Approximating smooth transfer in the Jacquet–Rallis trace formulae

Hang Xue

MPIM
&
University of Arizona

July 15th 2016

Overview

1. The Theorems

2. The Proof

Notation

- E/F quadratic extension of number fields;

Notation

- E/F quadratic extension of number fields;
- $W \subset V$ hermitian spaces over E of dimension n and $n+1$ respectively;

Notation

- E/F quadratic extension of number fields;
- $W \subset V$ hermitian spaces over E of dimension n and $n+1$ respectively;
- $\mathrm{U}(W) \subset \mathrm{U}(V)$ unitary groups;

Notation

- E/F quadratic extension of number fields;
- $W \subset V$ hermitian spaces over E of dimension n and $n+1$ respectively;
- $\mathrm{U}(W) \subset \mathrm{U}(V)$ unitary groups;
- $H = \mathrm{U}(W)$, $G = \mathrm{U}(W) \times \mathrm{U}(V)$, H embeds in G diagonally;

Notation

- E/F quadratic extension of number fields;
- $W \subset V$ hermitian spaces over E of dimension n and $n+1$ respectively;
- $\mathrm{U}(W) \subset \mathrm{U}(V)$ unitary groups;
- $H = \mathrm{U}(W)$, $G = \mathrm{U}(W) \times \mathrm{U}(V)$, H embeds in G diagonally;
- π an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$.

The Theorems

Theorem (W. Zhang, 2013)

Suppose that $\mathrm{Hom}_{H(\mathbb{A}_F)}(\pi, \mathbb{C}) \neq 0$*. Assume that*

1. *there are two split places* v_1, v_2 *so that* π_{v_1} *is tempered and* π_{v_2} *is supercuspidal,*
2. E/F *is split at all archimedean places.*

Then the following are equivalent.

(a) *There is a* $\varphi \in \pi$ *so that* $P(\varphi) = \int_{[H]} \varphi(h) \mathrm{d}h \neq 0$.

(b) $L(\frac{1}{2}, \Pi) \neq 0$*, where* $\Pi = \mathrm{BC}(\pi)$ *is the base change of* π.

The Theorems

Theorem (W. Zhang, 2013)

Suppose that $\mathrm{Hom}_{H(\mathbb{A}_F)}(\pi, \mathbb{C}) \neq 0$. *Assume that*

1. *there are two split places* v_1, v_2 *so that* π_{v_1} *is tempered and* π_{v_2} *is supercuspidal,*
2. E/F *is split at all archimedean places.*

Then the following are equivalent.

(a) *There is a* $\varphi \in \pi$ *so that* $P(\varphi) = \int_{[H]} \varphi(h)\mathrm{d}h \neq 0$.

(b) $L(\frac{1}{2}, \Pi) \neq 0$, *where* $\Pi = \mathrm{BC}(\pi)$ *is the base change of* π.

Theorem (X., 2015)

Assume only (1). The same conclusion holds.

Plan of attack

Plan of attack

- $\pi \rightsquigarrow J_\pi$, a distribution on $G(\mathbb{A}_F)$, i.e. a continuous linear map $\mathcal{S}(G(\mathbb{A}_F)) \to \mathbb{C}$.

$$J_\pi \neq 0 \Leftrightarrow (b) \text{ (i.e. period is not identically zero).}$$

Plan of attack

- $\pi \rightsquigarrow J_\pi$, a distribution on $G(\mathbb{A}_F)$, i.e. a continuous linear map $\mathcal{S}(G(\mathbb{A}_F)) \to \mathbb{C}$.

$$J_\pi \neq 0 \Leftrightarrow (b) \text{ (i.e. period is not identically zero).}$$

$(J_\pi(f) = \sum_\varphi P(\pi(f)\varphi)\overline{P(\varphi)})$.

Plan of attack

- $\pi \rightsquigarrow J_\pi$, a distribution on $G(\mathbb{A}_F)$, i.e. a continuous linear map $\mathcal{S}(G(\mathbb{A}_F)) \to \mathbb{C}$.

$$J_\pi \neq 0 \Leftrightarrow (b) \text{ (i.e. period is not identically zero).}$$

  $(J_\pi(f) = \sum_\varphi P(\pi(f)\varphi)\overline{P(\varphi)})$.
- Put $G' = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E}$. $\Pi = \mathrm{BC}(\pi) \rightsquigarrow I_\Pi$, a distribution on $G'(\mathbb{A}_F)$.

$$I_\Pi \neq 0 \Leftrightarrow (a)\ (L(\frac{1}{2}, \Pi) \neq 0).$$

Plan of attack

- $\pi \rightsquigarrow J_\pi$, a distribution on $G(\mathbb{A}_F)$, i.e. a continuous linear map $\mathcal{S}(G(\mathbb{A}_F)) \to \mathbb{C}$.

$$J_\pi \neq 0 \Leftrightarrow (b) \text{ (i.e. period is not identically zero).}$$

  $(J_\pi(f) = \sum_\varphi P(\pi(f)\varphi)\overline{P(\varphi)})$.
- Put $G' = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E}$. $\Pi = \mathrm{BC}(\pi) \rightsquigarrow I_\Pi$, a distribution on $G'(\mathbb{A}_F)$.

$$I_\Pi \neq 0 \Leftrightarrow (a)\ (L(\frac{1}{2}, \Pi) \neq 0).$$

- WANT: for all $f \in \mathcal{S}(G(\mathbb{A}_F))$, we may find $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ so that $J_\pi(f) = I_\Pi(f')$. Also need the converse direction.

Jacquet–Rallis relative trace formulae

- $$\sum_{\pi \text{ cuspidal auto. rep of } G(\mathbb{A}_F)} J_\pi(f)$$

Jacquet–Rallis relative trace formulae

- $$\sum_{\pi \text{ cuspidal auto. rep of } G(\mathbb{A}_F)} J_\pi(f) = \sum_{\gamma \in G(F) \text{ regular semisimple}} O(\gamma, f)$$

Jacquet–Rallis relative trace formulae

- $$\sum_{\pi \text{ cuspidal auto. rep of } G(\mathbb{A}_F)} J_\pi(f) = \sum_{\gamma \in G(F) \text{ regular semisimple}} O(\gamma, f)$$

- $$\sum_{\Pi \text{ cuspidal auto. rep of } G'(\mathbb{A}_F)} I_\Pi(f') = \sum_{\delta \in G'(F) \text{ regular semisimple}} O(\delta, f')$$

Jacquet–Rallis relative trace formulae

- $$\sum_{\pi \text{ cuspidal auto. rep of } G(\mathbb{A}_F)} J_\pi(f) = \sum_{\gamma \in G(F) \text{ regular semisimple}} O(\gamma, f)$$

- $$\sum_{\Pi \text{ cuspidal auto. rep of } G'(\mathbb{A}_F)} I_\Pi(f') = \sum_{\delta \in G'(F) \text{ regular semisimple}} O(\delta, f')$$

- Idea: Compare the orbital integrals.

Smooth transfer

Reduction steps leads to the following problem.

Smooth transfer

Reduction steps leads to the following problem.

- E/F quadratic extension of local fields, W a hermitian space over E of dimension n.

Smooth transfer

Reduction steps leads to the following problem.

- E/F quadratic extension of local fields, W a hermitian space over E of dimension n.
- $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n) \rightsquigarrow O(\gamma, f')$.

Smooth transfer

Reduction steps leads to the following problem.

- E/F quadratic extension of local fields, W a hermitian space over E of dimension n.
- $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n) \rightsquigarrow O(\gamma, f')$.
- $f \in \mathcal{S}(\mathfrak{u}(W) \times W) \rightsquigarrow O(\delta, f)$.

Smooth transfer

Reduction steps leads to the following problem.

- E/F quadratic extension of local fields, W a hermitian space over E of dimension n.
- $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n) \rightsquigarrow O(\gamma, f')$.
- $f \in \mathcal{S}(\mathfrak{u}(W) \times W) \rightsquigarrow O(\delta, f)$.
- There is a matching for the regular semisimple orbits $\gamma \leftrightarrow \delta$.

Smooth transfer

Reduction steps leads to the following problem.

- E/F quadratic extension of local fields, W a hermitian space over E of dimension n.
- $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n) \rightsquigarrow O(\gamma, f')$.
- $f \in \mathcal{S}(\mathfrak{u}(W) \times W) \rightsquigarrow O(\delta, f)$.
- There is a matching for the regular semisimple orbits $\gamma \leftrightarrow \delta$.
- PROBLEM: For all matching γ and δ and any $f \in \mathcal{S}(\mathfrak{u}(W) \times W)$, there is an $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n)$, so that $O(\delta, f) = \pm O(\gamma, f')$.

Smooth transfer

Reduction steps leads to the following problem.

- E/F quadratic extension of local fields, W a hermitian space over E of dimension n.
- $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n) \rightsquigarrow O(\gamma, f')$.
- $f \in \mathcal{S}(\mathfrak{u}(W) \times W) \rightsquigarrow O(\delta, f)$.
- There is a matching for the regular semisimple orbits $\gamma \leftrightarrow \delta$.
- PROBLEM: For all matching γ and δ and any $f \in \mathcal{S}(\mathfrak{u}(W) \times W)$, there is an $f' \in \mathcal{S}(\mathfrak{gl}_n(F) \times F_n \times F^n)$, so that $O(\delta, f) = \pm O(\gamma, f')$.
- We put $\mathcal{V} = \mathfrak{u}(W) \times W$. We call $f \in \mathcal{S}(\mathcal{V})$ with this desired property transferable.

Theorem:nonarchimedean case

Theorem (W. Zhang)

Suppose that F is nonarchimedean. Then all $f \in \mathcal{S}(\mathcal{V})$ are transferable.

Theorem:nonarchimedean case

Theorem (W. Zhang)

Suppose that F is nonarchimedean. Then all $f \in \mathcal{S}(\mathcal{V})$ are transferable.

Proof.

There are three Fourier transforms on $\mathcal{V}$, i.e. with respect to $\mathfrak{u}(W)$, W or $\mathcal{V}$. Denote them by $\mathcal{F}_{\mathfrak{u}(W)}$, $\mathcal{F}_W$ and $\mathcal{F}_{\mathcal{V}}$ respectively.

Theorem:nonarchimedean case

Theorem (W. Zhang)

Suppose that F is nonarchimedean. Then all $f \in \mathcal{S}(\mathcal{V})$ are transferable.

Proof.

There are three Fourier transforms on $\mathcal{V}$, i.e. with respect to $\mathfrak{u}(W)$, W or $\mathcal{V}$. Denote them by $\mathcal{F}_{\mathfrak{u}(W)}$, $\mathcal{F}_W$ and $\mathcal{F}_{\mathcal{V}}$ respectively.

1. If f is transferable, then so is $\mathcal{F}_* f$, $* = \mathfrak{u}(W), W$ or $\mathcal{V}$.

Theorem:nonarchimedean case

Theorem (W. Zhang)

Suppose that F is nonarchimedean. Then all $f \in \mathcal{S}(\mathcal{V})$ are transferable.

Proof.

There are three Fourier transforms on $\mathcal{V}$, i.e. with respect to $\mathfrak{u}(W)$, W or $\mathcal{V}$. Denote them by $\mathcal{F}_{\mathfrak{u}(W)}$, $\mathcal{F}_W$ and $\mathcal{F}_{\mathcal{V}}$ respectively.

1. If f is transferable, then so is $\mathcal{F}_* f$, $* = \mathfrak{u}(W), W$ or $\mathcal{V}$.
2. Let $\mathcal{N}$ be the nilpotent cone in $\mathcal{V}$. Any $f \in \mathcal{S}(\mathcal{V})$ can be written as $f = f_1 + f_2 + f_3 + f_4 + f_0$, where $f_1, \mathcal{F}_{\mathfrak{u}(W)} f_2, \mathcal{F}_W f_3, \mathcal{F}_{\mathcal{V}} f_4 \in \mathcal{S}(\mathcal{V} \backslash \mathcal{N})$ and any regular semisimple orbital integrals of f_0 vanish.

Theorem:nonarchimedean case

Theorem (W. Zhang)

Suppose that F is nonarchimedean. Then all $f \in \mathcal{S}(\mathcal{V})$ are transferable.

Proof.

There are three Fourier transforms on $\mathcal{V}$, i.e. with respect to $\mathfrak{u}(W)$, W or $\mathcal{V}$. Denote them by $\mathcal{F}_{\mathfrak{u}(W)}$, $\mathcal{F}_W$ and $\mathcal{F}_{\mathcal{V}}$ respectively.

1. If f is transferable, then so is $\mathcal{F}_* f$, $* = \mathfrak{u}(W), W$ or $\mathcal{V}$.
2. Let $\mathcal{N}$ be the nilpotent cone in $\mathcal{V}$. Any $f \in \mathcal{S}(\mathcal{V})$ can be written as $f = f_1 + f_2 + f_3 + f_4 + f_0$, where $f_1, \mathcal{F}_{\mathfrak{u}(W)} f_2, \mathcal{F}_W f_3, \mathcal{F}_{\mathcal{V}} f_4 \in \mathcal{S}(\mathcal{V} \backslash \mathcal{N})$ and any regular semisimple orbital integrals of f_0 vanish.
3. Induction: all $\mathcal{S}(\mathcal{V} \backslash \mathcal{N})$ is transferable.

□

Theorem: archimedean case

Theorem (X.)

Suppose that F is archimedean. The transferable test functions form a dense subset of $\mathcal{S}(\mathcal{V})$.

Theorem: archimedean case

Theorem (X.)

Suppose that F is archimedean. The transferable test functions form a dense subset of $\mathcal{S}(\mathcal{V})$.

This implies (the direction (a) $\Rightarrow$ (b) of) the main theorem since J_π is continuous!!

Theorem: archimedean case

Theorem (X.)

Suppose that F is archimedean. The transferable test functions form a dense subset of $\mathcal{S}(\mathcal{V})$.

This implies (the direction (a) $\Rightarrow$ (b) of) the main theorem since J_π is continuous!!

Proof.

Theorem: archimedean case

Theorem (X.)

Suppose that F is archimedean. The transferable test functions form a dense subset of $\mathcal{S}(\mathcal{V})$.

This implies (the direction (a) $\Rightarrow$ (b) of) the main theorem since J_π is continuous!!

Proof.

1. If f is transferable, then so is $\mathcal{F}_* f$, $* = \mathfrak{u}(W), W$ or $\mathcal{V}$.

Theorem: archimedean case

Theorem (X.)

Suppose that F is archimedean. The transferable test functions form a dense subset of $\mathcal{S}(\mathcal{V})$.

This implies (the direction (a) $\Rightarrow$ (b) of) the main theorem since J_π is continuous!!

Proof.

1. If f is transferable, then so is $\mathcal{F}_* f$, $* = \mathfrak{u}(W), W$ or $\mathcal{V}$.
2. Any $f \in \mathcal{S}(\mathcal{V})$ can be *approximated* by functions of the form $f_1 + f_2 + f_3 + f_4 + f_0$, where $f_1, \mathcal{F}_{\mathfrak{u}(W)} f_2, \mathcal{F}_W f_3, \mathcal{F}_{\mathcal{V}} f_4 \in \mathcal{S}(\mathcal{V} \backslash \mathcal{N})$ and any regular semisimple orbital integrals of f_0 vanish.

Theorem: archimedean case

Theorem (X.)

Suppose that F is archimedean. The transferable test functions form a dense subset of $\mathcal{S}(\mathcal{V})$.

This implies (the direction (a) $\Rightarrow$ (b) of) the main theorem since J_π is continuous!!

Proof.

1. If f is transferable, then so is $\mathcal{F}_* f$, $* = \mathfrak{u}(W), W$ or $\mathcal{V}$.
2. Any $f \in \mathcal{S}(\mathcal{V})$ can be *approximated* by functions of the form $f_1 + f_2 + f_3 + f_4 + f_0$, where $f_1, \mathcal{F}_{\mathfrak{u}(W)} f_2, \mathcal{F}_W f_3, \mathcal{F}_\mathcal{V} f_4 \in \mathcal{S}(\mathcal{V} \backslash \mathcal{N})$ and any regular semisimple orbital integrals of f_0 vanish.
3. Induction: transferable functions in $\mathcal{S}(\mathcal{V} \backslash \mathcal{N})$ form a dense subset.

$\square$

Thank You!!!