Propagation of boundary-induced discontinuity in stationary radiative transfer and its application to the optical tomographghy

Daisuke Kawagoe
joint work with I-Kun Chen

Graduate School of Informatics, Kyoto University

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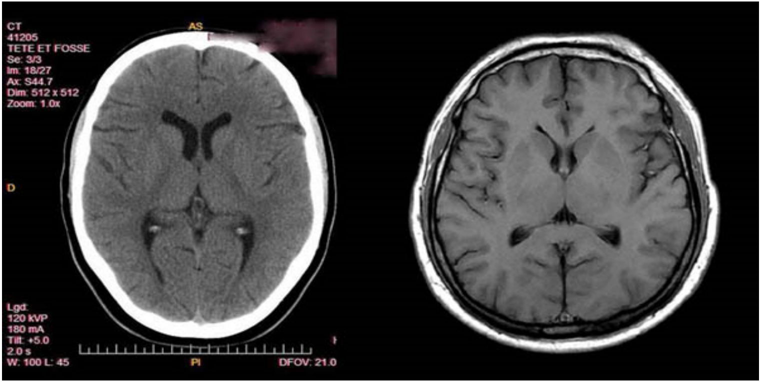
Contents

Contents

Optical Tomography

- A new technique of medical imaging
- Using near-infrared light
- Noninvasive



(cf. X-ray CT and MRI, http://www.fonar.com/news/100511.htm)

Stationary Transport Equation

Stationary Transport Equation(STE)

$$\xi \cdot \nabla_x f(x,\xi) + \mu_t(x) f(x,\xi) = \mu_s(x) \int_{S^{d-1}} p(x,\xi,\xi') f(x,\xi')\, d\sigma_{\xi'}.$$

- $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $d = 2$ or $d = 3$.
- $\xi = (\xi_1, \dots, \xi_d) \in S^{d-1}$ (unit sphere)
- $f(x,\xi)$: density of photon at position x with direction ξ.
- μ_t : attenuation coefficient / μ_s : scattering coefficient
- p : scattering phase function
- $\xi \cdot \nabla_x f(x,\xi) := \frac{d}{dt} f(x + t\xi, \xi)|_{t=0}$
- $d\sigma_{\xi'}$: Lebesgue measure on S^{d-1}

Incoming boundary condition

- $\Omega \subset \mathbb{R}^d$: convex, C^1 boundary $\partial\Omega$
- $\Gamma_\pm := \{(x,\xi) \in \partial\Omega \times S^{d-1} | \pm n(x) \cdot \xi > 0\}$
- $n(x)$: outer normal vector at $x \in \partial\Omega$
- f_0 : given function on Γ_-

Incoming boundary condition

$$f(x,\xi) = f_0(x,\xi), \quad (x,\xi) \in \Gamma_-.$$

Goal

Goal

To reconstruct the attenuation coefficient μ_t from "observed" data, f_0 and $f|_{\Gamma_+}$, with the coefficient μ_s and the phase function p unknown.

$$\begin{cases} \xi \cdot \nabla_x f(x,\xi) + \mu_t(x) f(x,\xi) = \mu_s(x) \int_{S^{d-1}} p(x,\xi,\xi') f(x,\xi')\, d\sigma_{\xi'}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,\xi) \in \Omega \times S^{d-1}, \\ f(x,\xi) = f_0(x,\xi), \quad (x,\xi) \in \Gamma_-. \end{cases}$$

- $\Omega \subset \mathbb{R}^d$: bounded, convex, C^1 boundary $\partial\Omega$

Propagation of boundary-induced disconituity

D. S. Anikonov, I. V. Prokhorov, A. E. Kovtanyuk, Investigation of scattering and absorbing media by the methods of X-ray tomography, *J. Inv. Ill-Posed Prob.*, **1**, no. 4, pp. 259–281, (1993)

- Discontinuities of boundary data with respect to ξ propagate along positive characteristic lines with exponential decay.
- In 3 dimentional bounded convex domain.
- With piecewise continuous coefficients and kernel.

X-ray transform

$$\begin{cases} \xi \cdot \nabla_x g(x,\xi) + \mu_t(x) g(x,\xi) = 0, & (x,\xi) \in \Omega \times S^{d-1}, \\ g(x,\xi) = g_0(x,\xi), & (x,\xi) \in \Gamma_-. \end{cases}$$

$\Omega \subset \mathbb{R}^d$: bounded, convex, C^1 boundary $\partial\Omega$

$\Rightarrow$

$$g(x,\xi) = \exp\left(- \int_0^{\tau_-(x,\xi)} \mu_t(x - r\xi)\, dr \right) g_0(x - \tau_-(x,\xi)\xi, \xi),$$

where $\tau_-(x,\xi) := \inf\{t > 0 | x - t\xi \notin \Omega\}$.

For $\mu_t \in L^2(\mathbb{R}^d)$ with $supp \mu_t \subset \Omega$,

$$X\mu_t(x,\xi) := \int_{-\infty}^{\infty} \mu_t(x - r\xi)\, dr$$

is called the X-ray transform of μ_t.

X-ray transform

Fact

We can reconstruct μ_t from its X-ray transform $X\mu_t$.

F. Natterer, *The Mathematics of Computerized Tomography*, SIAM, Germany, (2001)

Propagation of boundary-induced discontinuity

K. Aoki, C. Bardos, C. Dogbe, F. Golse, A note on the propagation of boundary induced discontinuities in kinetic theory, *Math. Models Methods Appl. Sci.* 11, no. 9 pp. 1581–1595, (2001).

- Discontinuities of boundary data with respect to x propagate along positive characteristic lines with exponential decay.
- In 2 dimentional half space.
- With $\mu_t = \mu_s = 1$ and $p = 1/(2\pi)$.

We would like to improve this result in our situation.

Contents

Stationary Transport Equation

Stationary Transport Equation(STE)

$$\xi \cdot \nabla_x f(x,\xi) + \mu_t(x) f(x,\xi) = \mu_s(x) \int_{S^{d-1}} p(x,\xi,\xi') f(x,\xi')\, d\sigma_{\xi'}. \quad (1)$$

- $x = (x_1, \ldots, x_d) \in \Omega \subset \mathbb{R}^d$, $d = 2$ or 3, bounded, convex, C^1
- $\xi = (\xi_1, \ldots, \xi_d) \in S^{d-1}$ (the unit sphere)
- f : unknown, bounded
- μ_t : attenuation coefficient / μ_s : scattering coefficient
- p : scattering phase function
- $\xi \cdot \nabla_x f(x,\xi) := \frac{d}{dt} f(x + t\xi, \xi)|_{t=0}$
- $d\sigma_{\xi'}$: Lebesgue measure on S^{d-1}

Incoming boundary condition

- $\Omega \subset \mathbb{R}^d$: bounded, convex, C^1 boundary $\partial\Omega$
- $\Gamma_\pm := \{(x,\xi) \in \partial\Omega \times S^{d-1} | \pm n(x)\cdot\xi > 0\}$
- $n(x)$: outer normal vector at $x \in \partial\Omega$
- f_0 : given function on Γ_-

Incoming boundary condition

$$f(x,\xi) = f_0(x,\xi), \quad (x,\xi) \in \Gamma_-. \tag{2}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$: $d = 2$ or 3, bounded, convex
- $\partial\Omega$: C^1
- $\overline{\Omega} = \cup_{j=1}^N \overline{\Omega_j}$, $\quad \Omega_j$, $1 \le j \le N$: disjoint subdomains of Ω
- $\partial\Omega_j$, $1 \le j \le N$: piecewise C^1
- $\Omega_0 := \cup_{j=1}^N \Omega_j$.
- For all $(x, \xi) \in \Omega \times S^{d-1}$, there exist positive integer $l(x, \xi)$ and real numbers $\{t_j(x, \xi)\}_{j=1}^{l(x,\xi)}$ such that

$$0 \le t_1(x, \xi) < t_2(x, \xi) < \cdots < t_{l(x,\xi)}, \quad x - t_j(x, \xi)\xi \in \partial\Omega_0$$

and

$$\sup_{(x,\xi) \in \Omega \times S^{d-1}} l(x, \xi) < \infty.$$

Assumptions

- μ_t, μ_s: nonnegative bounded functions on $\mathbb{R}^d$ such that
  - continuous on Ω_0,
  - $\mu_t(x) \geq \mu_s(x)$ for $x \in \Omega_0$,
  - $\mu_t(x)(= \mu_s(x)) = 0$ for $x \in \mathbb{R}^d \backslash \Omega_0$.

- p : nonnegative bounded function on $\mathbb{R}^d \times S^{d-1} \times S^{d-1}$ such that
  - continuous on $\Omega_0 \times S^{d-1} \times S^{d-1}$,
  - $\int_{S^{d-1}} p(x, \xi, \xi')\, d\sigma_{\xi'} = 1$ for all $(x, \xi) \in \Omega_0 \times S^{d-1}$,
  - $p(x, \xi, \xi') = 0$ for $(x, \xi, \xi') \in (\mathbb{R}^d \backslash \Omega_0) \times S^{d-1} \times S^{d-1}$.

Main results

Theorem 1 (Chen, K.)

Suppose that a boundary data f_0 is bounded and that it satisfies at least one of the following two conditions.

1. $f_0(\cdot, \xi)$ *is continuous on* $\Gamma_{-,\xi}$ *for almost all* $\xi \in S^{d-1}$.
2. $f_0(x, \cdot)$ *is continuous on* $\Gamma_{-,x}$ *for almost all* $x \in \partial\Omega$.

Then, there exists a unique solution f to (1)-(2) and f satisfies the following relation:

$$disc(f) = \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \leq t < \tau_+(x^*, \xi^*)\},$$

where $disc(f)$ is the set of discontinuous points for a function f.

- $\tau_+(x, \xi) := \inf\{t > 0 | x + t\xi \notin \Omega\}$
- $\Gamma_{-,\xi} := \{x \in \partial\Omega | n(x) \cdot \xi < 0\}$
- $\Gamma_{-,x} := \{\xi \in S^{d-1} | n(x) \cdot \xi < 0\}$

Definition of solution

Definition

Let $D := (\Omega \times S^{d-1}) \cup \Gamma_-$. A bounded function f on D is a solution to (1)-(2) if it satisfies the following conditions.

1. For all $(x, \xi) \in \Omega_0 \times S^{d-1}$, the directional derivative

$$\xi \cdot \nabla_x f(x, \xi) := \frac{d}{dt} f(x + t\xi, \xi)|_{t=0}$$

   is defined.

2. f satisfies the equation (1) in $\Omega_0 \times S^{d-1}$.
3. f satisfies the boundary condition (2) on Γ_-.
4. $f(\cdot, \xi)$ is continuous on the line $\{x + t\xi | t \in \mathbb{R}\} \cap (\Omega \cup \Gamma_{-,\xi})$ for all $(x, \xi) \in D$.

Main results

Theorem 2 (Chen, K.)

Let f be the solution to the boundary value problem (1)-(2). *Then, f can be extended upto Γ_+, denoted by $\overline{f}$, by*

$$\overline{f}(x,\xi) := \lim_{t \downarrow 0} f(x - t\xi, \xi), \quad (x,\xi) \in \Gamma_+.$$

Moreover, the following relation holds:

$$disc(\overline{f}) = \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \leq t \leq \tau_+(x_*, \xi_*)\}. \tag{3}$$

- $\tau_+(x,\xi) := \inf\{t > 0 | x + t\xi \notin \Omega\}$

Main results

- $d = 3$
- γ: simple closed curve in $\partial\Omega$
- A, B: two open surfaces such that $\partial\Omega = A \cup B \cup \gamma$ and $A \cap B = A \cap \gamma = B \cap \gamma = \emptyset$
- $$f_0(x,\xi) = \begin{cases} 1, & (x,\xi) \in ((A \cup \gamma) \times S^2) \cap \Gamma_-, \\ 0, & (x,\xi) \in (B \times S^2) \cap \Gamma_-. \end{cases}$$
- $$[f](\overline{x},\overline{\xi}) := \lim_{\substack{x \to \overline{x}, \\ P(x,\overline{\xi}) \in (A\cup\gamma)}} f(x,\overline{\xi}) - \lim_{\substack{x \to \overline{x}, \\ P(x,\overline{\xi}) \in B}} f(x,\overline{\xi})$$

  for $(\overline{x},\overline{\xi}) \in disc(f)$, where

  $$P(x,\xi) := x - \tau_-(x,\xi)\xi.$$

Main results

Theorem 3 (Chen, K.)

Let $(x^*, \xi^*) \in disc(\overline{f}|_{\Gamma_+})$. *Then,*

$$[\overline{f}](x^*, \xi^*) = \exp\left(-\int_0^{\tau_-(x^*,\xi^*)} \mu_t(x^* - r\xi^*)\, dr \right) = \exp\left(-X\mu_t(x^*, \xi^*)\right). \tag{4}$$

Contents

Strategy of proof

1. Reduction to an integral equation
2. Costruction of the unique solution to the integral equation
3. Regularity discussion
4. Trace theorem
5. Decay of jump discontinuity

Reduction to an integral equation

Integration STE (1) along the characteristic line $\{x - t\xi | t > 0\}$ with the incoming boundary condition (2).

$$f(x,\xi) = \exp\left(-M_t(x,\xi;\tau_-(x,\xi))\right) f_0(P(x,\xi),\xi) + \int_0^{\tau_-(x,\xi)} \mu_s(x - s\xi) \exp\left(-M_t(x,\xi;s)\right) \times \int_{S^{d-1}} p(x - s\xi,\xi,\xi') f(x - s\xi,\xi')\, d\sigma_{\xi'}\, ds. \tag{5}$$

- $\tau_-(x,\xi) = \inf\{t > 0 | x - t\xi \notin \Omega\}$
- $M_t(x,\xi;s) := \int_0^s \mu_t(x - r\xi)\, dr$
- $P(x,\xi) = x - \tau_-(x,\xi)\xi$

Costruction of the solution

$$f^{(0)}(x,\xi) := \exp\left(-M_t(x,\xi;\tau_-(x,\xi))\right) f_0(P(x,\xi),\xi) \qquad (6)$$

$$f^{(n+1)}(x,\xi) := \int_0^{\tau_-(x,\xi)} \mu_s(x-s\xi)\exp\left(-M_t(x,\xi;s)\right) \times \int_{S^{d-1}} p(x-s\xi,\xi,\xi')f^{(n)}(x-s\xi,\xi')\,d\sigma_{\xi'}\,ds. \qquad (7)$$

Then, $f(x,\xi) := \sum_{n=0}^{\infty} f^{(n)}(x,\xi)$ is the unique solution.

Lemma

$$\sup_{(x,\xi)\in D} |f^{(n+1)}(x,\xi)| \le M \sup_{(x,\xi)\in D} |f^{(n)}(x,\xi)|, \quad \forall n \ge 0.$$

- $M_t(x,\xi;s) := \int_0^s \mu_t(x-r\xi)\,dr$
- $M := \sup_{(x,\xi)\in D}(1-\exp(-M_t(x,\xi;\tau_-(x,\xi)))) < 1$
- $P(x,\xi) = x - \tau_-(x,\xi)\xi$

Regularity discussion

$$f(x,\xi) = F_0(x,\xi) + F_1(x,\xi)$$

- $F_0 := f^{(0)}$: discontinuous on D,

  $disc(F_0) = \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \leq t < \tau_+(x_*, \xi_*)\}$.

- $F_1 := \sum_{n=1}^{\infty} f^{(n)}$: continuous on D

$$f^{(0)}(x,\xi) := \exp\left(-M_t(x,\xi;\tau_-(x,\xi))\right) f_0(P(x,\xi),\xi)$$

$$f^{(n+1)}(x,\xi) := \int_0^{\tau_-(x,\xi)} \mu_s(x - s\xi) \exp\left(-M_t(x,\xi;s)\right) \times \int_{S^{d-1}} p(x - s\xi, \xi, \xi') f^{(n)}(x - s\xi, \xi')\, d\sigma_{\xi'}\, ds.$$

Discontinuity of F_0

$$F_0(x,\xi) = f^{(0)}(x,\xi) = \exp\left(-M_t(x,\xi;\tau_-(x,\xi))\right) f_0(x-\tau_-(x,\xi)\xi,\xi)$$

Proposition 1

$$disc(F_0) = \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \le t < \tau_+(x_*, \xi_*)\},$$

where $disc(f)$ is the set of discontinuity points for a function f.

Lemma

$M_t(x,\xi;\tau_-(x,\xi))$ is continuous on $\Omega \times S^{d-1}$.

$$(x,\xi) \in disc(F_0) \Leftrightarrow (x-\tau_-(x,\xi)\xi,\xi) \in disc(f_0),$$

- $\tau_\pm(x,\xi) := \inf\{t > 0 | x \pm t\xi \notin \Omega\}$
- $M_t(x,\xi;s) := \int_0^s \mu_t(x-r\xi)\,dr$

Continuity of $f^{(n)}$

$$f^{(n+1)}(x,\xi) := \int_0^{\tau_-(x,\xi)} \mu_s(x-s\xi)\exp\left(-M_t(x,\xi;s)\right) \times \int_{S^{d-1}} p(x-s\xi,\xi,\xi')f^{(n)}(x-s\xi,\xi')\,d\sigma_{\xi'}ds.$$

Proposition 2

$$f^{(n)} \in C_b(D) \Rightarrow f^{(n+1)} \in C_b(D)$$

Lemma

$M_t(x,\xi;s)$ is continuous on $\Omega \times S^{d-1}$ for all $s \geq 0$.

- $\tau_-(x,\xi) := \inf\{t > 0 | x - t\xi \notin \Omega\}$
- $M_t(x,\xi;s) := \int_0^s \mu_t(x-r\xi)\,dr$
- $C_b(D)$: the function space of all bounded functions on D

$$f_0 \to f^{(0)} \to f^{(1)} \to f^{(2)} \to f^{(3)} \to \dots$$

$$f_0 \to f^{(0)} \to f^{(1)} \to f^{(2)} \to f^{(3)} \to \dots$$

Q. $f^{(1)} \in C_b(D)$?

Regularity of $f^{(1)}$

$$\begin{aligned} f^{(1)}(x,\xi) &= \int_0^{\tau_-(x,\xi)} \mu_s(x-s\xi)\exp\left(-M_t(x,\xi;s)\right) \\ &\qquad \times \int_{S^{d-1}} p(x-s\xi,\xi,\xi')f^{(0)}(x-s\xi,\xi')\,d\sigma_{\xi'}\,ds \\ &= \int_0^{\tau_-(x,\xi)} \mu_s(x-s\xi)\exp\left(-M_t(x,\xi;s)\right)G(x-s\xi,\xi)\,ds. \end{aligned}$$

- $G(x,\xi) = \int_{S^{d-1}} p(x,\xi,\xi')\exp\left(-M_t(x,\xi';\tau_-(x,\xi'))\right) \times f_0(P(x,\xi'),\xi')\,d\sigma_{\xi'}$
- $f^{(0)}(x,\xi) := \exp\left(-M_t(x,\xi;\tau_-(x,\xi))\right) f_0(P(x,\xi),\xi)$
- $P(x,\xi) = x - \tau_-(x,\xi)\xi$

Proposition 3

Under the assumption in Theorem 1, $G \in C_b(\Omega_0 \times S^{d-1})$.

<u>Case 1.</u> $f_0(\cdot, \xi)$ is continuous on $\Gamma_{-,\xi}$ for almost all $\xi \in S^{d-1}$.

$$G(x,\xi) = \int_{S^{d-1}} p(x,\xi,\xi') \exp\left(-M_t(x,\xi';\tau_-(x,\xi'))\right) \times f_0(P(x,\xi'),\xi')\, d\sigma_{\xi'}.$$

- τ_- is continuous on $\Omega \times S^{d-1}$. (Guo, 2010)
- $|p(x,\xi,\xi') \exp\left(-M_t(x,\xi';\tau_-(x,\xi'))\right) f_0(x - \tau_-(x,\xi')\xi', \xi')|$
  $\leq \sup_{(x,\xi,\xi') \in \Omega \times S^{d-1} \times S^{d-1}} |p(x,\xi,\xi')| \sup_{(x,\xi) \in \Gamma_-} |f_0(x,\xi)|$.
- p is continuous on $\Omega_0 \times S^{d-1} \times S^{d-1}$

$\Rightarrow$ G is continuous on $\Omega_0 \times S^{d-1}$.
(by the dominated convergence theorem)

Regularity of G

Case 2. $f_0(x,\cdot)$ is continuous on $\Gamma_{-,x}$ for almost all $x \in \partial\Omega$.

Lemma

$$d\sigma_{\xi'} = \frac{|n(y)\cdot(x-y)|}{|x-y|^d}\, d\sigma_y \text{ via } y = P(x,\xi') = x - \tau_-(x,\xi')\xi'$$

$$\begin{aligned}
G(x,\xi) &= \int_{S^{d-1}} p(x,\xi,\xi') \exp\left(-M_t(x,\xi';\tau_-(x,\xi'))\right) \\
&\qquad\qquad\qquad \times f_0(P(x,\xi'),\xi')\, d\sigma_{\xi'} \\
&= \int_{\partial\Omega} p\left(x,\xi,\frac{x-y}{|x-y|}\right) \exp\left(-M_t\left(x,\frac{x-y}{|x-y|};|x-y|\right)\right) \\
&\qquad \times f_0\left(y,\frac{x-y}{|x-y|}\right) \frac{|n(y)\cdot(x-y)|}{|x-y|^d}\, d\sigma_y.
\end{aligned}$$

Regularity of G

Proposition 3

Under the assumption in Theorem 1, $G \in C_b(\Omega_0 \times S^{d-1})$.

Case 2. $f_0(x, \cdot)$ is continuous on $\Gamma_{-,x}$ for almost all $x \in \partial\Omega$.
Fix $\overline{x} \in \Omega_0$.

- $$p\left(x, \xi, \tfrac{x-y}{|x-y|}\right) \exp\left(-M_t\left(x, \tfrac{x-y}{|x-y|}; |x-y|\right)\right) f_0\left(y, \tfrac{x-y}{|x-y|}\right) \tfrac{|n(y)\cdot(x-y)|}{|x-y|^d}$$
  is continuous at $(\overline{x}, \xi) \in \Omega_0 \times S^{d-1}$.
- $$\left|p\left(x, \xi, \tfrac{x-y}{|x-y|}\right) \exp\left(-M_t\left(x, \tfrac{x-y}{|x-y|}; |x-y|\right)\right) f_0\left(y, \tfrac{x-y}{|x-y|}\right) \tfrac{|n(y)\cdot(x-y)|}{|x-y|^d}\right| \leq \frac{\sup_{(x,\xi,\xi')\in\Omega_0\times S^{d-1}\times S^{d-1}} |p(x,\xi,\xi')| \sup_{(x,\xi)\in\Gamma_-} |f_0(x,\xi)|}{(|\overline{x}-y| - \epsilon/2)^d}$$
  for all $x \in B_{\epsilon/2}(\overline{x})$, with $\epsilon = d(\overline{x}, \partial\Omega_0)$.

$\Rightarrow$ G is continuous on $\Omega_0 \times S^{d-1}$.
(by the dominated convergence theorem)

Regularity of $f^{(1)}$

- $G \in C_b(\Omega_0 \times S^{d-1})$
- $f^{(1)}(x,\xi) = \int_0^{\tau_-(x,\xi)} \mu_s(x - s\xi) \exp\left(-M_t(x,\xi;s)\right) G(x - s\xi, \xi)\, ds.$

$$\Rightarrow f^{(1)} \in C_b(D)$$

Regularity discussion

$$f(x,\xi) = F_0(x,\xi) + F_1(x,\xi).$$

- $F_0 = f^{(0)}$: discontinuous on D,

  $disc(F_0) = \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \le t < \tau_+(x_*, \xi_*)\}$.

- $F_1 = \sum_{n=1}^{\infty} f^{(n)}$: continuous on D

$$\begin{aligned} disc(f) =& disc(F_0) \\ =& \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \le t < \tau_+(x_*, \xi_*)\}. \end{aligned}$$

Trace theorem

Lemma

For all $(x, \xi) \in \Gamma_+$, the trace

$$\overline{f}(x, \xi) := \lim_{t \downarrow 0} f(x - t\xi, \xi)$$

exists.

By the fundamental theorem of calculus,

$$f(x, \xi) = f_0(x - \tau_-(x, \xi)\xi, \xi) - \int_0^{\tau_-(x,\xi)} \xi \cdot \nabla_x f(x - s\xi, \xi)\, ds.$$

Trace theorem

$$
\begin{aligned}
|f(x,\xi) - f(x - t\xi, \xi)| &= \left| \int_t^{\tau_-(x,\xi)} \xi \cdot \nabla_x f(x - s\xi, \xi)\, ds \right. \\
&\qquad \left. - \int_0^{\tau_-(x,\xi)} \xi \cdot \nabla_x f(x - s\xi, \xi)\, ds \right| \\
&= \left| \int_0^t \xi \cdot \nabla_x f(x - s\xi, \xi)\, ds \right| \\
&\le \left(\sup_{(x,\xi) \in \Omega \times S^{d-1}} |\xi \cdot \nabla_x f(x,\xi)| \right) t.
\end{aligned}
$$

- $\tau_-(x - t\xi, \xi) = \tau_-(x,\xi) - t$

Trace theorem

- $\{\overline{f}^{(n)}\}$: extended functions of $\{f^{(n)}\}$ upto Γ_+
- $\overline{F_0}(x,\xi) := \overline{f}^{(0)}(x,\xi)$
- $\overline{F_1}(x,\xi) := \sum_{n=1}^{\infty} \overline{f}^{(n)}$
- $\overline{D} := D \cup \Gamma_+$

Proposition 4

$$disc(\overline{F_0}) = \{(x_* + t\xi_*, \xi_*) | (x_*, \xi_*) \in disc(f_0), 0 \leq t \leq \tau_+(x_*, \xi_*)\}.$$

Proposition 5

Under the assumption in Theorem 1, $\overline{F_1}$ is bounded countinuous on $\overline{D}$.

Decay of jump discontinuity

Lemma

For $(x^*,\xi^*) \in disc(\overline{f}|_{\Gamma_+})$,

$$[\overline{F_0}](x^*,\xi^*) = \exp\Big(-M_t(x^*,\xi^*;\tau_-(x^*,\xi^*))\Big).$$

- $\overline{F_0}(x,\xi^*) = \exp\Big(-M_t(x,\xi^*;\tau_-(x,\xi^*))\Big), \quad x \in \overline{\Omega_{A,\xi^*}}$.
  $\overline{\Omega_{A,\xi^*}} = \{x \in \overline{\Omega} | P(x,\xi^*) \in (A \cup \gamma) \cap \Gamma_{-,\xi^*}\}$.
- $\overline{F_0}(x,\xi^*) = 0, \quad x \in \overline{\Omega_{B,\xi^*}}$.
  $\overline{\Omega_{B,\xi^*}} = \{x \in \overline{\Omega} | P(x,\xi^*) \in B \cap \Gamma_{-,\xi^*}\}$.

Decay of jump discontinuity

Lemma

For $(x^*, \xi^*) \in disc(\overline{f}|_{\Gamma_+})$,

$$[\overline{F_1}](x^*, \xi^*) = 0.$$

$$\begin{aligned} \therefore [\overline{f}](x^*, \xi^*) =& [F_0](x^*, \xi^*) + [F_1](x^*, \xi^*) \\ =& \exp\Big(-M_t(x^*, \xi^*; \tau_-(x^*, \xi^*))\Big). \\ =& \exp\Big(-X\mu_t(x^*, \xi^*)\Big). \end{aligned}$$

Albedo operater

Albedo operator

$$\mathcal{A} : L^1(\Gamma_-, |n(x)\cdot\xi|\, d\sigma_x\, d\sigma_\xi) \to L^1(\Gamma_+, |n(x)\cdot\xi|\, d\sigma_x\, d\sigma_\xi)$$

- M. Choulli, P. Stefanov, An inverse boundary value problem for the stationary transport equation, *Osaka J. Math.*, **36**(1), pp. 87–104, (1998)
- J. N. Wang, Stability estimates of an inverse problem for the stationary transport equation, *Ann. Inst. Henri Poincare*, **70**, pp. 473–495, (1999)
- G. Bal, A. Jollivet, Stability estimates in stationary inverse transport, *Inverse Probl. Imaging*, **2**:(4), pp. 427–454, (2008)

X-ray transform

Proposition 6

For each $\alpha \in \mathbb{R}$, there exist positive constants $c(\alpha, d)$, $C(\alpha, d)$ such that for $f \in C_0^\infty(B_R)$,

$$c(\alpha, d)\|f\|_{H_0^\alpha(B_R)} \leq \|Xf\|_{H^{\alpha+1/2}(T)} \leq C(\alpha, d)\|f\|_{H_0^\alpha(B_R)}.$$

- B_R: the ball in $\mathbb{R}^d$ with its center at the origin and the radius R.
- $\xi^\perp := \{x \in \mathbb{R}^d | x \cdot \xi = 0\}$.
- $T := \{(x, \xi) \in \mathbb{R}^d \times S^{d-1} | x \in \xi^\perp\}$.
- $\|f\|^2_{H^\alpha(T)} := \int_{S^{d-1}} \int_{\xi^\perp} (1 + |\eta|^2)^\alpha |\hat{f}(\eta)|^2 \, d\eta d\sigma_\xi$.
- $\hat{f}(\eta) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\eta} f(x) \, dx$.

Uniqueness of solutions to the integral equation

Proposition 7

The solution to the integral equation (5) bounded on D is unique, if it exists.

- f_1, f_2: two solutions.
- $\tilde{f} := f_1 - f_2$
- $$\tilde{f}(x,\xi) = \int_0^{\tau_-(x,\xi)} \mu_s(x - s\xi)\exp\left(-M_t(x,\xi;s)\right) \times \int_{S^{d-1}} p(x - s\xi, \xi, \xi')\tilde{f}(x - s\xi, \xi')\,d\sigma_{\xi'}\,ds$$
- $|\tilde{f}(x,\xi)| \leq M \sup_{(x,\xi)\in D} |\tilde{f}(x,\xi)|$
- $M := \sup_{(x,\xi)\in D}(1 - \exp(-M_t(x,\xi;\tau_-(x,\xi)))) < 1$
- $\therefore \tilde{f}(x,\xi) = 0$ for all $(x,\xi) \in X$. $(f_1 = f_2)$

Directional Differentiability

- $\xi \cdot \nabla_x f^{(0)}(x,\xi) = -\mu_t(x) f^{(0)}(x,\xi)$.

$$f^{(0)}(x,\xi) = \exp\left(-M_t(x,\xi;\tau_-(x,\xi))\right) f_0(P(x,\xi),\xi)$$

- $G \in C_b(\Omega_0 \times S^{d-1}) \Rightarrow$

$$\begin{aligned}\xi \cdot \nabla_x f^{(1)}(x,\xi) &= -\mu_t(x) f^{(1)}(x,\xi) + \mu_s(x) G(x,\xi)\\ &= -\mu_t(x) f^{(1)}(x,\xi)\\ &\quad + \mu_s(x) \int_{S^{d-1}} p(x,\xi,\xi') f^{(0)}(x,\xi')\, d\sigma_{\xi'}.\end{aligned}$$

- $f^{(n)} \in C_b(D) \Rightarrow$

$$\begin{aligned}\xi \cdot \nabla_x f^{(n+1)}(x,\xi) = &-\mu_t(x) f^{(n+1)}(x,\xi)\\ &+ \mu_s(x) \int_{S^{d-1}} p(x,\xi,\xi') f^{(n)}(x,\xi')\, d\sigma_{\xi'}.\end{aligned}$$