Validity of the Boltzmann equation beyond hard spheres

based on joint work with M. Pulvirenti and C. Saffirio

Sergio Simonella
Technische Universität München

Outline

Many-body classical system: $i = 1, \cdots, N, \quad z_i = (x_i, v_i) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad \varphi : \mathbb{R}^3 \to \mathbb{R}$

$$N = 10^{20}$$

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = -\sum_{j:j\neq i} \nabla\varphi(x_i - x_j) \\ z_i(0) = z_{i,0} \quad \longleftarrow (z_{i,0})_{i=1}^N \text{ random variable on } \left((\mathbb{R}^3 \times \mathbb{R}^3)^N, \mu_0^N\right) \end{cases} \qquad (*)$$

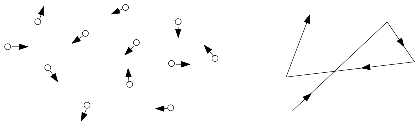
$\Longrightarrow$ *KINETIC THEORY:* $f \in \mathscr{P}\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)\,.$$

Chaos: $f_1^N(t), f_2^N(t)$ first and second marginal of $(*)$:

$$\partial_t f_1^N + v \cdot \nabla_x f_1^N = Q^N(f_2^N) \qquad\qquad f_2^N \to f_1^N f_1^N \ ?$$

- Mean-field (Vlasov)
- Collisions (Boltzmann)



Low density gas

$N =$ number of particles ; $\varepsilon =$ collision length

rate of coll. $\simeq N\varepsilon^2 = 1$; 'volume' density $\simeq N\varepsilon^3 = \varepsilon$

$\varepsilon \to 0$: low density limit (Boltzmann-Grad limit)

$$i = 1, \cdots, N, \quad (x_i, v_i) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad \varphi : \mathbb{R}^3 \to \mathbb{R}$$

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = -\frac{1}{\varepsilon} \sum_{j : j \neq i} \nabla\varphi\left(\frac{x_i - x_j}{\varepsilon}\right) \\ (x_i, v_i)(0) = z_{i,0} \end{cases}$$

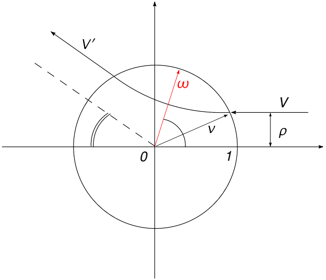
Grad's conjecture : the first marginal $f_1^N(t) \longrightarrow f(t)$ as $\varepsilon \to 0$

$$(\partial_t + v \cdot \nabla_x) f(x,v,t) = \int_{\mathbb{R}^3} dv_1 \int_{S^2} d\omega \; B(\omega, V) \times \left\{ f(x,v_1',t) f(x,v',t) - f(x,v_1,t) f(x,v,t) \right\}$$

$$\begin{cases} v' = v - \omega[\omega \cdot (v - v_1)] \\ v_1' = v_1 + \omega[\omega \cdot (v - v_1)] \end{cases}$$

$V = v_1 - v, \qquad V' = v_1' - v'$

$B(\omega, V)/|V| =$ differential cross–section



$f(x,v,t)\,dx\,dv =$ probability of finding a particle in position x with velocity v, at time t.

Setting

$\mathscr{M}_N$ canonical phase space of N hard spheres in $\mathbb{R}^3$ (diameter ε)

$\mathscr{M}_N = \left\{ \underline{z}_N = (z_1, \cdots, z_N),\ z_i = (x_i, v_i) \in \mathbb{R}^3 \times \mathbb{R}^3,\ \ |x_i - x_j| > \varepsilon \text{ for } i \neq j \right\}$

$\underline{z}_N \longrightarrow \underline{z}_N(t) =$ hard-sphere flow (a.e.-defined)

Initial distribution μ_0^N: symmetric density f_0^N s.t.:

– N particles "almost i.i.d."

– prob. density $f_0 \in \mathscr{P}(\mathbb{R}^3 \times \mathbb{R}^3)$

– $N\varepsilon^2 = 1$

– uniform bounds

Example: "Minimally correlated state"

$$f_0^N := \frac{1}{\mathcal{Z}_N} f_0^{\otimes N}, \qquad \| f_0 e^{\mu + \beta(v^2/2)} \|_{L^\infty} < +\infty \qquad \mu, \beta > 0, \qquad \mathcal{Z}_N = \int d\underline{z}_N f_0^{\otimes N}$$

Observables

$g_1, g_2, \cdots$ test functions over $\mathbb{R}^3 \times \mathbb{R}^3$, $F_1, F_2, \cdots$ observables over $\mathcal{M}_N$

$F_i(t)(z_1, \cdots, z_N) := \varepsilon^2 \sum_{j=1}^N g_i(z_j(t))$ (e.g. number of particles in a cell)

LLN: $F_i(t) \to \int g_i f(t)$ in the BG limit *(local Poisson)*

Theorem. There exists $t_0 > 0$ such that, if $t \in [0, t_0)$, $j = 1, 2, \cdots$

$$\left| \mathbb{E}\left[\prod_{i=1}^{j} \left(F_i(t) - \int g_i f(t) \right) \right] \right| \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0 \,.$$

OP:

1. $t_0 \to 2t_0$
2. $\varphi(x) \sim x^{-k}, \quad k > 0$

1. Some References

Short time, hard spheres [Lanford ('75)
See also: King, Spohn, Ilner, Pulvirenti, Uchiyama, Ukai ...]

Perturbation of vacuum. [Illner, Pulvirenti ('86)]

Perturbation of equilibrium. [van Beijeren, Lanford, Lebowitz, Spohn ('80),
Bodineau, Gallagher, Saint-Raymond ('16,'17)]

$M_{N,\beta}$ = Gibbs state (N hard spheres, inv.temp. β)

(i) $M_{N,\beta}(z_1,\cdots,z_N)h_0(z_1) \longrightarrow$ linear BE ($\longrightarrow$ Brownian motion)

(ii) $M_{N,\beta}(z_1,\cdots,z_N)\prod_{i=1}^N \left(1+\frac{1}{N}h_0(z_i)\right) \longrightarrow$ linearized BE

Quantitative chaos far from equilibrium. [Pulvirenti, S. ('17)]

Newton equation $\longleftrightarrow$ Liouville equation

kinetic equation $\longleftrightarrow$ kinetic hierarchy

$$\begin{cases}(\partial_t + v\cdot\nabla_x)f = Q(f,f)\\ f(0) = f_0 \in \mathscr{P}(\mathbb{R}^3\times\mathbb{R}^3)\end{cases} \qquad f_0 \longrightarrow f(t) = \mathcal{T}_t f_0$$

$\pi_0 \in \mathscr{P}\big(\mathscr{P}(\mathbb{R}^3\times\mathbb{R}^3)\big)$ $\qquad \pi(t,f) := \pi_0(\mathcal{T}_{-t}f)$ $\qquad$ *'Statistical solution'*

Moments: $\quad f_j(t) := \int_{\mathscr{P}(\mathbb{R}^6)} f^{\otimes j}\, d\pi(t,f) = \int_{\mathscr{P}(\mathbb{R}^6)} \big(\mathcal{T}_t f\big)^{\otimes j}\, d\pi_0(f)\,, \qquad j = 1,2,\cdots$

$$\Rightarrow \qquad (\partial_t + \sum_{i=1}^{j} v_i\cdot\nabla_{x_i})f_j = \mathscr{C}_{j+1}(f_{j+1})$$

Boltzmann hierarchy

$$\mathscr{C}_{j+1}(f_{j+1}) = \sum_{i=1}^{j} Q|_{i,j+1}(f_{j+1})$$

$$\Rightarrow \qquad (\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}) f_j = \mathscr{C}_{j+1}(f_{j+1})$$

Boltzmann hierarchy

$$\mathscr{C}_{j+1}(f_{j+1}) = \sum_{i=1}^{j} Q|_{i,j+1}(f_{j+1})$$

$$= \sum_{k=1}^{j} \int d\omega dv_{j+1} B(\omega, v_k - v_{j+1}) \Big[f_{j+1}(\cdots x_k, v'_k, \cdots, x_k, v'_{j+1}) - f_{j+1}(\cdots x_k, v_k, \cdots, x_k, v_{j+1}) \Big]$$

Uniqueness: Spohn ('84)

Remark. For any $(f_j)_{j\geq 1}$, $f_j = f_j(z_1, \cdots, z_j) \in \mathscr{P}(\mathbb{R}^{6j})$ (i) symmetric and (ii) compatible ($\int d f_{j+1}(z_{j+1}) = f_j$), $\exists!$ Borel π on $\mathscr{P}(\mathbb{R}^6)$ s.t. $f_j = \int_{\mathscr{P}(\mathbb{R}^6)} f^{\otimes j} d\pi(f)$

Propagation of chaos: $\pi_0 = \delta_{f_0} \Rightarrow \pi(t) = \delta_{f(t)}$

N particles, Hamiltonian H_N

Prob. density $f^N \in \mathscr{P}((\mathbb{R}^3 \times \mathbb{R}^3)^N)$ $\qquad \partial_t f^N = \{H_N, f^N\}$ $\qquad f^N(0) = f_0^N$

Marginals: $\qquad f_j^N := \int_{\mathbb{R}^{6(N-j)}} f^N dz_{j+1} \cdots dz_N\,, \qquad j = 1, 2, \cdots$

$$\Rightarrow \qquad \left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i} - \frac{1}{\varepsilon} \sum_{i,k=1}^{j} \nabla\varphi\left(\frac{x_i - x_k}{\varepsilon}\right) \cdot \nabla_{v_i}\right) f_j^N = \mathscr{C}_{j+1}^N(f_{j+1}^N) \qquad \text{BBGKY}$$

$$\mathscr{C}_{j+1}^N(f_{j+1}^N) = \tfrac{N-j}{\varepsilon} \textstyle\sum_{i=1}^{j} \int_{\mathbb{R}^6} \nabla\varphi\left(\tfrac{x_i - x_k}{\varepsilon}\right) \cdot \nabla_{v_i} f_{j+1}^N dx_{j+1} dv_{j+1}$$

Uniform bounds for $(f_j^N)_{1 \le j \le N}$: Lanford, King ('75)

Particle chaos: $f_j^N \to f_j$ as $N \to \infty$

$$\left(f_{0,j}^N \to f_0^{\otimes j} \Rightarrow f_j^N(t) \to f(t)^{\otimes j} \right)$$

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$$\left(f_{0,j}^N \to f_0^{\otimes j} \Rightarrow f_j^N(t) \to f(t)^{\otimes j} \right)$$

Remark: Formal comparison BBGKY

$$\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i} - \frac{1}{\varepsilon} \sum_{\substack{i,k=1 \\ i \neq k}}^{j} \nabla \varphi \left(\frac{x_i - x_k}{\varepsilon} \right) \cdot \nabla_{v_i} \right) f_j^N$$

$$= \frac{N-j}{\varepsilon} \sum_{i=1}^{j} \int dx_{j+1} \int dv_{j+1} \nabla \varphi \left(\frac{x_i - x_{j+1}}{\varepsilon} \right) \cdot \nabla_{v_i} f_{j+1}^N .$$

vs. Boltzmann hierarchy ? not so enlightening...

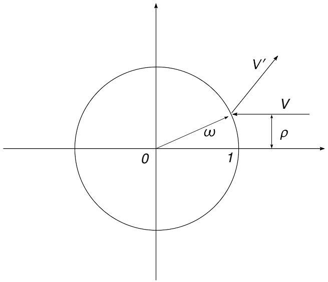
... unless for hard spheres (Cercignani '72):

$$(\partial_t + v\cdot\nabla_x) f_1^N(x,v,t) = (N-1)\varepsilon^2 \int_{\mathbb{R}^3} dv_1 \int_{S^2} d\omega\ B(\omega,V)$$
$$\times\left\{ f_2^N(x-\varepsilon\omega, v_1', x, v', t) - f_2^N(x+\varepsilon\omega, v_1, x, v, t)\right\}$$

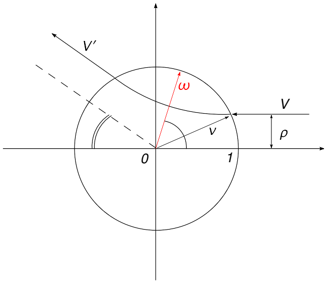
vs.

$$(\partial_t + v\cdot\nabla_x) f(x,v,t) = \int_{\mathbb{R}^3} dv_1 \int_{S^2} d\omega\ B(\omega,V)$$
$$\times\left\{ f(x,v_1',t) f(x,v',t) - f(x,v_1,t) f(x,v,t)\right\}$$

$B(\omega,V) = |\omega\cdot V|\ \mathbb{1}_{\{\omega\cdot V\le 0\}}$



$\Longrightarrow$



?

- Infinite range: poorly understood [Ayi ('17)]
- Finite range [Gallagher, Saint-Raymond, Texier ('14), Pulvirenti, Saffirio, S. ('14)]

'*Reduced marginals*':
$$\tilde f_j^N = \int_{S(\underline{x}_j)^{N-j}} dz_{j+1}\cdots dz_N f^N$$

$$S(\underline{x}_j) = \{|x - x_k| > \varepsilon \;\text{ for all } k = 1,\cdots,j\}$$

$$\Rightarrow \qquad \left(\partial_t + \sum_{i=1}^{j} v_i\cdot\nabla_{x_i} - \frac{1}{\varepsilon}\sum_{i,k=1}^{j}\nabla\varphi\left(\frac{x_i-x_k}{\varepsilon}\right)\cdot\nabla_{v_i}\right)\tilde f_j^N = \mathscr{C}^\varepsilon_{j+1}\tilde f^N_{j+1} + E_j^\varepsilon$$

Grad hierarchy

$$\mathscr{C}^\varepsilon_{j+1}\tilde f^N_{j+1}(\underline{z}_j,t) = \varepsilon^2(N-j)\sum_{k=1}^{j}\int_{S^2\times\mathbb{R}^3} d\nu\, dv_{j+1}\,\mathbb{1}_{\{\min_{\ell\neq k}|x_k+\nu\varepsilon-x_\ell|>\varepsilon\}}$$
$$\times(v_{j+1}-v_k)\cdot\nu\,\tilde f^N_{j+1}(\underline{z}_j, x_k+\nu\varepsilon, v_{j+1}, t)\,.$$

Error E_j^ε (finite range):

- Couplings to all $f_{j+2}, f_{j+3}, \cdots$
- $E_j^\varepsilon = O(\varepsilon)$
- does not worsen the uniform bounds on $(f_j^N)_{1\le j\le N}$ [King '75 ($\varphi\geq 0$)]

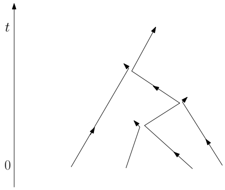
3. Difficulty

Is the Boltzmann equation valid for 'any' compactly supported φ*?*

Is the Boltzmann equation valid for 'any' compactly supported φ?

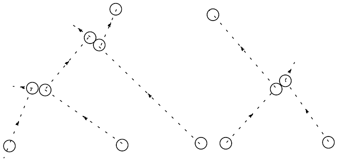
Caution:

geometrical estimates on recollision sets

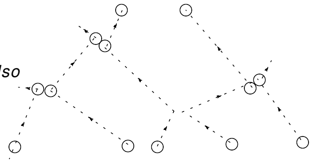


Explicit solution of the hierarchy: "weighted average" over backward clusters.

E.g. for two hard spheres with configuration (z_1, z_2), $z_i = (x_i, v_i)$ at time t:



, *but also*



(*Breakdown of chaos:* $f_j^N(t) \neq (f_1^N)^{\otimes j}(t)$)

Standard kinetics may fail (Uchiyama model ['88], external magnetic field [Bobylev et al '95])

Iterated Duhamel:

$$f_j(t) = \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \times \mathscr{S}_j(t-t_1)\mathscr{C}_{j+1}\mathscr{S}_{j+1}(t_1-t_2)\cdots\mathscr{C}_{j+n}\mathscr{S}_{j+n}(t_n) f_{j+n}(0)$$

$$\tilde{f}_j^N(t) = \sum_{n=0}^{N-j} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \times \mathscr{S}_j^\varepsilon(t-t_1)\mathscr{C}_{j+1}^\varepsilon\mathscr{S}_{j+1}^\varepsilon(t_1-t_2)\cdots\mathscr{C}_{j+n}^\varepsilon\mathscr{S}_{j+n}^\varepsilon(t_n)\tilde{f}_{j+n}^N(0) + O(\varepsilon)$$

$\mathscr{S}_j^\varepsilon(t) =$ flow operator for the $j-$body dynamics
$\mathscr{S}_j(t) =$ free flow operator
$O(\varepsilon) =$ multiple collisions

Two steps:

A. Absolute convergence of the series (uniform in ε) ;

B. Term by term convergence

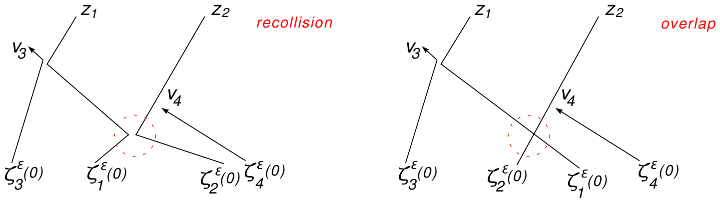
$\underline{\zeta}^\varepsilon(s) = (\zeta_1^\varepsilon(s), \zeta_2^\varepsilon(s), \cdots) =$ trajectory of backward clusters

$$\tilde{f}_2^N(z_1,z_2,t) - \tilde{f}_1^N(z_1,t)\tilde{f}_1^N(z_2,t) = \sum_{n\geq 2}\int d\Lambda_n^\varepsilon(\underline{\zeta}^\varepsilon)\, f_0^{\otimes n}(\underline{\zeta}^\varepsilon(0))\left[\chi_{1,2}^{rec} - \chi_{1,2}^{ov}\right] + O(\varepsilon)$$

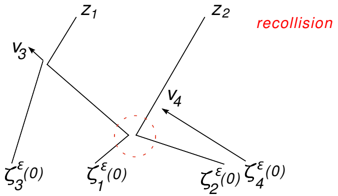
"$d\Lambda_n^\varepsilon$"= (signed) measure over trajectories of $n-$particle backward clusters

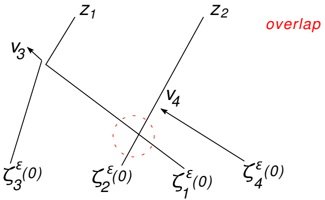
$\chi_{1,2}^{rec} = 1 \iff$ there exists a *recollision* among the clusters 1 and 2.

$\chi_{1,2}^{ov} = 1 \iff$ there exists an *overlap* among the clusters 1 and 2.



$$\underline{\zeta}^{\varepsilon}(s) \longleftarrow \begin{cases} \underline{z}_j & \text{(starting (time } t\text{) } j\text{-particle configuration)} \\ n & \text{(number of added particles)} \\ t_1, \cdots, t_n & \text{(times of creation of added particles)} \\ \nu_1, \cdots, \nu_n & \text{(impact vector of the added particles)} \\ v_{j+1}, \cdots, v_{j+n} & \text{(velocities of added particles)} \end{cases} .$$

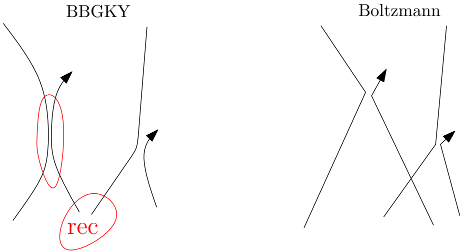




Smooth φ pathologies

1. interaction time

- high-energy
- grazing collisions
- trapping orbits
- ...

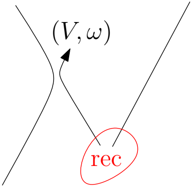


2. cross-section

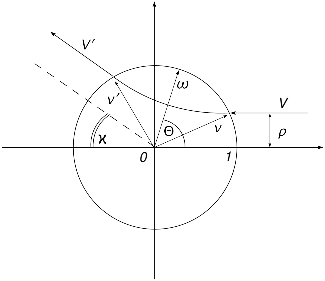
$R > 0$ energy cutoff

$$\int_{S^2} d\hat{V} \int_0^R dV V^2 \int d\omega B(\omega, V)\chi^{rec} = O(R^4 \|\sigma_\varphi\|_\infty \varepsilon^2)$$

σ_φ = differential cross-section. Typically $\|\sigma_\varphi\|_\infty = \infty$



3. EXPLORING SINGULARITIES: $\sigma_\varphi = \frac{\rho}{2|\sin(2\Theta)|}\left|\frac{d\rho}{d\Theta}\right|$



$$\Theta(\rho) = \arcsin\rho + \rho\int_{r_*}^{1} dr \frac{1}{r^2\sqrt{1-\frac{2\varphi(r)}{V^2}-\frac{\rho^2}{r^2}}} \qquad \frac{2\varphi(r)}{V^2}+\frac{\rho^2}{r^2}=\sin^2\alpha$$

$$\frac{d\Theta}{d\rho} = \frac{1}{\sqrt{1-\rho^2}}\left(1-\frac{1}{1-\frac{\varphi'(1^-)}{V^2\rho^2}}\right) + \int_{\arcsin\rho}^{\pi/2} d\alpha \frac{\sin\alpha}{\left(y-\frac{\rho}{V^2y^2}\varphi'(\frac{\rho}{y})\right)^3}\left[\frac{\rho}{V^2y^2}\varphi''\left(\frac{\rho}{y}\right)+\frac{2}{V^2y}\varphi'\left(\frac{\rho}{y}\right)+\frac{\rho}{V^4y^4}\left(\varphi'\left(\frac{\rho}{y}\right)\right)^2\right]$$

$\rho \to \Theta(\rho)$ monotonic if $|q|\varphi''(|q|)+2\varphi'(|q|)\geq 0$ [Sone, Aoki]

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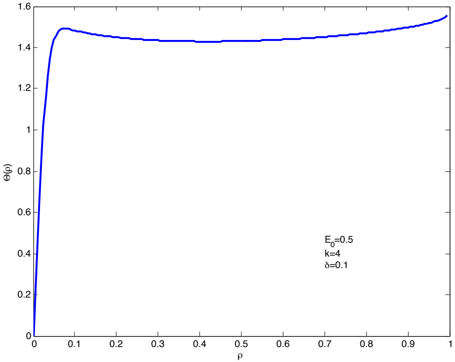

$$\varphi(q) = \begin{cases} \dfrac{\delta^{k+2}}{k|q|^k} + \delta - \delta^2(1+k^{-1}) & 0 < |q| < \delta \\ \delta(1-|q|) & \delta < |q| < 1 \\ 0 & |q| \geq 1 \end{cases}$$

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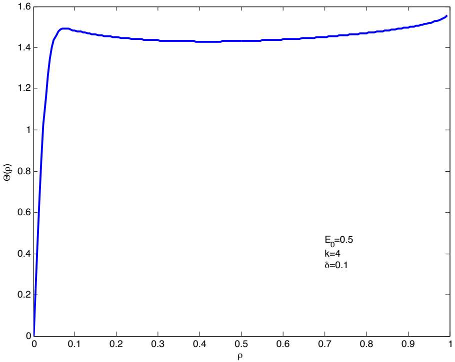

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Is the Boltzmann equation valid for 'any' compactly supported φ?

Hypothesis 1. The two–body potential $\varphi = \varphi(q)$ is radial, with support $|q| < 1$, class $C^2(\mathbb{R}^3 \setminus \{0\})$ and stable ($\sum_{i<k} \varphi(q_i - q_k) \geq -CN,\ C > 0$).
Hypothesis 2. The initial datum for Boltzmann is $f_0 \in C(\mathbb{R}^6),\ f_0(x,v) \leq Ce^{-(\beta/2)v^2}$, $C, \beta > 0$.
Hypothesis 3. The initial datum for the $N-$particle system is the symmetric probability density f_0^N, with marginals $f_{0,j}^N \leq e^{\alpha j} e^{-(\beta/2)\left[\sum_i v_i^2 + \sum_{i,k} \varphi\left(\frac{x_i - x_k}{\varepsilon}\right)\right]}$, $\alpha, \beta > 0$.
Hypothesis 4. $f_{0,j}^N \to f_0^{\otimes j}$ uniformly on compact sets outside the diagonals $(x_i = x_k)$.

THEOREM (PULVIRENTI, SAFFIRIO, S.)

Let $\Omega_j = \left\{ \underline{z}_j \text{ t.c. } |x_i - x_k - (v_i - v_k)s| > 0\ \forall s \geq 0 \right\}$. *In the hypotheses* $1-4$, *there exists* t_0 *s.t. for any* $t < t_0,\ j > 0$,

$$\lim_{\substack{\varepsilon \to 0 \\ N\varepsilon^2 = 1}} f_j^N(t) = f(t)^{\otimes j}$$

uniformly on compact sets in Ω_j.

Hypothesis 5. φ is non increasing. Moreover, for some $C', L > 0$,

$$\sup_{|x_i - x_k| > \varepsilon} e^{\frac{\beta}{2}\sum_{i=1}^{j} v_i^2} \left| f_0^{\otimes j} - f_{0,j}^N \right| \le (C')^j \varepsilon \qquad \text{(convergence)},$$

$$e^{\frac{\beta}{2} v^2} \left| f_0(x,v) - f_0(x',v) \right| \le L \left| x - x' \right| \qquad \text{(regularity)}.$$

Theorem (Pulvirenti, Saffirio, S.)

In the hypotheses 1–5, for any $t < t_0$, $j > 0$, $\underline{z}_j \in \Omega_j$ and ε small enough,

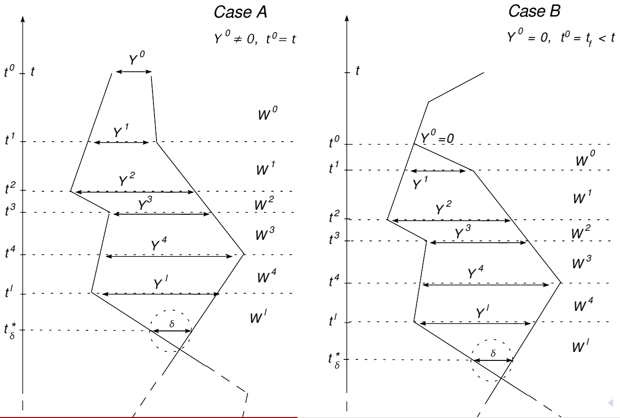
$$\left| f_j^N(\underline{z}_j, t) - \left(f(t)^{\otimes j} \right)(\underline{z}_j) \right| \le C^j \varepsilon^{\gamma},$$

with suitable $C, \gamma > 0$.

Remark. "$\underline{z}_j \in \Omega_j$" $\longleftrightarrow$ irreversibility.
Improved: 'one-sided convergence' [Bodineau, Gallagher, Saint-Raymond, S. '17]

Boltzmann equation emerging in the form

$$(\partial_t + v \cdot \nabla_x) f(x,v,t) = \int_{\mathbb{R}^3} dv_1 \int_{S^2_+} d\nu \, (v - v_1) \cdot \nu \times \Big\{ f(x,v_1',t) f(x,v',t) - f(x,v_1,t) f(x,v,t) \Big\}$$



Global approach:

– work on the Boltzmann flow ($|Y^l - W^l s| < \delta$)

– integrate over times;

– exploit the global structure of $\underline{\zeta}(s)$;

– keep using ν, not ω.

Open

- unstable interactions
- Vlasov-Boltzmann
- long range

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THANKS!