Asymptotic behavior of the linearized Boltzmann collision operator: from angular cutoff to non-cutoff

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Boltzmann equation

The Boltmzann equation reads:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \qquad (1.1)$$

where $f(t, x, v) \geq 0$ is the distribution function in the phase space of collision particles which at time $t \geq 0$ and point $x \in \mathbb{R}^3$ move with velocity $v \in \mathbb{R}^3$.

The Boltzmann collision operator Q is a bilinear operator which acts only on the velocity variables v, that is,

$$Q(g,f)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(g'_*f'-g_*f)d\sigma dv_*.$$

Here we use the standard shorthand $f=f(v)$, $g_*=g(v_*)$, $f'=f(v')$, $g'_*=g(v'_*)$ where v', v'_* are given by

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\,,\quad v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma\,,\qquad \sigma\in\mathbb{S}^2\,.$$

We stress that the representation follows the parametrization of the set of solutions of the physical law of elastic collision:

$$\begin{aligned} v+v_* &= v'+v'_*,\\ |v|^2+|v_*|^2 &= |v'|^2+|v'_*|^2. \end{aligned}$$

Assumptions on the Boltzmann collision kernel $B(v-v_*,\sigma)$

- (A-1) The cross-section $B(v-v_*,\sigma)$ takes a product form

$$B(v-v_*,\sigma)=B(|v-v_*|,\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle)=\Phi(|v-v_*|)b(\cos\theta),$$

  where both Φ and b are nonnegative functions and $\cos\theta=\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle=\langle\frac{v-v_*}{|v-v_*|},\frac{v'-v'_*}{|v'-v'_*|}\rangle$.

- (A-2) The angular function $b(t)$ is not locally integrable and it satisfies for $\theta\in[0,\pi/2]$,

$$K\theta^{-1-2s}\le\sin\theta b(\cos\theta)\le K^{-1}\theta^{-1-2s},\quad \text{with } 0<s<1,\ K>0.$$

- (A-3) The kinetic factor Φ takes the form

$$\Phi(|v-v_*|)=|v-v_*|^{\gamma}.$$

- (A-4) The parameter γ verifies $\gamma+2s>-1$.

Conservation and Entropy

Conservation of mass, momentum and the kinetic energy:

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)\phi(v)dvdx = \int_{\mathbb{T}^3\times\mathbb{R}^3} f(0,x,v)\phi(v)dvdx, \quad \phi(v)=1,v,|v|^2.$$

The Boltzmann's H theorem:

$$\begin{aligned} -\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} f\log f dvdx &= -\int_{\mathbb{R}^3} Q(f,f)\log f dvdx \\ &= \int B(v-v_*,\sigma)(f_*'f'-ff_*)\log\frac{f_*'f'}{ff_*}d\sigma dv_* dvdx \ge 0. \end{aligned}$$

Grad's cut off assumption

Suppose the collision operator can be rewritten by

$$Q(f,f) = Q^{+}(f,f) - L(f)f,$$

where

$$\begin{aligned} L(f) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{\gamma} b(\cos\theta) f(v_*) d\sigma dv_* \\ &= |\mathbb{S}^1| \Big(\int_{v_*} |v-v_*|^{\gamma} f_* dv_*\Big)\Big(\int_0^{\pi} b(\cos\theta)\sin\theta d\theta\Big). \end{aligned}$$

Then the condition

$$\int_0^{\pi} b(\cos\theta)\sin\theta d\theta < \infty,$$

is needed.

Grad's cutoff assumption: the deviation angle θ has a lower bound, i.e., $\theta \geq C\epsilon > 0$. In this case, the collision operator Q is turned to be Q^ϵ defined by:

$$Q^\epsilon(g,f)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^\epsilon(v-v_*,\sigma)(g'_*f' - g_*f)d\sigma dv_* \stackrel{\text{def}}{=} Q^\epsilon_+(g,f) - Q^\epsilon_-(g,f), \quad (1.2)$$

where $B^\epsilon(v-v_*,\sigma) = b^\epsilon(\cos\theta)|v-v_*|^\gamma$ with $b^\epsilon(\cos\theta) = b(\cos\theta)(1-\psi)((\sin\frac{\theta}{2})/\epsilon)$. The bump function ψ with support around 0. Then the Boltzmann equation with angular cutoff and initial data f_0 is written as

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = Q^\epsilon(f,f), \\ f|_{t=0} = f_0. \end{cases} \quad (1.3)$$

Summary

I. Two equations:

♦ Boltzmann equation with angular cutoff: Hyperbolic equation with damping,

♦ Boltzmann equation without angular cutoff: Quasi-Linear Parabolic equation with Fractional regularity,

II. Relationship between the equations:

♦ Boltzmann equation with angular cutoff($\theta \geq \epsilon$) $\stackrel{\epsilon \to 0}{\longrightarrow}$ Boltzmann equation without angular cutoff;

Existing results

I. Boltzmann equation with angular cutoff:

♦ DiPerna-Lions, Mischler: Existence of renormalized solution and weak stability;

♦ Classical solutions near equilibrium or near vacuum.

II.Boltzmann equation without angular cutoff:

♦ Alexandre-Villani: Renormalized solution with defect measure;
♦ Alexandre-Morimotto-Ukai-Xu-Yang, Gressman-Strain: Classical solution near equilibrium;

III. Global dynamics for Boltzmann equation:
♦ Desvillettes-Villani: Entropy method to derive the polynomial decay rate to the equilibrium;
♦ AMUXY, G-S, Duan . . . : Energy method to get the optimal decay rate to the equilibrium when the solution is near equilibrium.

Problems

Find a framework to consider

Asymptotics: Boltzmann equation with angular cutoff($\theta \geq \epsilon$) $\stackrel{\epsilon\to 0}{\longrightarrow}$ Boltzmann equation without angular cutoff.

♦ to give a complete description on the linearized Boltzmann collision operator in the process of the limit $\epsilon \to 0$;

♦ to establish an asymptotic formula between the solutions;

♦ to give the mathematical explanation on the jump phenomena, i.e, for the soft potential($\gamma + 2s \geq 0$) the spectral gap of the linearized Boltzmann operator does not exist for the cutoff case but it does for the non-cutoff case.

Denote

$$\Gamma^{\epsilon}(g,h) = \mu^{-1/2} Q^{\epsilon}(\mu^{1/2} g, \mu^{1/2} h).$$

Then the linearized cut-off Boltzmann operator is defined by

$$\mathcal{L}^{\epsilon} g \overset{\text{def}}{=} \underbrace{(-\Gamma^{\epsilon}(\mu^{1/2}, g))}_{=\mathcal{L}_1^{\epsilon} g} + \underbrace{(-\Gamma^{\epsilon}(g, \mu^{1/2}))}_{=\mathcal{L}_2^{\epsilon} g}.$$

It is not difficult to check that $\mathcal{L}^{\epsilon} \geq 0$. BUT, what is the behavior of $\mathcal{L}^{\epsilon}$, i.e.

$$\langle \mathcal{L}^{\epsilon} g, g \rangle_v \sim ???$$

For simplicity, we only address the estimates in the maxwellian molecular case, that is, $\gamma = 0$.

♦ In a physical paper due to Wang-Uhlenbeck, the authors show that $\mathcal{L}_B = \mathcal{L}^0$ is a self-adjoint operator and has explicit eigenvalues and eigenfunctions. In particular, the eigenfunction $E(v)$ takes the form of

$$E(v) = f(|v|^2)Y(\sigma),$$

where $v = |v|\sigma$, f is a radical function and Y is a real spherical harmonic.

♦ Later Pao used this fact to give a first description on the behavior of the operator for $\gamma + 2s \geq 0$.

♦ Mathematically the first attempt to capture the anisotropic structure of the operator were due to AMUXY and G-S. In fact, they introduce two types of anisotropic norms to describe the behavior of the operator which are

$$|||f|||^2 \stackrel{\text{def}}{=} \|f\|^2_{L^2_s} + \underbrace{\int_{v,v_*,\sigma} b(\cos\theta)\mu_*(f' - f)^2 d\sigma dv_* dv}_{\mathcal{E}^0_\mu(f)} \tag{3.4}$$

and

$$\|f\|^2_{N^s} \stackrel{\text{def}}{=} \|f\|^2_{L^2_s} + \underbrace{\int_{v,v'} \langle v\rangle^{s+1/2}\langle v'\rangle^{s+1/2} \frac{|f - f'|^2}{d(v,v')^2} 1_{d(v,v')\le 1} dv dv'}_{\mathcal{A}(f)}, \tag{3.5}$$

where $d(v,v') = \sqrt{|v - v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2}$.

♦ Very recently, two groups gave an explicit description of the anisotropic behavior of the linearized operator. In fact, we have

$$\mathcal{L}_L \sim (-\triangle + |v|^2/4) + (-\triangle_{\mathbb{S}^2}) \sim (-\triangle + |v|^2/4) + |D_v \times v|^2,$$

recalling that $-\triangle_{\mathbb{S}^2} = \sum_{1\le i<j\le 3} (v_i\partial_j - v_j\partial_i)^2$.

(1). In A-H-L, the authors show that

$$\langle \mathcal{L}_B f, f\rangle_v + \|f\|_{L^2}^2 \sim \|f\|_{L^2_s}^2 + \|f\|_{H^s}^2 + \||D_v \times v|^s f\|_{L^2}^2, \tag{3.6}$$

where $|D_v \times v|^s$ is a pseudo-differential operator with the symbol $|\xi \times v|^s$.

(2). By comparing the eigenvalues between Boltzmann and Landau operators, in L-M-P-X , the authors show that

$$\mathcal{L}_B \sim \mathcal{L}_L^s, \tag{3.7}$$

$$\langle \mathcal{L}_B f, f\rangle_v + \|f\|_{L^2}^2 \sim \|f\|_{L^2_s}^2 + \|f\|_{H^s}^2 + \|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2}^2. \tag{3.8}$$

The short review can be summarized as follows:

1. Two anisotropic norms introduced by AMUXY or G-S are useful but are given in an implicit way which helps less to understand the anisotropic property of the operator. BUT they are stable in the limit $\epsilon \to 0$.
2. The anisotropic norms introduce by A-H-L and L-M-P-X are given in a explicit way but they are not stable in the limit $\epsilon \to 0$.

It means that the anisotropic structure hidden in the operator is still mysterious and not captured well.

Theorem

For $\gamma > -3$, we have

$$\langle \mathcal{L}^{\epsilon} f, f\rangle + |f|^2_{L^2_{\gamma/2}} \sim |f|^2_{\epsilon,\gamma}, \tag{4.9}$$

where $|f|^2_{\epsilon,\gamma} \stackrel{def}{=} |W^{\epsilon}((-\Delta_{\mathbb{S}^2})^{1/2})W_{\gamma/2}f|^2_{L^2} + |W^{\epsilon}(D)W_{\gamma/2}f|^2_{L^2} + |W^{\epsilon}W_{\gamma/2}f|^2_{L^2}$ *and*

$$W^{\epsilon}(x) = \langle x\rangle^{s}\phi(\epsilon x) + \epsilon^{-s}(1-\phi(\epsilon x)) \sim \langle x\rangle^{s}1_{|x|\leq 1/\epsilon} + \epsilon^{-s}1_{|x|\geq 1/\epsilon}.$$

Remark

1. It is a unified description for the behavior of the linearized collision operators with and without angular cutoff.

2. It shows that for the local frequency part, the operator behaviors like the fractional Laplacian operator while in the high frequency part the operator still remains the hyperbolic structure.

3. The asymptotical behavior of the weighted L^2 norm explains that why in the limit the operator will have the spectrum gap.

4. The operator $W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})$ is defined by

$$(W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})f)(v) = \sum_{[l(l+1)]^{1/2}\leq 1/\epsilon} \sum_{m=-l}^{l} [l(l+1)]^{s/2} Y_l^m(\sigma) f_l^m(r) + \sum_{[l(l+1)]^{1/2}> 1/\epsilon} \sum_{m=-l}^{l} \epsilon^{-s} Y_l^m(\sigma) f_l^m(r), \qquad v = r\sigma,$$

where $f_l^m(r) = \int_{\mathbb{S}^2} Y_l^m(\sigma) f(r\sigma) d\sigma$.

5. The symbol W^ϵ comes from

$$\iint_{|x-y|\geq\epsilon} \frac{|f(x)-f(y)|^2}{|x-y|^{3+2s}} dxdy + \|f\|^2_{L^2} \sim \|W^\epsilon(D)f\|^2_{L^2}.$$

Reduction of the problem (I)

According to the previous work, we first have

$$\langle \mathcal{L}^{\epsilon} f, f\rangle_v + \|f\|^2_{L^2_{\gamma/2}} \sim \mathcal{E}_{\mu}^{\epsilon,\gamma}(f) + \mathcal{J}^{\epsilon,\gamma}(f)$$

where

$$\mathcal{E}_{g}^{\epsilon,\gamma}(f) \stackrel{\text{def}}{=} \int b^{\epsilon}(\cos\theta)|v-v_*|^{\gamma} g_*(f'-f)^2 d\sigma dv dv_*$$

and

$$\mathcal{J}^{\epsilon,\gamma}(f) \stackrel{\text{def}}{=} \int B^{\epsilon,\gamma} f_*^2(\mu'^{1/2}-\mu^{1/2})^2 d\sigma dv dv_*,$$

We only need to focus on the estimates on $\mathcal{E}_{\mu}^{\epsilon,\gamma}(f)$ and $\mathcal{J}^{\epsilon,\gamma}(f)$

Reduction of the problem (II)

1. For generalized case, we notice the fact

$$\mathcal{E}_\mu^{\epsilon,\gamma}(f) + |f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{E}_\mu^{\epsilon,0}(W_{\gamma/2}f).$$

2. By direct calculation, we have

Lemma

$$\mathcal{J}^{\epsilon,\gamma}(f) + |f|^2_{L^2_{\gamma/2}} \sim |W^\epsilon f|^2_{L^2_{\gamma/2}},$$

Now it suffices to consider the estimates on $\mathcal{E}_\mu^{\epsilon,0}(W_{\gamma/2}f)$.

We have the following lemma:

Lemma

There hold

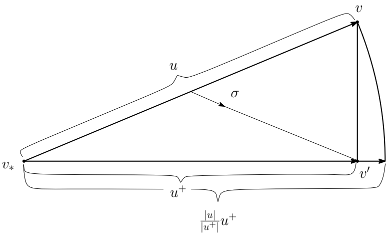
$$\mathcal{E}_\mu^{\epsilon,0}(f) + |W^\epsilon f|_{L^2}^2 \sim |W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})f|_{L^2}^2 + |W^\epsilon(D)f|_{L^2}^2 + |W^\epsilon f|_{L^2}^2, \quad (4.10)$$

and

$$\mathcal{E}_\mu^{\epsilon,0}(f) + |f|_{L^2}^2 \gtrsim |W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})f|_{L^2}^2 + |W^\epsilon(D)f|_{L^2}^2 - |W^\epsilon f|_{L^2}^2. \quad (4.11)$$

Idea of the proof

We look for a new decomposition for the term $f' - f$ contained in $\mathcal{E}_{\mu}^{\epsilon,0}(f)$. Set $u = v - v_*$, then $v' = v_* + u^+$ and $v = v_* + u$, where $u^+ = \frac{u+|u|\sigma}{2}$.



Now assuming $u = r\tau$ with $r = |u|$ and $\tau \in \mathbb{S}^2$, we infer

$$v = v_* + r\tau, v' = v_* + r\frac{\tau+\sigma}{2}.$$

Let $\varsigma = \frac{\tau+\sigma}{|\tau+\sigma|} \in \mathbb{S}^2$. Then we have the following decomposition:

$$\begin{aligned} f(v') - f(v) &= \Big(f(v_* + \frac{|\tau+\sigma|}{2} r\varsigma) - f(v_* + r\varsigma)\Big) + \Big(f(v_* + r\varsigma) - f(v_* + r\tau)\Big) \\ &= \Big(f(v_* + u^+) - f(v_* + |u|\frac{u^+}{|u^+|})\Big) + \Big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\Big). \quad (4.12) \end{aligned}$$

By Bobylev's formula, we have

$$\begin{aligned}\mathcal{E}^{\epsilon,0}_{\mu}(f) &= \frac{1}{(2\pi)^3}\int b^{\epsilon}(\frac{\xi}{|\xi|}\cdot\sigma)(\hat{\mu}(0)|\hat{f}(\xi)-\hat{f}(\xi^+)|^2+2\Re((\hat{\mu}(0)-\hat{\mu}(\xi^-))\hat{f}(\xi^+)\bar{\hat{f}}(\xi))d\xi d\sigma \\ &\overset{\text{def}}{=} \frac{\hat{\mu}(0)}{(2\pi)^3}\mathcal{I}_1+\frac{2}{(2\pi)^3}\mathcal{I}_2,\end{aligned}$$

where $\xi^+=\frac{\xi+|\xi|\sigma}{2}$ and $\xi^-=\frac{\xi-|\xi|\sigma}{2}$.
Applying (4.12) to $\mathcal{I}_1$, we have

$$\begin{aligned}\mathcal{I}_1 &= \int b^{\epsilon}(\frac{\xi}{|\xi|}\cdot\sigma)|\hat{f}(\xi)-\hat{f}(\xi^+)|^2 d\xi d\sigma \\ &\geq \frac{1}{2}\int b^{\epsilon}(\frac{\xi}{|\xi|}\cdot\sigma)|\hat{f}(\xi)-\hat{f}(|\xi|\frac{\xi^+}{|\xi^+|})|^2 d\xi d\sigma \\ &\quad -\int b^{\epsilon}(\frac{\xi}{|\xi|}\cdot\sigma)|\hat{f}(|\xi|\frac{\xi^+}{|\xi^+|})-\hat{f}(\xi^+)|^2 d\xi d\sigma \\ &\overset{\text{def}}{=} \frac{1}{2}\mathcal{I}_{1,1}-\mathcal{I}_{1,2}.\end{aligned}$$

Let $\xi = r\tau$. Then

$$\xi^+ = \frac{\xi + \xi\sigma}{2} = \frac{r(\tau + \sigma)}{2}.$$

We introduce $\varsigma \stackrel{\text{def}}{=} \frac{\tau+\sigma}{|\tau+\sigma|}$. Then

$$\frac{\xi}{|\xi|} \cdot \sigma = 2(\tau \cdot \varsigma)^2 - 1, \quad |\xi| \frac{\xi^+}{|\xi^+|} = r\varsigma.$$

For the change of variable $(\xi, \sigma) \to (r, \tau, \varsigma)$, one has

$$d\xi d\sigma = 4(\tau \cdot \varsigma) r^2 dr d\tau d\varsigma.$$

The fact $b^\epsilon(\cos\theta) = b^\epsilon(2(\tau\cdot\varsigma)^2-1) \sim |\tau-\varsigma|^{-2-2s}\mathbf{1}_{\{\epsilon\le|\tau-\varsigma|\le\sqrt{2-\sqrt{2}}\}}$, we have

$$\begin{aligned}
\mathcal{I}_{1,1} + |f|^2_{L^2} &= 4\int b^\epsilon(2(\tau\cdot\varsigma)^2-1)|\hat{f}(r\tau)-\hat{f}(r\varsigma)|^2(\tau\cdot\varsigma)r^2 dr d\tau d\varsigma + |f|^2_{L^2} \\
&\sim \int \frac{|\hat{f}(r\tau)-\hat{f}(r\varsigma)|^2}{|\tau-\varsigma|^{2+2s}} 1_{|\tau-\varsigma|\ge\epsilon} r^2 dr d\tau d\varsigma + |f|^2_{L^2} \\
&\sim |W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})\hat{f}|^2_{L^2} + |\hat{f}|^2_{L^2} \\
&\sim |W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2} + |f|^2_{L^2}.
\end{aligned}$$

It is not difficult to see that

$$\mathcal{I}_{1,2} + \mathcal{I}_2 \lesssim |W^\epsilon(D)f|^2_{L^2} + |W^\epsilon f|^2_{L^2}.$$

Recall that following the computation of ADWV, it holds

$$\mathcal{E}_{\mu}^{\epsilon,0}(f) + |f|_{L^2}^2 \gtrsim |W^{\epsilon}(D)f|_{L^2}^2.$$

We derive that

$$\mathcal{E}_{\mu}^{\epsilon,0}(f) + |W^{\epsilon}f|_{L^2}^2 \sim |W^{\epsilon}((-\Delta_{\mathbb{S}^2})^{1/2})f|_{L^2}^2 + |W^{\epsilon}(D)f|_{L^2}^2 + |W^{\epsilon}f|_{L^2}^2.$$

Now the proof is complete.

We arrive at

Theorem

For $\gamma = 0$, we have

$$\langle \mathcal{L}^\epsilon f, f\rangle + |f|_{L^2}^2 \sim |W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})f|_{L^2}^2 + |W^\epsilon(D)f|_{L^2}^2 + |W^\epsilon f|_{L^2}^2,$$

To conclude the main theorem, the final step is give the upper bounds for $\mathcal{E}_\mu^{\epsilon,\gamma}(f)$.

Observe that

$$\begin{aligned}\langle \Gamma^\epsilon(g,h),f\rangle &= \langle Q^\epsilon(\mu^{1/2}g,h),f\rangle + \int b^\epsilon(\cos\theta)|v-v_*|^\gamma(\mu_*'^{1/2}-\mu_*^{1/2})g_*hf'd\sigma dv_*dv\\ &\overset{\text{def}}{=} \langle Q^\epsilon(\mu^{1/2}g,h),f\rangle + \mathcal{I}(g,h,f).\end{aligned}$$

Theorem

We have

- *when* $\gamma > -\frac{3}{2}$,

$$|\langle \Gamma^\epsilon(g,h),f\rangle| \lesssim |g|_{L^2}|h|_{\epsilon,\gamma}|f|_{\epsilon,\gamma};$$

- *when* $-3 < \gamma \le -\frac{3}{2}$,

$$|\langle \Gamma^\epsilon(g,h),f\rangle| \lesssim |g|_{L^2}|h|_{\epsilon,\gamma}|f|_{\epsilon,\gamma} + |\langle D\rangle^{s_1}\mu^{1/8}g|_{L^2}|\langle D\rangle^{s_2}W^\epsilon(D)\langle\cdot\rangle^{\gamma/2}h|_{L^2}|f|_{\epsilon,\gamma}.$$

Here $s_1, s_2 \in [0, -\gamma-3/2]$ *with* $s_1+s_2 = -\gamma-3/2$.

Idea of the proof

Applying (4.12) to the functional $\langle Q^\epsilon(g,h), f\rangle$, we then have

$$\langle Q^\epsilon(g,h), f\rangle = \underbrace{\int_{\sigma\in\mathbb{S}^2, v, v_*\in\mathbb{R}^3} b^\epsilon(\cos\theta) g_*(T_{v_*}h)(u)\Big(f(v_*+u^+) - f(v_* + |u|\frac{u^+}{|u^+|})\Big) d\sigma dv_* du}_{=\langle Q_g^\epsilon h, f\rangle}$$
$$+ \underbrace{\int_{\sigma\in\mathbb{S}^2, v, v_*\in\mathbb{R}^3} b^\epsilon(\cos\theta) g_*(T_{v_*}h)(u)\Big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\Big) d\sigma dv_* du}_{=\langle \mathcal{G}_g^\epsilon h, f\rangle_v}.$$

Notice that $|u^+ - |u|\frac{u^+}{|u^+|}| \sim \theta^2|u|$. Then by technical argument, the operator Q_g^ϵ behaves like

$$Q_g^\epsilon \sim C_{|g|} W^\epsilon(v) W^\epsilon(D_v).$$

which may be regarded as the lower order term.

Now we concentrate on the functional $\langle \mathcal{G}_g^\epsilon h, f\rangle_v$. By the change of variables and the symmetric property of the structure, we get

$$\begin{aligned}\langle \mathcal{G}_g^\epsilon h, f\rangle_v &= \frac{1}{2}\int_{\varsigma\in\mathbb{S}^2, v, v_*\in\mathbb{R}^3} H(\varsigma\cdot\tau) g_*\Big((T_{v_*}h)(r\tau) - (T_{v_*}h)(r\varsigma)\Big) \\ &\quad \times\Big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\Big) r^2 d\varsigma d\tau dr dv_*,\end{aligned}$$

where $H(\varsigma\cdot\tau) = b^\epsilon\Big(2(\varsigma\cdot\tau)^2 - 1\Big)4(\varsigma\cdot\tau)1_{\varsigma\cdot\tau\geq\sqrt{2}/2} \sim |\varsigma-\tau|^{-2-2s}1_{\epsilon^2\leq|\varsigma-\tau|^2\leq 2-\sqrt{2}}$. By Cauchy-Schwartz inequality, we finally obtain that

$$|\langle \mathcal{G}_g^\epsilon h, f\rangle_v| \leq \langle \mathcal{G}_{|g|} h, h\rangle_v^{\frac{1}{2}} \langle \mathcal{G}_{|g|} f, f\rangle_v^{\frac{1}{2}}.$$

Observe that $\langle \mathcal{G}_{|g|}^\epsilon h, h\rangle_v = \langle Q^\epsilon(|g|, h), h\rangle_v - \langle \mathcal{Q}_{|g|}^\epsilon h, h\rangle_v$, then we get

$$\langle \mathcal{G}_{|g|}^\epsilon h, h\rangle_v \lesssim \mathcal{E}_{|g|}^{\epsilon,\gamma}(h) + |(W^\epsilon)^2 g|_{L^2}|h|_{\epsilon,\gamma}^2 \lesssim |(W^\epsilon)^2 g|_{L^2}(\mathcal{E}_\mu^{\epsilon,0}(W_{\gamma/2}h) + |h|_{\epsilon,\gamma}^2) \lesssim |(W^\epsilon)^2 g|_{L^2}|h|_{\epsilon,\gamma}^2.$$

Lemma

When $\gamma \geq -2$,

$$|\langle \Gamma^{\epsilon}(g, W_l h) - W_l \Gamma^{\epsilon}(g,h), f\rangle| \lesssim |g|_{L^2} |W_{l+\gamma/2} h|_{L^2} |f|_{\epsilon,\gamma},$$

when $-3 < \gamma < -2$,

$$\begin{aligned} & |\langle \Gamma^{\epsilon}(g, W_l h) - W_l \Gamma^{\epsilon}(g,h), f\rangle| \\ \lesssim \quad & |g|_{L^2} |W_{l+\gamma/2} h|_{L^2} |f|_{\epsilon,\gamma} + |\mu^{1/32} g|_{H^{s_1}} |\mu^{1/32} h|_{H^{s_2}} |f|_{\epsilon,\gamma}, \end{aligned}$$

where $s_1, s_2 \in [0, -\gamma/2 - 1]$ *with* $s_1 + s_2 = -\gamma/2 - 1$.

We consider the Boltzmann equation defined by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^\epsilon f = \Gamma^\epsilon(f, f), & t > 0; \\ f|_{t=0} = f_0. \end{cases} \tag{5.13}$$

We first have the following global existence result:

Theorem

When $-3/2 < \gamma < 0, N \geq 2, q \geq 0$, there exists $\delta_0, \lambda_0, C_0 > 0$ uniformly in ϵ, such that if

$$\|f_0\|_{H^N L^2} \leq \delta_0,$$

then the Cauchy problem (5.13) *admits a unique global solution f^ϵ such that*

$$\|f^\epsilon(t)\|^2_{H^N L^2_q} + \lambda_0 \int_0^t \|f^\epsilon(s)\|^2_{\mathcal{P}^{N,0,\epsilon,q+\gamma/2}} ds \leq C_0 \|f_0\|^2_{H^N L^2_q},$$

for any $t \geq 0$.

For the propagation of the full regularity:

Theorem

Additionally, if $N \geq k+2$, then there exist constants $\lambda_k, C_k > 0$ and a sequence $\{q_j\}_{0\leq j\leq k}$ with $q_k \geq 2$ and $q_j \geq q_{j+1} - \gamma$ for any $0 \leq j \leq k-1$, uniformly in ϵ, such that for any $t \geq 0$, the solution f^ϵ satisfies

$$\sum_{0\leq j\leq k} \|f(t)\|^2_{H_x^{N-j}H^j_{q_j}} + \lambda_k \int_0^t \sum_{0\leq j\leq k} \|f(s)\|^2_{\mathcal{P}^{N-j,j,\epsilon,q_j+\gamma/2}} ds$$

$$\leq \quad C_k(\{\|f_0\|^2_{H_x^{N-j}H^{j-1}_{q_j}}\}_{1\leq j\leq k}) \sum_{0\leq j\leq k} \|f_0\|^2_{H_x^{N-j}H^j_{q_j}}.$$

We first derive polynomial decay of the $H_x^N L_l^2$ norm.

Theorem

When $-3/2 < \gamma < 0, N \geq 2, l \geq 0$, *fix* $p > 0$, *take* $q = l - \gamma p/2$, *let* f^ϵ *be the solution to the Cauchy problem* (5.13) *as in theorem 5.1, then it enjoys the following polynomial decay*

$$\|f^\epsilon(t)\|^2_{H_x^N L_l^2} \lesssim C^2(f_0; \epsilon, l, p)(1 + \frac{t}{p})^{-p}, \tag{5.14}$$

where $C^2(f_0; \epsilon, l, p) = \|f_0\|^2_{H_x^N L_l^2} + \epsilon^{2sp}\|f_0\|^2_{H_x^N L^2_{l-\gamma p/2}}$.

We next derive polynomial decay of high order $v-$regularity.

Theorem

When $-3/2 < \gamma < 0, k \geq 1, N \geq k+2$, *let* $\{q_i\}_{0\leq i\leq k}$ *be a sequence of numbers such that* $q_i \geq q_{i+1} - \gamma$ *for any* $0 \leq i \leq k-1$ *with* $q_k \geq 2$. *Fix* $p > 0$. *Suppose the initial datum satisfy the condition in theorem 5.1, let* f^ϵ *be the solution to the Cauchy problem* (5.13), *then it enjoys the following polynomial decay estimate*

$$\sum_{0\leq i\leq k} \|f^\epsilon(t)\|^2_{H^{N-i}_x H^i_{q_i}} \lesssim C_k C^2(f_0;\epsilon,p,\{q_i\}_{0\leq i\leq k})(1+\frac{t}{p})^{-p},$$

where $C^2(f_0;\epsilon,p,\{q_i\}_{0\leq i\leq k}) = \sum_{0\leq i\leq k} \|f_0\|^2_{H^{N-i}_x H^i_{q_i}} + \epsilon^{2sp} \sum_{0\leq i\leq k} \|f_0\|^2_{H^{N-i}_x H^i_{q_i-\gamma p/2}}$ *and* C_k *is a constant depending on* $\{\|f_0\|^2_{H^{N-i}_x H^{i-1}_{q_i}}\}_{1\leq i\leq k}$.

Finally we have

Theorem

When $-3/2 < \gamma < 0, N \geq 3, p > 0$, *let* f^ϵ *be global solution to the Cauchy problem* (5.13) *as in theorem. Suppose* $\|f_0\|_{H_x^{N-1}H^1_{\gamma/2+s+1-\gamma p/2}}, \|f_0\|_{H_x^N L^2_{s-\gamma/2+1-\gamma p/2}} < \infty$. *Then there exists* $\delta_0 > 0$ *uniformly in* ϵ, *such that if*

$$\|f_0\|_{H_x^N L^2} \leq \delta_0,$$

there holds

$$\|f^\epsilon(t)\|_{H_x^{N-1}L^2} \lesssim \|f_0\|_{H_x^{N-1}L^2} e^{-\lambda t} + C^2(f_0; \epsilon, s, \gamma, p)\epsilon^2(1 + \frac{t}{p})^{-p}, \tag{5.15}$$

where

$$\begin{aligned} C^2(f_0; \epsilon, s, \gamma, p) &= \|f_0\|^2_{H_x^{N-1}H^1_{s+\gamma/2+1}} + \epsilon^{2sp}\|f_0\|^2_{H_x^{N-1}H^1_{s+\gamma/2+1-\gamma p/2}} + \|f_0\|^2_{H_x^N L^2_{s-\gamma/2+1}} \\ &\quad + \epsilon^{2sp}\|f_0\|^2_{H_x^N L^2_{s-\gamma/2+1-\gamma p/2}}. \end{aligned}$$

Idea of the proof

- Step 1: Set $f^l = \chi_\epsilon f$ and $f^h = \chi^\epsilon f$ where $\chi^\epsilon = 1 - \chi_\epsilon$. Then we may prove that

$$\begin{aligned}\mathcal{E}_N(f^l(t)) &\le e^{-\lambda t}\mathcal{E}_N(f^l(0)) + C\int_0^t e^{-\lambda(t-s)}\|f(s)\cdot \mathbf{1}_{|v|\ge 1/\sqrt{2}\epsilon}\|^2_{\mathcal{P}^{N,\epsilon,\gamma/2}}ds \\ &\le e^{-\lambda t}\mathcal{E}_N(f^l(0)) + C^2(f_0;\epsilon,s,\gamma,k)\epsilon^2(1+\frac{t}{k})^{-k}.\end{aligned}$$

- Step 2:

$$\begin{aligned}\mathcal{E}_N(f(t)) &\le \mathcal{E}_N(f^l(t)) + \mathcal{E}_N(f^h(t)) \\ &\le \mathcal{E}_N(f^l(t)) + \epsilon^2\|f^\epsilon(t)\|^2_{H^N_x H^1_{s+\gamma/2+1}} \\ &\le e^{-\lambda t}\mathcal{E}^M(f^l(0)) + C^2(f_0;\epsilon,s,\gamma,k)\epsilon^2(1+\frac{t}{k})^{-k}.\end{aligned}$$

Conclusions

♦ We solve the Boltzmann equation with and without angular cutoff in a unified framework in the close-to-equilibrium setting. The uniform estimate will imply the asymptotical formula: $f^\epsilon = f + O(\epsilon^{2-2s})$.

♦ We show that for the soft potential $\gamma \in [-2s, 0]$, the rate of the convergence is continuous with respect to the parameter ϵ. In other words, there is no jump for the rate from Polynomial decay to Exponential decay as $\epsilon \to 0$. In some sense we give the explanation to the jump phenomenon of the spectral gap of the linearized collision operator in the same limit.

Open questions

♦ Can we give a quantitative estimates for the jump phenomena of the spectrum gap of the linearized operator in the limit?

♦ What happened for the linearized Boltzmann collision operator in the grazing collsions limit?

Thank you for your attention!