A new construction of the second order master equation in mean field games with common noise

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Mean Field Games (MFG) study

- **Evolution of populations** = infinitely many agents having individually a negligible influence on the global system

- **Consisting of rational agents** = each agent acts on his state which evolves in continuous time and has a payoff depending on the other’s position (stochastic optimal control)

**Pioneering works:**


- Similar models in the economic literature: heterogeneous agent models (Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

1. Crash course on MFG models
   - The MFG system
   - The limit of $N$–player game
   - The master equation

2. Construction of a solution for (M2)

3. Uniqueness
Outline

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2. Construction of a solution for \( (M2) \)

3. Uniqueness
The main equation in MFG is the second order master equation:

\[
\begin{aligned}
&\left\{
\begin{array}{l}
-\partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) \\
\quad - (1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_y(y, D_x U, m) \, dm(y) \\
\quad - 2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \, dm(y) - \beta \int_{\mathbb{R}^{2d}} \text{Tr} \left[ D^2_{mm} U \right] \, dm \otimes dm = 0
\end{array}
\right.
\end{aligned}
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

\[U(T, x, m) = G(x, m)\quad \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\]

where

- The unknown \(U = U(t, x, m)\) is scalar,
- \(H = H(x, p, m) : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) is a standard Hamiltonian in \((x, p)\), non local and smoothing in \(m\),
- \(\beta \geq 0\) is the level of common noise,
- the coupling function \(G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) is nonlocal and smoothing.

**Aim of our work**: Provide a new construction of solutions for \((M2)\).
The main equation in MFG is the second order master equation:

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\begin{align*}
-\partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) \\
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- 2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \, dm(y) - \beta \int_{\mathbb{R}^{2d}} \text{Tr} \left[ D^2_{mm} U \right] \, dm \otimes dm = 0
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**Aim of our work**: Provide a new construction of solutions for (M2).
The MFG system

Formally and in the simplest case, an MFG equilibrium should take the form of an MFG system

\[
\begin{align*}
\begin{cases}
-\partial_t u - \varepsilon \Delta u + H(x, Du, m(t)) &= 0 \text{ in } [0, T] \times \mathbb{R}^d \\
\partial_t m - \varepsilon \Delta m - \text{div}(mD_p H(x, Du, m(t))) &= 0 \text{ in } [0, T] \times \mathbb{R}^d \\
u(T, x) &= G(x, m(T)), \ m(0, \cdot) = m_0 \text{ in } \mathbb{R}^d
\end{cases}
\end{align*}
\]

where

- \( \varepsilon \geq 0 \),
- The unknowns \((u, m) = (u(t, x), m(t, x))\) are scalar, \(m(t, \cdot)\) being a probability density,
- \(H = H(x, p, m) : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}\) is a standard Hamiltonian in \((x, p)\),
- the coupling function \(G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}\).

Remark: The Benamou-Brenier system in optimal transport corresponds to \(\varepsilon = 0\), \(H(x, p, m) = |p|^2/2\):

\[
\begin{align*}
\begin{cases}
-\partial_t u + |Du|^2/2 &= 0 \text{ in } [0, T] \times \mathbb{R}^d \\
\partial_t m - \text{div}(mDu) &= 0 \text{ in } [0, T] \times \mathbb{R}^d \\
m(0, \cdot) &= m_0, \ m(T, \cdot) = m_T \text{ in } \mathbb{R}^d
\end{cases}
\end{align*}
\]
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- \(H = H(x, p, m) : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) is a standard Hamiltonian in \((x, p)\),
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m(0, \cdot) &= m_0, \quad m(T, \cdot) = m_T \text{ in } \mathbb{R}^d
\end{align*}
\]
Interpretation of the (MFG) system

(MFG) correspond to Nash equilibria in an optimal control problem of the form:

$$u(t, x) := \inf_{\alpha} \mathbb{E} \left[ \int_t^T L(X(s), \alpha(s), m(s)) ds + G(X(T), m(T)) \right]$$

where $L(x, \alpha, m) = \sup_{\alpha} \{ -\alpha \cdot p - H(x, p, m) \}$ and where $(X = X^\alpha)$ solves

$$dX(s) = \alpha(s) ds + \sqrt{2\varepsilon} dB(s), \quad X(t) = x.$$ 

Then $u$ solves the HJ equation

$$\begin{cases} 
-\partial_t u - \varepsilon \Delta u + H(x, Du, m(t)) = 0 & \text{in } [0, T] \times \mathbb{R}^d \\
u(T, x) = G(x, m(T)), & \text{in } \mathbb{R}^d 
\end{cases}$$

The optimal feedback control is $\alpha^* = -D_p H(x, Du(t, x), m(t))$ and the optimal trajectory is given by

$$dX^*(s) = \alpha^*(s, X^*(s)) ds + \sqrt{2\varepsilon} dB(s), \quad X^*(t) = x.$$ 

If the noise of the players are independent, $m(t) = \mathcal{L}(X^*_t)$ evolves according to the McKean-Vlasov equation:

$$\begin{cases} 
\partial_t m - \varepsilon \Delta m - \text{div}(m D_p H(x, Du, m(t))) = 0 & \text{in } [0, T] \times \mathbb{R}^d \\
m(0, \cdot) = m_0 & \text{in } \mathbb{R}^d 
\end{cases}$$
Basic results of the MFG system

For the **MFG equilibrium system**:

\[
\begin{align*}
(i) & \quad -\partial_t u - \varepsilon \Delta u + H(x, Du, m) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d \\
(ii) & \quad \partial_t m - \varepsilon \Delta m - \text{div}(m D_p H(x, Du, m)) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d \\
(iii) & \quad m(0, \cdot) = m_0, \ u(T, x) = g(x, m(T)) \quad \text{in } \mathbb{T}^d
\end{align*}
\]

**Existence of solutions** : (Lasry-Lions)
- for \( \varepsilon > 0 \), holds under "general conditions"
- for \( \varepsilon = 0 \) and smoothing coupling: viscosity solution for (HJ) and distributional solutions for (FP)
- for \( \varepsilon = 0 \) and local monotone coupling: associated to the optimal control of the continuity equation

**Uniqueness** cannot be expected in general, but holds
- either under **smallness condition on** \( T \)
- or under a **structure and monotonicity conditions** on \( H \) (Lasry-Lions):

\[
H = H(x, p) - F(x, m), \quad \int_{\mathbb{T}^d} (F(x, m) - F(x, m')) d(m - m') \geq 0, \\
\int_{\mathbb{T}^d} (G(x, m) - G(x, m')) d(m - m') \geq 0.
\]
The (MFG) system is now well-studied and well-understood.

Many applications: complex socio-economical models (heterogenous agent models), mathematical finance, crowd models, epidemiology, ...

Various generalizations of the model
(different boundary conditions, stationary models, jump diffusions, several population case,...)

Extensions not so much studied:
- Kinetic MFG models
  (motivated by traffic flow models, e.g.)
- Boltzmann type MFG models
  (motivated by knowledge growth models - Lucas-Moll)
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3. Uniqueness
The limit of $N$–player game

We consider a game with $N$ symmetric players

- Player $i \in \{1, \ldots, N\}$ controls a dynamics of the form

$$dX^i(t) = \alpha^i(t)dt \ + \ \sqrt{2}dB^i(t) \ + \ \sqrt{2}\beta dW(t), \ t \in [0, T] \quad X^i_0 = \bar{X}^i_0$$

where $\bar{X}^i_0$ is fixed, $(\alpha^i)$ is her control and $(B^i)$ and $W$ are i.i.d. BM and $\beta \geq 0$ is the level of common noise. She aims at minimizing

$$J^i(\alpha^i, (\alpha^j)_{j \neq i}) = \mathbb{E} \left[ \int_0^T L(X^i(s), \alpha^i(s), m^{N,i}_{(X(s))}) \ ds + G^{N,i}(X^i(T), m^{N,i}_{(X(T))}) \right].$$

where $m^{N,i}_{(X(s))} := \frac{1}{N-1} \sum_{j \neq i} \delta x^i(s)$.

- The value functions $(v^{N,i})_{i=1,\ldots,N}$ associated with a Nash equilibrium is a solution to

$$
\begin{cases}
\begin{aligned}
-\partial_t v^{N,i}(t, x) &- \sum_{j=1}^N \Delta x_j v^{N,i}(t, x) - \beta \sum_{j,k=1}^N \text{Tr}D^2_{x_j,x_k} v^{N,i}(t, x) + H(x_i, D_{x_i} v^{N,i}(t, x), m^{N,i}_{x_i}) \\
&+ \sum_{j \neq i} D_p H^{N,i}(x_j, D_{x_j} v^{N,j}(t, x), m^{N,j}_{x_j})) \cdot D_{x_j} v^{N,i}(t, x) = 0 \quad \text{in } [0, T] \times (\mathbb{R}^d)^N, \\
v^{N,i}(T, x) & = G(x, m^{N,i}_x) \quad \text{in } (\mathbb{R}^d)^N.
\end{aligned}
\end{cases}
$$
Let $v^{N,i}(t, x_i, (x_j)_{-i}) = V^N(t, x_i, m^{N,i}_x)$ be the solution to

\[
\begin{cases}
-\partial_t v^{N,i}(t, x) - \sum_{j=1}^N \Delta x_j v^{N,i}(t, x) - \beta \sum_{j,k=1}^N \text{Tr} D^2 x_j x_k v^{N,i}(t, x) + H(x, D x_i v^{N,i}(t, x), m^{N,i}_x)) \\
+ \sum_{j \neq i} D_p H^{N,i}(x_j, D x_j v^{N,j}(t, x), m^{N,i}_x)) \cdot D x_j v^{N,i}(t, x) = 0 \quad \text{in } [0, T] \times (\mathbb{R}^d)^N,
\end{cases}
\]

\[v^{N,i}(T, x) = G(x, m^{N,i}_x)) \quad \text{in } (\mathbb{R}^d)^N.\]

and assume that $V^N(t, x, m^{N,i}_x) \to U(t, x, m)$ as $N \to +\infty$, $m^{N,i}_x \to m$. Then $U$ formally solves the master equation:

\[
\begin{cases}
-\partial_t U - (1 + \beta) \Delta x U + H(x, D x U, m) \\
-(1 + \beta) \int_{\mathbb{R}^d} \text{div} y [D m U] \ dm(y) + \int_{\mathbb{R}^d} D m U \cdot H_p(y, D x U, m) \ dm(y) \\
-2\beta \int_{\mathbb{R}^d} \text{div} x [D m U] \ dm(y) - \beta \int_{\mathbb{R}^{2d}} \text{Tr} \left[ D^2_{mm} U \right] \ dm \otimes dm = 0 \\
\text{in } [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d),
\end{cases}
\]

\[U(T, x, m) = G(x, m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).\]
Limit of $N$–player game (continued)

Let $v^{N,i}(t, x, (x_j)_{j \neq i}) = V^N(t, x, m^N_x, i)$ be the solution to

\[
\begin{cases}
-\partial_t v^{N,i}(t, x) - \sum_{j=1}^N \Delta x_j v^{N,i}(t, x) - \beta \sum_{j, k=1}^N \text{Tr}D^2_{x_j, x_k} v^{N,i}(t, x) + H(x, D_x v^{N,i}(t, x), m^N_x, i)) \\
+ \sum_{j \neq i} D_p H^{N,i}(x_j, D_{x_j} v^{N,j}(t, x), m^N_x, i))) \cdot D_{x_j} v^{N,i}(t, x) = 0 \quad \text{in } [0, T] \times (\mathbb{R}^d)^N,
\end{cases}
\]

\[v^{N,i}(T, x) = G(x, m^N_x, i)) \quad \text{in } (\mathbb{R}^d)^N.\]

and assume that $V^N(t, x, m^N_x, i) \to U(t, x, m)$ as $N \to +\infty$, $m^N_x, i \to m$. Then $U$ formally solves the master equation:

\[
\begin{cases}
-\partial_t U - (1 + \beta)\Delta x U + H(x, D_x U, m) \\
- (1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \ dm(y) \\
- 2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \ dm(y) - \beta \int_{\mathbb{R}^{2d}} \text{Tr} \left[ D^2_{mm} U \right] \ dm \otimes dm = 0 \\
in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d),
\end{cases}
\]

\[U(T, x, m) = G(x, m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).\]
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3. Uniqueness
Derivatives in the space of measures

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$ with finite second order moment, endowed for the Wasserstein distance

$$d_2^2(m, m') = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \ d\pi(x, y),$$

where the infimum is taken over coupling between $m$ and $m'$.

Derivatives

A map $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is $C^1$ if there exists a continuous and bounded map

$$\frac{\delta U}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$$

such that, for any $m, m' \in \mathcal{P}_1(\mathbb{R}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m} ((1 - s)m + sm', y)d(m' - m)(y)ds.$$

We set

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$
Note that $\frac{\delta U}{\delta m}$ is defined up to an additive constant. We adopt the normalization convention

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, y) dm(y) = 0.$$ 

$D_m U$ corresponds to the derivative in the space of measures as introduced by Ambrosio-Gigli-Savaré.

$D_m U$ controls the Lipschitz norm of $U$:

$$|U(m_1) - U(m_2)| \leq \sup_{\mu \in \mathcal{P}_1(\mathbb{R}^d)} \|D_m U(\mu, \cdot)\|_{L^2_\mu} \mathbf{d}_2(m_1, m_2) \quad \forall m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^d).$$
The first order master equation ($\beta = 0$)

The first order master equation is the backward equation

\[
\begin{aligned}
-\partial_t U(t, x, m) - \Delta x U(t, x, m) &+ H(x, D_x U(t, x, m), m) \\
- \int_{\mathbb{R}^d} \text{div}_y [D_m U] (t, x, m, y) \, dm(y) &+ \int_{\mathbb{R}^d} D_m U(t, x, m, y) \cdot D_p H(y, D_x U(t, y, m), m) \, dm(y) = 0
\end{aligned}
\]

for $(t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$,

$U(T, x, m) = G(x, m)$, \quad for $(x, m) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

Proposition

If $U$ solves (M1) and $(u, m)$ solves the MFG system

\[
\begin{aligned}
-\partial_t u - \Delta u + H(x, Du, m(t)) &= 0 \quad \text{in } [t_0, T] \times \mathbb{R}^d \\
\partial_t m - \Delta m - \text{div}(m D_p H(x, Du, m(t))) &= 0 \quad \text{in } [t_0, T] \times \mathbb{R}^d \\
u(T, x) &= G(x, m(T)), \quad m(t_0, \cdot) = m_0 \quad \text{in } \mathbb{R}^d
\end{aligned}
\]

then

$U(t, x, m(t)) = u(t, x)$
Parenthesis: Finite dimensional analogue

The master equation \((\mathbf{M1})\) in finite state space reads (Lasry-Lions, Gomes):

\[
\begin{aligned}
-\partial_t U^x(t, \eta) &= \sum_{y,z} \partial_{\eta z} U^x(t, \eta) \eta_y a^*_z(y, \eta, \Delta_y U(t, \eta)) + H^x(\eta, \Delta_x U(t, \eta)) \\
U^x(T, \eta) &= U^x_T(\eta)
\end{aligned}
\]

\(x \in \{1, \ldots, I\}\), \(\eta \in \mathcal{P}(\{1, \ldots, I\})\)

or, in vectorial form, the hyperbolic system

\[
\begin{aligned}
-\partial_t U(t, \eta) &= (F(\eta, U) \cdot \nabla U) + H(\eta, U) \\
U(T, \eta) &= U_T(\eta)
\end{aligned}
\]

Well-posedness under monotonicity condition (Lasry-Lions):

\[
\langle (F, H)(\eta, U) - (F, H)(\eta', U'), (\eta, U) - (\eta', U') \rangle \geq 0,
\]

\(\exists \alpha > 0\), \(\langle U_T(\eta) - U_T(\eta'), \eta - \eta' \rangle \geq \alpha |\eta - \eta'|^2\).

(Proof: method of characteristics.)

- Lipschitz continuous solutions
- Preservation of the monotony:

\[
\langle U(t, \eta) - U(t, \eta'), \eta - \eta' \rangle \geq 0 \quad \forall t \geq 0.
\]
For the second order master equation

\[
\begin{cases}
-\partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) \\
-(1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \, dm(y) \\
-2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \, dm(y) - \beta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Tr} \left[ D_{mm}^2 U \right] \, dm \otimes dm = 0
\end{cases}
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

\[U(T, x, m) = G(x, m)\quad \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\]

The characteristics become the stochastic MFG system:

\[
\begin{cases}
d_t u_t = \left\{- (1 + \beta) \Delta u_t + H(x, D_u t, m_t) - \sqrt{2\beta} \text{div}(v_t) \right\} dt \\
+ v_t \cdot dW_t \quad \text{in } [t_0, T] \times \mathbb{R}^d,
\end{cases}
\]

\[
\begin{cases}
d_t m_t = \left[ (1 + \beta) \Delta m_t + \text{div} \left( m_tD_p H(x, D_u t, m_t) \right) \right] dt - \sqrt{2\beta} \text{div}(m_t dW_t) \\
\quad \text{in } [t_0, T] \times \mathbb{R}^d
\end{cases}
\]

\[m_{t_0} = m_0, \quad u_T(x) = G(x, m_T) \quad \text{in } \mathbb{R}^d.\]

where \((v_t)\) is a vector field which ensures \((u_t)\) to be adapted to the filtration \((\mathcal{F}_t)_{t \in [t_0, T]}\) generated by the M.B. \((W_t)_{t \in [0, T]}\).
Motivations to study (M1) and (M2)

- Allows to prove the convergence in the $N$–player problem, (under suitable conditions - C., Delarue, Lasry, Lions (2015))
- Gives a better understanding of more complex MFG models. (Major-Minor MFG)

Difficulties:

- (M1) and (M2) are infinite dimensional, nonlocal, nonlinear transports equation without maximum principle.
- For (M2), the characteristics are the stochastic MFG systems, which are difficult to manipulate.
- Discontinuities appear in finite time.
- So far, no notion of generalized solution (even for the finite dimension analogue).

Previous existence/uniqueness results: Lasry-Lions (’13), Buckdahn-Li-Peng-Rainer (’14), Gangbo-Swiech (’14), Bessi (’15), Chassagneux-Crisan-Delarue (’15), C.-Delarue-Lasry-Lions (2015), Lacker-Webster (’15), Ahuja (’16), Carmona-Delarue’s monograph (2017),...

→ Need of a new construction for the solution of (M2):
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Method of proof for the existence of a solution of (M2)

We see the second order master equation

\[
(M2) \begin{cases}
-\partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) \\
-(1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \ dm(y) \\
-2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \ dm(y) - \beta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Tr} \left[D_{mm}^2 U\right] \ dm \otimes dm = 0
\end{cases}
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

\[U(T, x, m) = G(x, m) \quad \text{in} \ \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\]

as the superposition of (M1) and (L2)

\[
(M1) \begin{cases}
-\partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) \\
-(1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \ dm(y) \\
-2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \ dm(y) - \beta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Tr} \left[D_{mm}^2 U\right] \ dm \otimes dm = 0
\end{cases}
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

\[U(T, x, m) = G(x, m) \quad \text{in} \ \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\]
Method of proof for the existence of a solution of (M2)

We see the second order master equation

(M2) \[
\begin{align*}
- \partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) & \\
- (1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \ dm(y) & \\
- 2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \ dm(y) - \beta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Tr} \left[ D_{mm}^2 U \right] \ dm \otimes dm = 0 & \\
\end{align*}
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

as the superposition of (M1) and (L2)

(L2) \[
\begin{align*}
- \partial_t U - (1 + \beta) \Delta_x U + H(x, D_x U, m) & \\
- (1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m U] \ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \ dm(y) & \\
- 2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m U] \ dm(y) - \beta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Tr} \left[ D_{mm}^2 U \right] \ dm \otimes dm = 0 & \\
\end{align*}
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

\[
U(T, x, m) = G(x, m) \quad \text{in} \ \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)
\]
The first order master equation (M1)

It is the backward equation

\[
\begin{aligned}
- \partial_t U(t, x, m) - \Delta_x U(t, x, m) + H(x, D_x U(t, x, m), m) \\
- \int_{\mathbb{R}^d} \text{div}_y[D_m U](t, x, m, y) \, dm(y) \\
+ \int_{\mathbb{R}^d} D_m U(t, x, m, y) \cdot D_p H(y, D_x U(t, y, m), m) \, dm(y) = 0
\end{aligned}
\]

for \((t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\),

\[U(T, x, m) = G(x, m), \quad \text{for} \ (x, m) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\]

Theorem (Chassagneux-Crisan-Delarue)

Under the suitable assumptions, there exists \(T > 0\) such that the first order master equation (M1) has a unique classical solution on \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\).

See also C.-Delarue-Lasry-Lions for a more “PDE like” construction.
The proof of Theorem 1 relies on the method of characteristics in infinite dimension.

Given \((t_0, m_0) \in [0, T) \times P_2(\mathbb{R}^d)\), let \((u, m) = (u(t, x), m(t, x))\) be the solution of the MFG system:

\[
\begin{aligned}
-\partial_t u - \Delta u + H(x, Du, m(t)) &= 0 \text{ in } [t_0, T] \times \mathbb{R}^d \\
\partial_t m - \Delta m - \text{div}(mD_pH(x, Du, m(t))) &= 0 \text{ in } [t_0, T] \times \mathbb{R}^d \\
&\quad \text{in } [0, T] \times \mathbb{R}^d \times P_2(\mathbb{R}^d) \\
u(T, x) &= G(x, m(T)), \quad m(t_0, \cdot) = m_0 \text{ in } \mathbb{R}^d
\end{aligned}
\]

If \(T > 0\) is small or under some monotonicity assumptions on \(F\) and \(G\), the (MFG) system is well-posed. (Lasry-Lions, 2007)

We define \(U\) by

\[
U(t_0, \cdot, m_0) := u(t_0, \cdot)
\]

Claim: \(U\) is a solution to the first order master equation.
Note that, for any \( h \in [0, T - t_0] \), \( u(t_0 + h, x) = U(t_0 + h, x, m(t_0 + h)) \).

So

\[
\partial_t u(t_0, x) = \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(t_0, x, m_0, y) \partial_t m(t_0, y) dy
\]

\[
= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_0, y) (\Delta m_0 + \text{div}(m_0 D_p H(x, Du, m_0))) dy
\]

\[
= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \Delta_y \left[ \frac{\delta U}{\delta m} \right] (m_0, y) m_0(y) dy
\]

\[
- \int_{\mathbb{R}^d} D_y \left[ \frac{\delta U}{\delta m} \right] (m_0, y) \cdot D_p H(x, Du, m_0) m_0(y) dy
\]

\[
= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \text{div}_y [D_m U] (m_0, y) m_0(y) dy
\]

\[
- \int_{\mathbb{R}^d} D_m U(m_0, y) \cdot D_p H(x, Du, m_0) m_0(y) dy
\]

Then \( U \) satisfies \((M1)\) because

\[
\partial_t u(t_0, x) = -\Delta u + H(x, Du, m_0)
\]

\[
= -\Delta x U(t_0, x, m_0) + H(x, D_x U(t_0, x, m_0), m_0).
\]

In the actual proof, one has to show that \( U \) is regular in \( m \): this relies on linearization of the MFG system.
Construction of a solution for (M2)

The linear second order equation (L2)

Let $\Gamma = \Gamma(t, x)$ be the heat kernel. For a map $G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ of class $C^2$, we set

$$U(t, x, m) = \int_{\mathbb{R}^d} G(\xi, (id - x + \xi)^\#m) \Gamma(T - t, x - \xi) d\xi.$$  

Proposition

The map $U$ solves the second order equation

$$(L2) \begin{cases}
-\partial_t U - \Delta U - \int_{\mathbb{R}^d} \text{div}_y [D_m U] dm - 2 \int_{\mathbb{R}^d} \text{Tr}[D_{xm}^2 U] dm \\
- \int_{\mathbb{R}^{2d}} \text{Tr}[D_{mm}^2 U] dm dm = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \\
U(T, x, m) = G(x, m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)
\end{cases}$$

Proof: Computation.
The short time existence for $(M2)$

Let
- $(S^1_t)$ be the backward semi-group associated with $(M1)$,
- $(S^2_t)$ be the backward semi-group associated with $(L2)$.

For $h > 0$ small and $T - t = 2kh$ ($k \in \mathbb{N}$), we set

$$S^h_{T-t} := (S^1_h \circ S^2_h)^k.$$

**Theorem**

For $M > 0$ there exists $T_M > 0$ such that, if $T \leq T_M$ and

$$\|D^2_{xx} G\|_\infty \leq M \quad \text{and} \quad \|D^2_{xm} G\|_\infty \leq M,$$

then $(S^h_t G)_{t \in [0, T]}$ converges to a solution of $(M2)$ on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

**Remarks :**
- The above Theorem gives the existence of a solution on a short time interval.
- The length of the interval depends on $\|D^2_{xx} G\|_\infty$ and $\|D^2_{xm} G\|_\infty$ only.
Idea of proof: relies of the estimates.

- For (M1): Fix $M > 0$ and $n \geq 2$. There exists $C_{M,n} > 0$ and $T_{M,n} > 0$ such that, if

$$\|D^2_{xx} G\|_\infty \leq M, \quad \|D^2_{xm} G\|_\infty \leq M \quad \text{and} \quad T \in (0, T_{M,n}],$$

then the solution $U := (S^1_t G)_{t \in [0, T]}$ to (M1) satisfies

$$\sup_{t \in [0, T]} \left( \|U(t)\|_{n+1} + \left\| \frac{\delta U}{\delta m}(t) \right\|_{n} + \left\| \frac{\delta^2 U}{\delta m^2}(t) \right\|_{n-1} + \Lip_{n-2} \left( \frac{\delta^2 U}{\delta m^2}(t) \right) \right)$$

$$\leq \left( \|G\|_{n+1} + \left\| \frac{\delta G}{\delta m} \right\|_{n} + \left\| \frac{\delta^2 G}{\delta m^2} \right\|_{n-1} + \Lip_{n-2} \left( \frac{\delta^2 G}{\delta m^2} \right) \right) (1 + C_{M,n} T) + C_{M,n} T.$$

- For (L2): Similar estimates for (L2) are straightforward.
Outline

1. Crash course on MFG models
   - The MFG system
   - The limit of $N$–player game
   - The master equation

2. Construction of a solution for (M2)

3. Uniqueness
Goal: prove the uniqueness by PDE arguments,

by using a maximum principle.

Difficulty: (M2) is a nonlocal equation, without maximum principle.
Optimality conditions

Let $U : \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be smooth and have a local maximum point at $(\hat{x}, \hat{m}) \in \times \mathcal{P}_2(\mathbb{R}^d)$. We have

- $D_x U(\hat{x}, \hat{m}) = 0$,
- $\delta U / \delta m(\hat{x}, \hat{m}, y) \leq 0$, $\forall y \in \mathbb{R}^d$, and $\delta U / \delta m(\hat{x}, \hat{m}, y) = 0$, $\hat{m} - \text{a.e. } y \in \mathbb{R}^d$,
- for any $(v, \phi) \in \times L^2_m(\mathbb{R}^d, )$,

$$D^2_{xx} U(\hat{x}, \hat{m}) v \cdot v + 2 \int_{\mathbb{R}^d} D^2_{xm} U(\hat{x}, \hat{m}, y) \phi(y) \cdot v d\hat{m}(y)$$
$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D^2_{mm} U(\hat{x}, \hat{m}, y, z) \phi(y) \cdot \phi(z) d\hat{m}(y) d\hat{m}(z) \leq 0.$$

In particular:

- $D_m U(\hat{x}, \hat{m}, y) = 0$, $D^2_{ym} U(\hat{x}, \hat{m}, y) \leq 0$ $\hat{m} - \text{a.e. } y \in \mathbb{R}^d$,
- and

$$\Delta_x U(\hat{x}, \hat{m}) + 2 \int_{\mathbb{R}^d} \text{Tr}[D^2_{xm} U](\hat{x}, \hat{m}, y) d\hat{m}(y)$$
$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Tr}[D^2_{mm} U](\hat{x}, \hat{m}, y, z) d\hat{m}(y) d\hat{m}(z) \leq 0.$$
A maximum principle

Let \( W = W(t, x, m) \) satisfy in \((0, T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\) the backward inequality:

\[
\mathcal{L}(W) := -\partial_t W - (1 + \beta) \Delta_x W + v_1(t, x, m) \cdot D_x W \\
- (1 + \beta) \int_{\mathbb{R}^d} \text{div}_y [D_m W] \ m(dy) + \int_{\mathbb{R}^d} D_m W \cdot v_2(t, y, m) \ m(dy) \\
- 2\beta \int_{\mathbb{R}^d} \text{div}_x [D_m W] \ dm(y) - \beta \int_{\mathbb{R}^{2d}} \text{Tr} \left[ D_{mm}^2 W \right] \ dm \otimes dm \leq f(t, x, m)
\]

in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

Proposition

Assume that \( v_1 \) and \( v_2 \) are continuous and bounded vector fields and \( f \) is continuous and bounded. If \( W \) is bounded, then

\[
W \leq \sup_{x, m} |W(T, x, m)| + T \|f\|_{\infty}
\]
Uniqueness for (M2)

**Theorem**

(M2) has at most one classical solution.

**Remarks**

- Standard maximum principle cannot work because of the nonlocal term
  \[
  \int_{\mathbb{R}^d} D_m U(t, x, m) \cdot H_p(y, D_x U(t, y, m), m) m(dy)
  \]

- Usual proof by methods of characteristics
  (C.-Delarue-Lasry-Lions, Carmona-Delarue)

**Sketch of proof** : Let $U_1$ and $U_2$ be two solutions.

- Key step : show that $D_x U_1 = D_x U_2$ by using Bernstein method.
- Indeed, $V = |D_x(U_1 - U_2)|^2$ satisfies $\mathcal{L}(V) \leq C\|V\|_{\infty}$.
- Equality $U_1 = U_2$ then follows again by maximum principle.
Conclusion

In this work:
- We understood how to build a short time solution of the second order master equation with general Hamiltonians,
- obtained uniqueness results without the use of characteristics,

by PDE methods.

Extensions:
- diffusions terms depending on \((x, m)\),
- major/minor MFG problem.

Open problem:
- Existence on large time intervals.
- Regularizing effects of the equation.