Solving linearized Boltzmann equation pointwisely

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Outline

Boltzmann equation

Two decompositions

New Mixture Lemma

Conclusion

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- Boltzmann equation
- Two decompositions
- New Mixture Lemma
- Conclusion

Boltzmann with cutoff $(-2 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,G) = \int_{\mathbb{R}^3 \times S^2} B(|\xi - \xi_*|, \theta) \left\{ F(\xi_*') G(\xi') - F(\xi_*) G(\xi) \right\} d\xi_* d\omega \,.$$

- Cross section: $B(V, \theta) = |V|^\gamma b(\theta)$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf \,, \quad f(0, x, \xi) = f_0$
- f_0 cpt. supp. in x and ξ weight $e^{\alpha|\xi|^p}$ (α small, $0 < p \leq 2$)
- $Lf = -\nu(\xi) f + Kf$
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.

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Wave-Remainder decomposition

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0,x,\xi) = f_0(x,\xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0,x,\xi) = 0 \end{cases}$$

We can define the wave part and remainder part:

$$W^{(6)} = \sum_{j=0}^{6} h^{(j)}\,, \quad \mathcal{R}^{(6)} = f - W^{(6)}$$

Note that $\mathcal{R}^{(6)}$ solves the equation

$$\begin{cases} \partial_t \mathcal{R}^{(6)} + \xi \cdot \nabla_x \mathcal{R}^{(6)} = L\mathcal{R}^{(6)} + K h^{(6)}\,, \\ \mathcal{R}^{(6)}(0,x,\xi) = 0 \end{cases}$$

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Let $0 < p \leq 2$ and α small,

- $0 \leq \gamma < 1$,

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \leq \sup_y e^{-c_0\left(t + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right)} \left|h_0(y, \cdot)\right|_{L^\infty_\xi\left(e^{\alpha|\xi|^p}\right)}$$

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Long-Short wave decomposition

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = Lf \,, & (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \,, \\ f(0,x,\xi) = f_0(x,\xi) \end{cases}$$

$$f(t,x,\xi) = \int_{\mathbb{R}^3} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0(\eta,\xi)\, d\eta$$

$$f_L(t,x,\xi) = \int_{|\eta|\le 1} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0(\eta,\xi)\, d\eta$$

$$f_S(t,x,\xi) = \int_{|\eta|> 1} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0(\eta,\xi)\, d\eta$$

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Fluid structure $0 \le \gamma < 1$ (Liu-Yu)

Let $v = \sqrt{5/3}$ be the sound speed associated with normalized global Maxwellian, and any given Mach number $\mathbb{M} > 1$, there exists positive constant C such that for $|x| \le (\mathbb{M}+1)\,ct$,

$$|f_L|_{L^2_\xi} \le C \left[(1+t)^{-2} \left(1 + \frac{(|x| - vt)^2}{1+t} \right)^{-N} + (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-N} + \mathbf{1}_{|x| \le ct} (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-3/2} \right] \|f_0\|_{L^1_x L^2_\xi} .$$

$$\|f_S\|_{L^2} \lesssim e^{-ct} \|f_0\|_{L^2}$$

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& + \mathbf{1}_{|x| \le ct} (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-3/2} \Bigg] \|f_0\|_{L^1_x L^2_\xi} .
\end{aligned}
$$

$$
\|f_S\|_{L^2} \lesssim e^{-ct} \|f_0\|_{L^2}
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Time asymptotic $-2 < \gamma < 0$ (Caflisch and Strain-Guo)

$$\|f_L\|_{L^\infty_x L^2_\xi} \lesssim t^{-3/2} \|f_0\|_{L^1_x L^2_\xi(e^{\alpha|\xi|^p})} .$$

$$\|f_S\|_{L^2} \lesssim e^{-c\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}} \|f_0\|_{L^2(e^{\alpha|\xi|^p})} .$$

Time-like region: $|x| < Mt$

- $f = f_L + f_S = W^{(6)} + \mathcal{R}^{(6)}$
- Define the kinetic part $f_K = W^{(6)}$, and tail part $f_R = \mathcal{R}^{(6)} - f_L$
- $f_R = \mathcal{R}^{(6)} - f_L = f_S - W^{(6)}$
- $f = f_L + W^{(6)} + f_R$
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Time-like region: $|x| < Mt$

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- (Weighted energy estimate for $\mathcal{R}^{(6)}$, $-2 < \gamma < 1$)

$$w(t,x,\xi) = e^{\frac{\varepsilon\rho(t,x,\xi)}{2}}, \quad \mu(x,\xi) = e^{\varepsilon\rho(0,x,\xi)},$$

where

$$\rho(t,x,\xi) = 5\Big(\delta(\langle x\rangle - Mt)\Big)^{\frac{p}{p+1-\gamma}}(1-\chi(X)) + \Big[\Big((1-\chi(X))\delta(\langle x\rangle - Mt)\Big)\langle\xi\rangle^{\gamma-1} + 3\langle\xi\rangle^{p}\Big]\chi(X)$$

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Outline

- Boltzmann equation
- Two decompositions
- New Mixture Lemma
- Conclusion

Regularization: Estimate of $h^{(6)}$

- Key observation: $|\nabla_\xi K g_0|_{L^2_\xi} \leq |g_0|_{L^2_\xi}$. $(\gamma > -2)$
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0$
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Let $\mathbb{T}^3 = [0,t] \times [0,s_1] \times [0,s_2]$ and $dS_3 = ds_3 ds_2 ds_1$

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\begin{aligned}
&\nabla_x h^{(3)}(t) \\
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&= \int_{\mathbb{T}^3} \frac{1}{s_1 - s_3} \mathbb{S}^{t-s_1} K \left(\mathcal{D}_{s_1-s_2} - \nabla_\xi\right) \mathbb{S}^{s_1-s_2} K \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3}\, g_0(s_3)\, dS_3 \\
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&\le e^{-c_\gamma t} \int_0^t \int_0^{s_1} \int_{s_3}^{s_1} \frac{1}{s_1 - s_3} \|g_0\|_{L^2(\mu)}\, ds_2 ds_3 ds_1 \\
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Outline

- Boltzmann equation
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Conclusion

Theorem

Let f be a solution to the Boltzmann equation with initial data compactly supported in the x-variable and bounded in L^2_ξ space with some weight

$$f_0(x,\xi) \equiv 0 \text{ for } |x| \geq 1.$$

If $0 \leq \gamma < 1$

1. *For $\langle x \rangle \leq Mt$,*

$$|f|_{L^2_\xi} \lesssim \left[e^{-c_0\left(t+|x|^{\frac{p}{p+1-\gamma}}\right)} + e^{-Ct} \right] \|f_0\|_{L^2(e^{7\alpha|\xi|^p})} + \textit{Fluid}$$

2. *For $\langle x \rangle \geq Mt$,*

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Conclusion

Theorem

If $-2 < \gamma < 0$

1. *For* $\langle x \rangle \leq Mt$,

$$|f|_{L^2_\xi} \lesssim \left[(1+t)^{\frac{-3}{2}} + e^{-c_0\left(t^{\frac{p}{p-\gamma}} + |x|^{\frac{p}{p+1-\gamma}}\right)} + e^{-t^{\frac{p}{p-\gamma}}} \right] \|f_0\|_{L^2(e^{7\alpha|\xi|^p})}$$

2. *For* $\langle x \rangle \geq Mt$,

$$|f|_{L^2_\xi} \lesssim \left[e^{-c_0\left(t^{\frac{p}{p-\gamma}} + |x|^{\frac{p}{p+1-\gamma}}\right)} + e^{-c_1(t+|x|)^{\frac{p}{p+1-\gamma}}} \right] \|f_0\|_{L^2(e^{7\alpha|\xi|^p})}$$

Remarks

- This idea can be applied to Fokker-Planck equation with potential or Landau equation (but without detailed description of initial singularity).

- Compare Liu-Yu's results.

- Singular wave

- Refined weighted energy estimate

- Mixture Lemma

- Compare Caflisch and Strain-Guo's results.

- How to describe fluid part for $-2 < \gamma < 0$ more precisely?

Remarks

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Epilog

THANK YOU