ of $\mathfrak{c}$, we take the unique Gross point $x_{(\mathfrak{c}')} \in X$ of conductor $\mathfrak{c}'$ over K which occurs in $T_{\mathfrak{c}/\mathfrak{c}'} x_{(\mathfrak{c})}$. It is observed that for $\mathfrak{p} \mid \mathfrak{c}'$,

$$N_{\mathfrak{c}'/(\mathfrak{c}'/\mathfrak{p})}(x_{(\mathfrak{c}')}) = \begin{cases} T_{\mathfrak{p}} x_{(\mathfrak{c}'/\mathfrak{p})} - x_{(\mathfrak{c}'/\mathfrak{p})}, & \text{if } \mathfrak{p} \mid (\mathfrak{c}'/\mathfrak{p}) \text{ in } K, \\ T_{\mathfrak{p}} x_{(\mathfrak{c}'/\mathfrak{p})} - (x_{(\mathfrak{c}'/\mathfrak{p})}^{\sigma_{\mathfrak{q}_1}} + x_{(\mathfrak{c}'/\mathfrak{p})}^{\sigma_{\mathfrak{q}_2}}), & \text{if } \mathfrak{p} \nmid (\mathfrak{c}'/\mathfrak{p}) \text{ and split in } K, \\ T_{\mathfrak{p}} x_{(\mathfrak{c}'/\mathfrak{p})} - x_{(\mathfrak{c}'/\mathfrak{p})}^{\sigma_{\mathfrak{q}}}, & \text{if } \mathfrak{p} \nmid (\mathfrak{c}'/\mathfrak{p}) \text{ and ramified in } K, \\ T_{\mathfrak{p}} x_{(\mathfrak{c}'/\mathfrak{p})}, & \text{if } \mathfrak{p} \nmid (\mathfrak{c}'/\mathfrak{p}) \text{ and inert in } K. \end{cases}$$

Gross points

We also assumed that Π has trivial central character. For each prime $\mathfrak{p} \mid \mathfrak{c}$, take $\alpha_{\mathfrak{p}}$ to be a root of $X^2 - a_{\mathfrak{p}}(\Pi)X + \|\mathfrak{p}\|$, where $a_{\mathfrak{p}}$ is the "Hecke eigenvalue of ϕ_{Π} at $\mathfrak{p}$." For each divisor $\mathfrak{c}'$ of $\mathfrak{c}$, put

$$\alpha_{\mathfrak{c}'} := \prod_{\mathfrak{p} \mid \mathfrak{c}'} \alpha_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}')}.$$

Let

$$z_{(\mathfrak{c})} := \alpha_{\mathfrak{c}}^{-1} \cdot \sum_{\mathfrak{c}' \mid \mathfrak{c}} \mu(\mathfrak{c}')\alpha_{\mathfrak{c}'}^{-1} \cdot [x_{(\mathfrak{c}/\mathfrak{c}')}] \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$

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Gross points

Then:

Proposition

For $\mathfrak{c}' \mid \mathfrak{c}$, we have

$$\phi_{\prod \mathcal{D}}\left(N_{\mathfrak{c}/\mathfrak{c}'}(z_{(\mathfrak{c})})\right) = \phi_{\prod \mathcal{D}}\left(\Big(\prod_{\substack{\mathfrak{p}\mid\mathfrak{c},\\ \mathfrak{p}\nmid\mathfrak{c}'}} e_{\mathfrak{p}}\Big) \cdot z_{(\mathfrak{c}')} \right),$$

where $e_{\mathfrak{p}} \in \mathbb{C}[G_{\mathfrak{c}'}]$ is defined by

$$e_{\mathfrak{p}} := \begin{cases} (1-\alpha_{\mathfrak{p}}^{-1}\cdot\sigma_{\mathfrak{q}_1})(1-\alpha_{\mathfrak{p}}^{-1}\cdot\sigma_{\mathfrak{q}_2}), & \text{if } \mathfrak{p} \text{ splits in } K,\\ (1-\alpha_{\mathfrak{p}}^{-1}\cdot\sigma_{\mathfrak{q}}), & \text{if } \mathfrak{p} \text{ is ramified in } K,\\ (1-\alpha_{\mathfrak{p}}^{-1})\cdot(1+\alpha_{\mathfrak{p}}^{-1}), & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

Gross points

For each character χ on $G_{\mathfrak{c}'}$ and $\mathfrak{p} \nmid \mathfrak{c}'$, it is observed that

$$\chi(e_\mathfrak{p}) \cdot \overline{\chi(e_\mathfrak{p})} = L_\mathfrak{p}(\frac{1}{2}, \Pi \times \chi)^{-1}.$$

Therefore by Theorem 2, we obtain:

Proposition

Suppose Π has trivial central character and square-free conductor $\mathfrak{n}\infty$, with $(\mathfrak{n}, D) = 1$. For each ideal $\mathfrak{c}$ of A coprime to $\mathfrak{n}$, we have that for each character $\chi : G_\mathfrak{c} \to \mathbb{C}^\times$,

$$\frac{\left|\sum_{\sigma \in G_\mathfrak{c}} \phi_{\Pi^\mathcal{D}}(z_{(\mathfrak{c})}^\sigma)\chi(\sigma)\right|^2}{\|\phi_{\Pi^\mathcal{D}}\|_{\mathrm{Pet}} \cdot |D|^{1/2}} = \frac{L^\mathfrak{c}(1/2, \Pi \times \chi)}{\|\phi_\Pi\|_{\mathrm{Pet}}}.$$

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Theta element

Recall that we let $\mathcal{L}^{\mathfrak{c}}_{\Pi,K} \in \mathbb{C}[G_{\mathfrak{c}}]$ be the unique element so that for each character $\chi : G_{\mathfrak{c}} \to \mathbb{C}^{\times}$, we have

$$\chi(\mathcal{L}^{\mathfrak{c}}_{\Pi,K}) = \frac{L^{\mathfrak{c}}(1/2, \Pi \times \chi)}{\|\phi_{\Pi}\|_{\mathrm{Pet}}}.$$

From the above Proposition, we may express $\mathcal{L}^{\mathfrak{c}}_{\Pi,K}$ as follows:

Theorem 3 (Chuang-W.)

Suppose Π has trivial central character and square-free conductor $\mathfrak{n}\infty$, with $(\mathfrak{n}, D) = 1$. For each ideal $\mathfrak{c}$ of A coprime to $\mathfrak{n}$, we have

$$\mathcal{L}^{\mathfrak{c}}_{\Pi,K} = \frac{\left(\sum_{\sigma \in G_{\mathfrak{c}}} \phi_{\Pi^{\mathcal{D}}}(z^{\sigma}_{(\mathfrak{c})}) \cdot \sigma\right)\left(\sum_{\sigma \in G_{\mathfrak{c}}} \overline{\phi_{\Pi^{\mathcal{D}}}(z^{\sigma}_{(\mathfrak{c})})} \cdot \sigma^{-1}\right)}{\|\phi_{\Pi^{\mathcal{D}}}\|_{\mathrm{Pet}} \cdot |D|^{1/2}}.$$

Waldspurger formula over function fields

Recall the main theorem:

Theorem 1 (Chuang-W.)

Let Π be an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, with a unitary central character ω_Π. Given a unitary Hecke character $\chi : K^\times \backslash K_\mathbb{A}^\times \to \mathbb{C}^\times$, suppose $\omega_\Pi \cdot \chi\big|_{\mathbb{A}^\times} \equiv 1$. Then

$$\mathbb{P}_\chi = L(\frac{1}{2}, \Pi \times \chi) \cdot \mathscr{P}_\chi.$$

Proof of Theorem 1: Rankin-Selberg method

Write $\mathcal{D} = K + Kj$, where $j^2 = \gamma \in k^\times$, we may decompose the quadratic space $(\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k})$ into:

$$(\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k}) = (V_1, Q_1) \oplus (V_2, Q_2),$$

where $(V_1, Q_1) = (K, \mathrm{N}_{K/k})$ and $(V_2, Q_2) = (K, -\gamma \cdot \mathrm{N}_{K/k})$. Given $\varphi \in S(\mathcal{D}_{\mathbb{A}})$, we may write $\varphi = \sum_j \varphi_{1,j} \oplus \varphi_{2,j}$ where $\varphi_{i,j} \in S(V_i(\mathbb{A}))$. For $f \in \Pi$, put

$$\begin{aligned} &\mathcal{Z}(f, \varphi, s) \\ = &\sum_j \int_{\mathrm{GL}_2^{+K}(k)\mathbb{A}^\times \backslash \mathrm{GL}_2^{+K}(\mathbb{A})} f(g)\theta_\chi^{V_1}(g, \varphi_{1,j}) E(g, s, \varphi_{2,j}) dg \end{aligned}$$

Rankin-Selberg method

Applying Rankin-Selberg method, we have:

Proposition

Suppose φ and f are pure tensors, then

$$\mathcal{Z}(f, \varphi, s) = \prod_v \mathcal{Z}_v(f_v, \varphi_v, s),$$

where $\mathcal{Z}_v(f_v, \varphi_v, s)$ is equal to

$$\int_{K_v^\times} \int_{\mathrm{SL}_2(O_v)} W_{f_v}\left(\begin{pmatrix} \mathrm{N}_{K/k}(a) & 0 \\ 0 & 1 \end{pmatrix} \kappa_v^1\right) \left(\omega_v^{\mathcal{D}}(\kappa_v^1)\varphi_v\right)(\bar{a})\, d\kappa_v^1$$

$$\cdot \chi_v(a) |\mathrm{N}_{K/k}(a)|_v^{s-\frac{1}{2}} d^\times a$$

Rankin-Selberg method

Remark: when v is "good," one has

$$\mathcal{Z}_v(f_v, \varphi_v, s) = \frac{L_v(s, \Pi \times \chi)}{L_v(2s, \xi_{K/k})}.$$

Thus

$$L(2s, \xi_{K/k}) \cdot \mathcal{Z}(f, \varphi, s) = L(s, \Pi \times \chi) \cdot \prod_v \mathcal{Z}_v^o(f_v, \varphi_v, s),$$

where

$$\mathcal{Z}_v^o(f_v, \varphi_v, s) := \frac{L_v(2s, \xi_{K/k})}{L_v(s, \Pi \times \chi)} \cdot \mathcal{Z}_v(f_v, \varphi_v, s).$$

- Siegel-Weil formula:

$$E(g, \frac{1}{2}, \varphi_2) = L(1, \xi_{K/k})^{-1} \cdot \theta^{V_2}_{\mathbf{1}_K}(g, \varphi_2), \quad \forall \varphi_2 \in S(V_2(\mathbb{A})).$$

- Seesaw identity:

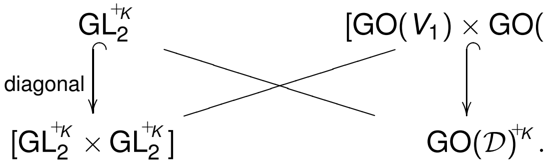
$$\begin{array}{ccc} \mathrm{GL}_2^{+K} & & [\mathrm{GO}(V_1) \times \mathrm{GO}(V_2)] \\ \text{diagonal} \downarrow & \times & \downarrow \\ [\mathrm{GL}_2^{+K} \times \mathrm{GL}_2^{+K}] & & \mathrm{GO}(\mathcal{D})^{+K}. \end{array}$$

$L(1, \xi_{K/k}) \cdot \mathcal{Z}(f, \varphi, 1/2) \leftrightarrow \mathbb{P}_\chi$

- Siegel-Weil formula:

$$E(g, \frac{1}{2}, \varphi_2) = L(1, \xi_{K/k})^{-1} \cdot \theta_{\mathbf{1}_K}^{V_2}(g, \varphi_2), \quad \forall \varphi_2 \in S(V_2(\mathbb{A})).$$

- Seesaw identity:



Therefore we get

$$\begin{aligned} & L(1, \xi_{K/k}) \cdot \mathcal{Z}(f, \varphi, 1/2) \\ = & \int_{K^\times \mathbb{A}^\times \backslash \mathbb{A}_K^\times} \int_{K^\times \mathbb{A}^\times \backslash \mathbb{A}_K^\times} \theta^{\mathcal{D}}(h_1, h_2; f, \varphi) \cdot \chi(h_1 h_2^{-1}) dh_1 dh_2, \end{aligned}$$

where for $b_1, b_2 \in \mathcal{D}_{\mathbb{A}}^\times$,

$$\begin{aligned} & \theta^{\mathcal{D}}(b_1, b_2; f, \varphi) \\ := & \int_{\mathrm{SL}_2(k) \backslash \mathrm{SL}_2(\mathbb{A}_k)} f\big(g_1 \alpha(b_1 b_2^{-1})\big) \cdot \theta^{V_{\mathcal{D}}}\big(g_1 \alpha(b_1 b_2^{-1}), [b_1, b_2]; \varphi\big) dg_1. \end{aligned}$$

Here $\alpha(b) := \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Nr}_{\mathcal{D}/k}(b) \end{pmatrix}$ for every $b \in \mathbb{A}_{\mathcal{D}}^\times$.

Shimizu correspondence

Put

$$\Theta^{\mathcal{D}}(\Pi) := \left\{\theta^{\mathcal{D}}(\cdot,\cdot;f,\varphi) \mid f \in \Pi, \varphi \in S(\mathcal{D}_{\mathbb{A}})\right\}.$$

Then

Shimizu correspondence

$$\Theta^{\mathcal{D}}(\Pi) = \Pi^{\mathcal{D}} \otimes \widetilde{\Pi}^{\mathcal{D}}.$$

In particular,

$$L(1,\xi_{K/k}) \cdot \mathcal{Z}(f,\varphi,1/2) = \mathbb{P}_{\chi}\left(\theta^{\mathcal{D}}(\cdot,\cdot;f,\varphi)\right).$$

Local Shimizu correspondence

Given $f_v \in \Pi_v$ and $\varphi_v \in S(\mathcal{D}_v)$, put

$$\theta_v(b_v, b_v'; f_v, \varphi_v) = \int_{\mathrm{U}(k_v)\backslash \mathrm{SL}_2(k_v)} W_{f_v}(g_v^1 \alpha(b_v b_v'^{-1})) \cdot (\omega_v^{\mathcal{D}}(g_v^1 \alpha(b_v b_v'^{-1}), [b_v, b_v'])\varphi_v)(1) dg_v^1$$

for $b_v, b_v' \in \mathcal{D}_v^\times$, and

$$\theta_v^o(b_v, b_v'; f_v, \varphi_v) := \frac{\zeta_v(2)}{L_v(1, \Pi_v, \mathrm{Ad})} \cdot \theta_v(b_v, b_v'; f_v, \varphi_v).$$

Local Shimizu correspondence

Proposition

1. $\theta_v^o(b_v, b'_v; f_v, \varphi_v) = 1$ when v is "good."
2. For $b_1, b_2 \in \mathbb{A}_{\mathcal{D}}^{\times}$,

$$\int_{\mathcal{D}^\times \mathbb{A}^\times \backslash \mathcal{D}_{\mathbb{A}}^\times} \theta_f^{\mathcal{D}}(bb_1, bb_2; \varphi) db = \frac{2L(1, \Pi, \mathrm{Ad})}{\zeta_k(2)} \cdot \prod_v \theta_v^o(b_{1,v}, b_{2,v}; f_v, \varphi_v).$$

3. Let $\Theta_v^{\mathcal{D}}(\Pi_v)$ be the space consisting of $\theta_v^o(\cdot, \cdot; f_v, \varphi_v)$ for $f_v \in \Pi_v$ and $\varphi_v \in S(\mathcal{D}_v)$. Then

$$\Theta_v^{\mathcal{D}}(\Pi_v) \cong \Pi_v^{\mathcal{D}} \otimes \widetilde{\Pi}_v^{\mathcal{D}}.$$

It is observed that

$$\mathcal{Z}_v(\frac{1}{2}; f_v, \varphi_v) = \int_{K_v^\times/k_v^\times} \theta_v(h_v, 1; f_v, \varphi_v)\chi_v(h_v)d^\times h_v.$$

Therefore

$$\begin{aligned}\mathcal{Z}_v^o(f_v, \varphi_v, 1/2) &= \frac{L_v(1, \xi_{K/k}) \cdot L_v(1, \Pi, \mathrm{Ad})}{L_v(\frac{1}{2}, \Pi \times \chi) \cdot \zeta_{k,v}(2)} \\ &\quad \cdot \int_{K_v^\times/k_v^\times} \theta_v^o(h_v, 1; f_v, \varphi_v)\chi_v(h_v)d^\times h_v \\ &= \mathscr{P}_{\chi,v}\Big(\theta_v^o(\cdot, \cdot; f_v, \varphi_v)\Big).\end{aligned}$$

Q.E.D.

The end. Thank you for your attention!