Asymptotic normality in combinatorics

Xi Chen

School of Mathematical Sciences
Dalian University of Technology
chenxi@dlut.edu.cn

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Normal distribution (Gaussian distribution)

A random variable $X$ is normally distributed, write $X \sim \mathcal{N}(\mu, \sigma^2)$, if

$$
\text{Prob}(X \leq x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, dt.
$$

Having achieved fame with his calculation on the orbit of asteroid Ceres (discovered in 1801), Gauss later published a treatise on the subject of celestial orbits. In this work he propounded the method of least squares, and provided theoretical justification by assuming that observational measurements are normally distributed about their mean.

probability density function of $\mathcal{N}(\mu, \sigma^2)$

Carl F. Gauss (1777–1855)
De Moivre-Laplace theorem

Let $X_i, 1 \leq i \leq n$, be a sequence of independent 0,1 random variables with $\text{Prob}(X_i = 1) = p$ and $\text{Prob}(X_i = 0) = q = 1 - p$. Let $S_n = \sum_{i=1}^{n} X_i$.

De Moivre-Laplace theorem

For any fixed $x$,

$$\text{Prob}(S_n < np + x\sqrt{npq}) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$$

as $n \to \infty$.

- The mean and variance of $S_n$ are $np$ and $npq$, respectively.
- $\text{Prob}(S_n < np + x\sqrt{npq}) = \sum_{k < np + x\sqrt{npq}} \binom{n}{k} p^k q^{n-k}$. 

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Generalization of de Moivre-Laplace theorem

Let \( X_i, 1 \leq i \leq n \), be a sequence of independent 0,1 random variables with \( \text{Prob}(X_i = 1) = p_i \). Let \( S_n = \sum_{i=1}^{n} X_i \) with mean and variance

\[
\mu_n = \sum_{i=1}^{n} p_i, \quad \sigma_n^2 = \sum_{i=1}^{n} p_i(1 - p_i).
\]

De Moivre-Laplace theorem: generalization

Provided \( \sigma_n \to \infty \), then for any fixed \( x \),

\[
\text{Prob}(S_n < \mu_n + x\sigma_n) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt,
\]

as \( n \to \infty \).
Let $X_i$, $1 \leq i \leq n$, be independent random variables with means $\mu_i$, variances $\sigma^2_i$, and absolute third central moments $\rho_i = \mathbb{E}|X_i - \mu_i|^3$. Let $S_n = \sum_{i=1}^n X_i$ with mean $\mu = \sum_i \mu_i$ and variance $\sigma^2 = \sum_i \sigma^2_i$.

**Berry-Esseen theorem**

There exist a constant $C$ such that

$$
\left| \text{Prob}(S_n < \mu + x\sigma) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right| \leq \frac{C \sum_{i=1}^n \rho_i}{\sigma^3}.
$$

A.C. Berry, The accuracy of the Gaussian approximation to the sum of independent variates, Trans. AMS, 1941.

Asymptotic normality

Let \( a(n, k) \) be a double-indexed sequence of nonnegative numbers and 
\[ p(n, k) = a(n, k) / \sum_{j=0}^{n} a(n, j) \] 
the normalized probabilities. We say that \( a(n, k) \) is asymptotically normal by a central limit theorem with mean \( \mu_n \) and variance \( \sigma_n^2 \) if

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sum_{k \leq \mu_n + x\sigma_n} p(n, k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0. \tag{1}
\]

Say that \( a(n, k) \) is asymptotically normal by a local limit theorem on \( S \) if

\[
\lim_{n \to \infty} \sup_{x \in S} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \tag{2}
\]

When \( S = \mathbb{R} \), the validity of (2) implies that of (1) and

\[
a(n, k) \sim \frac{e^{-x^2/2} \sum_{j=0}^{n} a(n, j)}{\sqrt{2\pi \sigma_n^2}}, \quad n \to \infty,
\]

where \( k = \mu_n + x\sigma_n \) for some fixed \( x \).
Outline

1. Asymptotic normality
2. Methods and techniques
3. Narayana numbers
4. Problems and conjectures
Method 1: direct approach

Sometimes the enumeration problem, suitably normalized, can be viewed as the distribution of a sum of independent 0, 1 variables. Then the generalization of de Moivre-Laplace theorem is applicable.

Example

- number of cycles in a permutation;
- number of left-to-right minima in a permutation;
- number of inversions in a permutation;
- number of distinct prime divisors (Erdős-Kac theorem).

P. Erdős, M. Kac, The Gaussian law of errors in the theory of additive number theoretic functions, American Journal of Mathematics, 1940.
Method 2: RZness

Let \([a(n, k)]_{n,k \geq 1}\) be an infinite lower triangular matrix with nonnegative entries. Let \(A_n(x) = \sum_{k=0}^{n} a(n, k)x^k\) and \(p(n, k) = \frac{a(n,k)}{\sum_j a(n,j)}\). Then

\[
\mu_n = \sum_k kp(n, k) = \frac{A'_n(1)}{A_n(1)}
\]

and

\[
\sigma^2_n = \sum_k k^2 p(n, k) - \mu^2_n = \frac{A''_n(1)}{A_n(1)} + \mu_n - \mu^2_n.
\]

In particular, if \(A_n(x) = \prod_{i=1}^{n} (x + r_{n,i})\) have only negative zeros, then

\[
\mu_n = \sum_{i=1}^{n} \frac{1}{1 + r_{n,i}}, \quad \sigma^2_n = \sum_{i=1}^{n} \frac{r_{n,i}}{(1 + r_{n,i})^2}.
\]

It follows that \(\sigma^2_n \leq \mu_n, \ \forall n\).
Central limit theorem

**Theorem**

Let $A_n(x) = \sum_k a(n, k)x^k$ have only real zeros, and let $\mu_n, \sigma_n^2$ be the associated means and variances. If $\sigma_n^2 \to +\infty$, then $a(n, k)$ are asymptotically normal by a central limit theorem.


Local limit theorem

**Theorem (Bender, 1973)**

Suppose that $a(n, k)$ satisfy a central limit theorem, and $\sigma_n^2 \to +\infty$.

1. If $a(n, k)$ is unimodal in $k$ for each $n$, then $a(n, k)$ satisfy a local limit theorem on $S = \{ x : |x| \geq \varepsilon \}$, $\forall \varepsilon > 0$;

2. if $a(n, k)$ is log-concave in $k$ for each $n$, then $a(n, k)$ satisfy a local limit theorem on $\mathbb{R}$.

E.A. Bender, Central and local limit theorems applied to asymptotic enumeration, JCTA, 1973.
Strongly asymptotic normality

Newton’s inequality

If $\sum_{k=0}^{n} a_k x^k$ has only real zeros, then

$$a_k^2 \geq a_{k-1} a_{k+1} \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),$$

and the sequence $a_0, a_1, \ldots, a_n$ is therefore log-concave and unimodal.

Definition

If $A_n(x) = \sum_k a(n, k)x^k$ have only real zeros and $\sigma_n^2 \to +\infty$, then we say that $a(n, k)$ are strongly asymptotically normal.

In this case, $a(n, k)$ satisfy both central and local limit theorems.
Polynomials with only real zeros

**Theorem (Wang-Yeh, 2005)**

Suppose that $A_n(x) = \sum_{k=0}^{n} a(n, k)x^k$ and $a(n, k)$ satisfy

$$a(n, k) = (rn + sk + t)a(n - 1, k - 1) + (an + bk + c)a(n - 1, k).$$

If $rb \geq as$ and $(r + s + t)b \geq (a + c)s$, then $A_n(x)$ have only real zeros.

**Theorem (Liu-Wang, 2007)**

Suppose that $A_n(x)$ satisfy $\deg A_{n-1}(x) \leq \deg A_n(x) \leq \deg A_{n-1}(x) + 1$ and

$$A_n(x) = a_n(x)A_{n-1}(x) + b_n(x)A'_{n-1}(x) + c_n(x)A_{n-2}(x).$$

If $b_n(x), c_n(x) \leq 0$ for $x \leq 0$, then $A_n(x)$ have only real zeros.

Wang, Yeh, Polynomials with real zeros and Pólya frequency sequences, JCTA, 2005.

Examples of strongly asymptotically normal numbers

Example

- binomial coefficients \( \binom{n}{k} \)
- signless Stirling numbers of the first kind \( c(n, k) \)
- Stirling numbers of the second kind \( S(n, k) \)
- Eulerian numbers \( A(n, k) \)
- matching numbers of a graph
- Laplacian coefficients of a graph


C.D. Godsil, Matching behaviour is asymptotically normal, Combinatorica, 1981.

Method 3: moments

For a random variable $X$, the associated moment generating function is defined by

$$M_X(t) := \mathbb{E}e^{tx}.$$ 

If $X \sim \mathcal{N}(0,1)$, then $M_X(t) = e^{t^2/2}$.

**Theorem (Curtiss, 1942)**

Suppose $X_n$ is a sequence of random variables with distribution functions $F_n(x)$. If $M_{X_n}(t) \to e^{t^2/2}$ for all $t$, then $F_n(x) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ for all $x$.

Method 3: moments

Example

- number of partitions into distinct parts
- number of distinct parts in a random partition
- coefficients of polynomials of binomial type
- $q$-Catalan numbers
- $q$-derangements

- E.R. Canfield, Central and local limit theorems for the coefficients of polynomials of binomial type, JCTA, 1977.
Method 4: singularity analysis

Theorem (Bender, 1973)

Let \( f(x, y) = \sum_{n,k} a(n, k) x^n y^k \) with \( a(n, k) \geq 0 \). Suppose there exist

1. a function \( A(s) \) continuous and nonzero near 0,
2. a function \( r(s) \) with bounded third derivative near 0,
3. a nonnegative integer \( m \), and
4. positive numbers \( \varepsilon \) and \( \delta \) such that

\[
\left( 1 - \frac{x}{r(s)} \right)^m f(x, e^s) - \frac{A(s)}{1 - x/r(s)}
\]

is analytic and bounded for \( |s| < \varepsilon \), and \( |x| < r(0) + \delta \).

Put \( \mu = -r'(0)/r(0) \) and \( \sigma^2 = \mu^2 - r''(0)/r(0) \). If \( \sigma^2 \neq 0 \), then \( a(n, k) \) are asymptotically normal with mean \( n\mu \) and variance \( n\sigma^2 \).

Bender, Central and local limit theorems applied to asymptotic enumeration, JCTA, 1973.
Outline

1. Asymptotic normality
2. Methods and techniques
3. Narayana numbers
4. Problems and conjectures
Narayana numbers

The Narayana numbers $N(n, k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}$, $1 \leq k \leq n$.

Question (Shapiro, 2001)

Do the rows of the Narayana triangle approach a normal distribution?

\[
N(n, k)_{n, k \geq 1} = \begin{bmatrix}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 6 & 6 & 1 \\
1 & 10 & 20 & 10 & 1 \\
\vdots & \ddots & \ddots & \ddots \end{bmatrix}
\]

Narayana polynomials

The Narayana numbers $N(n, k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}$, $1 \leq k \leq n$.

Narayana polynomials

$$N_n(x) = \sum_{k=1}^{n} N(n, k)x^k = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k-1} \binom{n}{k}x^k$$

satisfy the recurrence

$$(n + 1)N_n(x) = (2n - 1)(1 + x)N_{n-1}(x) - (n - 2)(1 - x)^2N_{n-2}(x)$$

with $N_1(x) = x$ and $N_2(x) = x + x^2$.

It is well known that $N_n(x)$ have only real zeros.
Asymptotic normality of Narayana numbers

\[ N_n(x) = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k-1} \binom{n}{k} x^k \Rightarrow N_n(1) = \frac{1}{n+1} \binom{2n}{n} \]

\[ N'_n(x) = \frac{1}{n} \sum_{k=1}^{n} k \binom{n}{k-1} \binom{n}{k} x^{k-1} \Rightarrow N'_n(1) = \binom{2n-1}{n-1} \]

\[ N''_n(x) = \frac{1}{n} \sum_{k=1}^{n} k(k-1) \binom{n}{k-1} \binom{n}{k} x^{k-2} \Rightarrow N''_n(1) = (n-1) \binom{2n-2}{n-1} \]

\[ \mu_n = \frac{N'_n(1)}{N_n(1)} = \frac{n+1}{2}, \quad \sigma_n^2 = \frac{N''_n(1)}{N_n(1)} + \mu_n - \mu_n^2 = \frac{(n-1)(n+1)}{4(2n-1)} \rightarrow +\infty. \]

Theorem (C-Mao-Wang, 2016)

The Narayana numbers are strongly asymptotically normal.
Narayana numbers $N(n, k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}$ have row sums $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Consider

$$\frac{N(n, k)}{C_n} = \frac{(n+1) \binom{n}{k-1} \binom{n}{k}}{n \binom{2n}{n}}.$$

To show the strong asymptotical normality of $N(n, k)$, it suffices to show that

$$\frac{N(n, k)}{C_n} \sim \frac{1}{\sqrt{2\pi \sigma_n^2}} e^{-\frac{(k-\mu_n)^2}{2\sigma_n^2}}$$

for some $\mu_n$ and $\sigma_n^2$. Assume that $k = \mu_n + x \sigma_n$ for some fixed $x$, where $\mu_n$ and $\sigma_n$ are the mean and standard variance of $N(n, k)$ respectively.

Then $\mu_n = (n+1)/2$ since the symmetry of each row. Recall that $\sigma_n^2 \leq \mu_n$, $\forall n$, since $N_n(x) = \sum_{k=1}^{n} N(n, k)x^k$ have only real zeros. Then $k \rightarrow (n+1)/2$ as $n \rightarrow \infty$. 
Asymptotic normality of Narayana numbers II

It is well known that the binomial coefficients \( \binom{n}{k} \) is strongly asymptotically normal with mean \( n/2 \) and variance \( n/4 \).

Then

\[
\binom{n}{k} \sim \frac{2^n}{\sqrt{2\pi n/4}} e^{-\frac{(k-n/2)^2}{2n/4}},
\]

\[
\binom{n}{k-1} \sim \frac{2^n}{\sqrt{2\pi n/4}} e^{-\frac{(k-1-n/2)^2}{2n/4}},
\]

and

\[
\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.
\]

Hence

\[
\frac{N(n, k)}{C_n} = \frac{(n+1)\binom{n}{k-1} \binom{n}{k}}{n\binom{2n}{n}} \sim \frac{1}{\sqrt{2\pi n/8}} e^{-\frac{(k-(n+1)/2)^2}{2n/8}},
\]

which implies that \( N(n, k) \) is strong asymptotically normal with mean \( (n+1)/2 \) and variance \( n/8 \).
Stack sorting problem

- The stack sorting problem introduced by Knuth in the 1960’s was a founding inspiration in the study of permutation patterns.
- Simultaneously he introduced the notion of pattern containment, defining a class of permutations by a forbidden set, and the enumeration of permutations in such classes.

Stack sorting problem

- A **stack** is a last-in, first-out linear sequence accessed at one end called the top.
- Items are added and removed from the top end by **push** and **pop** operations.

A stack sorting makes the output permutation to be $12 \cdots n$. 
Stack sorting problem

**Greedy algorithm**: only pops a stack if pushing the next input would have violated the increasing property of the stack (read from the top).

Ex.

```
132  32  1  32  1  2
132  32  1  32  1  2
132  32  1  32  1  2
123  123  12  3  1  2
123  123  12  3  1  2
123  123  12  3  1  2
```

**Theorem (Knuth, 1968)**

Let $p = LnR$ with $L$ and $R$ respectively denoting the string on the left and right of $n$. The stack-sorting operation $s$ can be defined recursively by

$s(p) = s(L)s(R)n$. 

In the above example $p = 132$ and $s(p) = 123$. 
Stack sorting problem

Example (Knuth, 1968)

A permutation $\pi$ can be sorted by a stack if and only if $\pi$ avoids 231.

The number of sortable $n$-permutations is $C_n = \frac{1}{n+1} \binom{2n}{n}$. 

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Given stacks $S_1, S_2, \ldots, S_t$. At any point in the sorting process we may push the next input onto one of the stacks or pop one of the stacks.

West’s RULE: examine stacks from $S_1$ to $S_t$, locate the first stack on which a push can be applied; if there is no such stack then $S_t$ is popped.

$p$ is $t$-stack sortable if and only if $s^t(p) = 12 \cdots n$.

All $n$-permutations are $(n - 1)$-stack sortable in this model.

Special cases of $t$-stack sortable permutations

Let $W_t(n) = \# \ t$-stack sortable $n$-permutations.

- $W_1(n) = \frac{1}{n+1} \binom{2n}{n}$;
- $W_{n-1}(n) = n!$;
- $W_2(n) = \frac{2(3n)!}{(n+1)!(2n+1)!}$;
- $W_{n-2}(n) = \# \ all \ n$-permutations that do not end in the string $n1$.


D. Zeilberger, A proof of Julian West’s conjecture that the number of two-stack-sortable permutations of length $n$ is $2(3n)!/((n + 1)!(2n + 1)!)$, Discrete Math. 102 (1992) 85–93.
Special cases of $t$-stack sortable permutations

Let $W_t(n, k) = \# \ t$-stack sortable $n$-permutations with $k$ descents.

- $W_1(n, k) = N(n, k)$;
- $W_{n-1}(n, k) = A(n, k)$;
- $W_2(n, k) = \frac{(n+k-1)(2n-k)!}{k!(n-k+1)(2k-1)(2n-2k+1)!}$;
- $W_{n-2}(n, k) = A(n, k) - A(n - 2, k - 1)$.

We can show that these numbers are all strongly asymptotically normal.

Conjecture (Bóna, 2002)

The polynomial $W_n,t(x) = \sum_{k=1}^{n} W_t(n, k)x^k$ has only real zeros for fixed $n \geq 2$ and $1 \leq t \leq n$.

Conjecture

The $t$-stack sortable numbers are strongly asymptotically normal.

M. Bóna, Symmetry and unimodality in $t$-stack sortable permutations, JCTA, 2002.
Shapiro’s problems

**Theorem (de Moivre-Laplace)**

\[
\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \text{ is asymptotically normal.}
\]

**Theorem (Godsil, 1981)**

\[
\binom{n}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \ldots \text{ is asymptotically normal.}
\]

**Conjecture (Shapiro, 2001)**

\[
\binom{n}{0}, \binom{n-a}{b}, \binom{n-2a}{2b}, \ldots \text{ is asymptotically normal if } 0 \leq a \leq b.
\]

- This conjecture has been settled by Q.-H. Hou recently.
Thanks for your attention!