Regularity for Diffuse Reflection Boundary Problem to the Stationary Linearized Boltzmann Equation in a Convex Domain

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Base on a joint work with Hsia and Kawagoe
**Boltzmann Equation**

$g(X, \xi, t)$ is the density of mass at position $X$ and time $t$ with velocity $\xi$. In other words, $g(X, \xi, t)\Delta X\Delta \xi$ is the mass inside the small volume $\Delta X\Delta \xi$ in the phase space.

$$\frac{\partial g}{\partial t} + \sum_{i=1}^{3} \xi_i \frac{\partial g}{\partial X_i} = J(g, g),$$

(1)

$$J(g, g) =$$

$$\int_{\mathbb{R}^3} \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} (g(\xi')g(\xi'_*) - g(\xi)g(\xi_*)) B(|\xi - \xi_*|, \theta) d\theta d\phi d\xi_*.$$  

(2)

$J(g, g)$ is called the collision operator.
Elastic collision:

\[ \zeta + \zeta^* = \zeta' + \zeta^*, \]
\[ |\zeta|^2 + |\zeta^*|^2 = |\zeta'|^2 + |\zeta^*|^2. \]  

Let

\[ \alpha = \cos \theta \frac{\zeta^* - \zeta}{|\zeta^* - \zeta|} + \sin \theta \cos \epsilon e_2 + \sin \theta \sin \epsilon e_3. \]  

Then,

\[ \zeta' = \zeta + [(\zeta^* - \zeta) \cdot \alpha] \alpha \]
\[ \zeta_*' = \zeta_* - [(\zeta^* - \zeta) \cdot \alpha] \alpha \]
**Linearized Boltzmann equation**

We consider the standard Maxwellian

\[ E(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}. \]  \hspace{1cm} (6)

We express \( g \) as a perturbation of the Maxwellian.

\[ g = E + E^\frac{1}{2} f. \]  \hspace{1cm} (7)

We have the equation for \( f \).

\[ \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \zeta_i \frac{\partial f}{\partial x_i} = L(f) + \Gamma(f), \text{ where} \] \hspace{1cm} (8)

\[ L(f) = E^{-\frac{1}{2}} \left( J(E^\frac{1}{2} f, E) + J(E, E^\frac{1}{2} f) \right), \] \hspace{1cm} (9)

\[ \Gamma(f) = E^{-\frac{1}{2}} J(E^\frac{1}{2} f, E^\frac{1}{2} f). \] \hspace{1cm} (10)

We ignore \( \Gamma \) and get the linearized Boltzmann equation.
We consider the cross-section:

\[ B(\,|\zeta^* - \zeta|, \theta) = |\zeta^* - \zeta|^\gamma \beta(\theta), \]

where \(0 \leq \gamma \leq 1\) and \(0 \leq \beta(\theta) \leq C \cos \theta \sin \theta\).

Compare with Grad’s angular cutoff:

\[ 0 \leq B \leq C|\zeta^* - \zeta|^\gamma \cos \theta \sin \theta. \]
Properties of the collision operator

\[ L(f) = -\nu(|\zeta|)f + K(f), \]
\[ K(f)(x, \zeta) = \int_{\mathbb{R}^3} k(\zeta, \zeta^*) f(x, \zeta^*) d\zeta^*, \]
\[ \nu(|\zeta|) = \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2} |\eta - \zeta|^{\gamma} d\eta \]
\[ \nu_0(1 + |\zeta|)^\gamma \leq \nu(|\zeta|) \leq \nu_1(1 + |\zeta|)^\gamma. \]
Estimates for Kernel

Let $0 < \delta < 1$.

(Caflisch 1980):

$$\left| k(\zeta, \zeta^*) \right| \leq C_1 |\zeta - \zeta^*|^{-1} \left( 1 + |\zeta| + |\zeta^*| \right)^{-1} \left( 1 - \gamma \right) e^{-\frac{1}{4}(1-\delta) \left( |\zeta - \zeta^*|^2 + \left( \frac{|\zeta|^2 - |\zeta^*|^2}{|\zeta - \zeta^*|^2} \right)^2 \right)},$$

(C., Hsia 2015):

$$\left| \nabla_{\zeta} k(\zeta, \zeta^*) \right| \leq C_2 \frac{1 + |\zeta|}{|\zeta - \zeta^*|^2} \left( 1 + |\zeta| + |\zeta^*| \right)^{-1} \left( 1 - \gamma \right) e^{-\frac{1}{4}(1-\delta) \left( |\zeta - \zeta^*|^2 + \left( \frac{|\zeta|^2 - |\zeta^*|^2}{|\zeta - \zeta^*|^2} \right)^2 \right)}.$$
Stationary linearized Boltzmann equation in a convex domain in $\mathbb{R}^3$

We consider

$$
\zeta \cdot \nabla f(x, \zeta) = L(f),
$$

$x \in \Omega,$

$$
\zeta \in \mathbb{R}^3.
$$

(12)
Boundary Condition

Diffuse reflection boundary condition (proposed by Maxwell):

▶ The velocity distribution function leaving the boundary is in thermal equilibrium with boundary temperature.

▶ There is no net flux cross the boundary.
Diffuse reflection boundary condition for linearized Boltzmann equation:

Let $T(x)$ be the temperature on the boundary.

For $x \in \partial \Omega$ and $\zeta \cdot n(x) < 0$,

$$f(x, \zeta) = \sigma(x)M^{\frac{1}{2}} + T(x)(|\zeta|^2 - \frac{3}{2})M^{\frac{1}{2}},$$  \hspace{1cm} (13)

where

$$M = M(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}.$$

$$\sigma(x) = -\frac{1}{2} T(x) + 2\sqrt{\pi} \int_{\zeta \cdot n > 0} f(x, \zeta)|\zeta \cdot n| M^{\frac{1}{2}} d\zeta.$$  \hspace{1cm} (14)
Existence of solutions:

- Convex domain: Guiraud (1970 J. de Mc.)
- General domain: Esposito, Guo, Kim, and Marra (2013 CMP)

Regularity:

- Continuous alway from the grazing set: Esposito, Guo, Kim, and Marra (2013 CMP)
- (C. 2018 SIMA ) Local Hölder continuity for incoming boundary value problem.

Key idea: An analogy to the Velocity Averaging and Mixture Lemmas.
Regularity for the time evolitional problem (weakly nonlinear):

$$\frac{\partial}{\partial t} f(x, \zeta, t) + \zeta \cdot \nabla_x f(x, \zeta, t) = L(f) + \Gamma(f, f),$$  \hspace{1cm} (15)


These results are not uniform in time.
In this talk, we assume $\Omega$ to be a convex bounded domain in $\mathbb{R}^3$ such that $\partial \Omega$ is $C^2$ and of positive Gaussian curvature.
Definition of solution

We define

\[ p(x, \zeta) : \text{backward trajectory} \cap \partial \Omega, \]
\[ \tau_-(x, \zeta) : \text{traveling time}. \]

Notice that under our assumption,

\[ L(f) = -\nu(|\zeta|)f + K(f). \]

We write

\[ \zeta \cdot \nabla f(x, \zeta) + \nu(|\zeta|)f(x, \zeta) = K(f). \tag{16} \]

Integral equation:

\[ f(x, \zeta) = f(p(x, \zeta), \zeta)e^{-\nu \tau_-(x, \zeta)} + \int_0^{\tau_-(x, \zeta)} e^{-\nu s}K(f)(x - \zeta s, \zeta)ds. \tag{17} \]

We say \( f(x, \zeta) \) is a solution to the stationary linearized Boltzmann equation if the integral equation is satisfied a. e.
Main Theorem

Theorem (C., Hsia, Kawagoe, 2018)

Let \( \Omega \) be a \( C^2 \) convex bounded domain in \( \mathbb{R}^3 \) such that \( \partial \Omega \) is of positive Gaussian curvature. Let \( f \in L_{x,\zeta}^\infty \) be a stationary solution to the diffuse reflection boundary problem on \( \Omega \), (12), for hardsphere, cutoff hard potential, or Maxwellian molecular gases, (11). Suppose the derivative of the boundary temperature is bounded. Then, for \( \epsilon > 0 \),

\[
\sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} f(x, \zeta) \right| + \sum_{i=1}^{3} \left| \frac{\partial}{\partial \zeta_i} f(x, \zeta) \right| \leq C (1 + d_x^{-1})^{4/3} + \epsilon, \tag{18}
\]

where \( d_x \) is the distance between \( x \) and \( \partial \Omega \).
Two difficulties:

- The implicit boundary condition.
- How to improve regularity from Hölder continuity to differentiability.
Sketch of proof

Let
\[ \psi(x) := 2\sqrt{\pi} \int_{\zeta \cdot n > 0} f(x, \zeta) |\zeta \cdot n| M_1^2 d\zeta. \quad (19) \]

Substitute \( f \) above by the integral equation (17) and boundary condition (13) and (14).

\[
\begin{align*}
\psi(x) &= 2\sqrt{\pi} \int_{\zeta \cdot n > 0} T(p(x, \zeta))(|\zeta|^2 - 2)M(\zeta)e^{-\nu(|\zeta|)\tau-(x,\zeta)}|\zeta \cdot n| d\zeta \\
&\quad + 2\sqrt{\pi} \int_{\zeta \cdot n > 0} \psi(p(x, \zeta))M(\zeta)e^{-\nu(|\zeta|)\tau-(x,\zeta)}|\zeta \cdot n| d\zeta \\
&\quad + 2\sqrt{\pi} \int_{\zeta \cdot n > 0} \int_{0}^{\tau-(x,\zeta)} e^{-\nu(|\zeta|)s}K(f)(x - s\zeta, \zeta)M_1^2(\zeta)|\zeta \cdot n| dsd\zeta \\
&=: B_T + B_\psi + D_f.
\end{align*}
\]  
\( (20) \)
Sketch of proof:

- $\psi$ is bounded provided $f \in L^\infty_{x,\zeta}$
- First derivatives of $B_T, B_\psi$ are bounded provided $T, \psi$ are bounded.
- $D_f$ is Hölder continuous provided $f \in L^\infty_{x,\zeta}$.

Now, we can conclude $f$ is locally Hölder continuous by using analysis in (C. 2016). In this research, we further improve the regularity to differentiability.
- $D_f$ is bounded differentiable provided $f$ is locally Hölder up to boundary.
- We have the desired estimate for first derivatives of $f$ provided derivatives of $\psi, T$ are bounded and $f$ is locally Hölder.
Recall

\[ D_f(x) = 2\sqrt{\pi} \int_{\zeta \cdot n > 0} \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} K(f)(x - s\zeta, \zeta) M^{\frac{1}{2}}(\zeta)|\zeta \cdot n|dsd\zeta. \]  

(21)
Proposition (C. Hsia 2015)

Let $1 \leq p \leq \infty$.

\[
\| \nabla_\zeta K(f)(x, \zeta) \|_{L^p_\zeta} \leq C \| f(x, \zeta) \|_{L^p_\zeta}.
\] (22)

Notice that $\| \nabla_\zeta k(\zeta, \zeta^*) \|_{L^\infty_{\zeta} L^1_{\zeta^*}}$ and $\| \nabla_\zeta k(\zeta, \zeta^*) \|_{L^\infty_{\zeta^*} L^1_{\zeta}}$ are bounded. By an argument similar to the proof of Young’s inequality, we have the proposition.
Transfer regularity from velocity to space

Idea: Combination of averaging or collision and transport can transfer regularity in velocity to space. For time evaluational problem in whole space,

- Velocity averaging lemma (Golse, Perthame, Sentis 1985)
- Mixture lemma (Liu, Yu 2004)

In present research, we realize this effect for stationary problem in a convex domain by an interplay between velocity and space.
Let $n(x)$, $e_2$, $e_3$ be an orthonormal basis. Let

$$\zeta' = \rho \cos \theta n(x) + \rho \sin \theta \cos \phi e_2 + \rho \sin \theta \sin \phi e_3,$$

$$r = \rho s,$$

$$\hat{\zeta}' = \frac{\zeta'}{|\zeta'|}.$$  

(23)

Then,

$$D_f = 2\pi^{-\frac{1}{4}} \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|x-p(x,\zeta)|} \rho^2 \sin \theta dr d\phi d\theta d\rho.$$

$$e^{-\frac{\nu(\rho)}{\rho} r} K(f)(x - r\hat{\zeta}, \zeta) |\hat{\zeta} \cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin \theta dr d\phi d\theta d\rho.$$  

(24)
\[ D_f = 2\pi^{-\frac{1}{4}} \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^\infty \rho \left| x - p(x, \zeta) \right| \hat{\zeta} \cdot n(x) e^{-\frac{1}{\rho} |\zeta|^2} \rho^2 \sin \theta \, d\theta d\phi d\rho. \] (25)
Let \( y = x - r\hat{\zeta} \)

\[
D_f = 2\pi^{-\frac{1}{4}} \int_0^\infty \int_\Omega \int_{\Omega} e^{-\frac{\nu(\rho)}{\rho} |x-y|} K(f)(y, \rho \frac{x-y}{|x-y|}) \frac{(x-y) \cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 dyd\rho.
\]

(27)
Definition

Let \( x, \eta \in \mathbb{R}^3 \) and \( D \) be a \( C^1 \) surface in \( \mathbb{R}^3 \) and \( f : D \subset \mathbb{R}^3 \to \mathbb{R} \).

Suppose \( \phi : (-\epsilon, \epsilon) \to D \) is a smooth space curve such that

\[
\phi(0) = x, \quad \left. \frac{d}{dt} \phi(t) \right|_{t=0} = \eta. \tag{28}
\]

We define

\[
\nabla^x_\eta f(x) := \left. \frac{d}{dt} f(\phi(t)) \right|_{t=0} \tag{29}
\]

when the limit at right-hand-side exists.
Notice that the velocity is represented by space variable. However, if we differentiate directly, the integrant has a singularity of order \( \frac{1}{|x - y|^3} \), which is not integrable in \( \Omega \subset \mathbb{R}^3 \). This is the reason the result of (C. 2016) is restricted to Hölder continuity.
Lemma

Under the assumptions in the main theorem, suppose $x \in \partial \Omega$ and $y \in \Omega$. Then,

$$|f(y) - f(x)| \leq C|x - y|^{1-\epsilon} \left(1 + \frac{1}{|\zeta|}\right).$$  \hspace{1cm} (30)
\[ \nabla^x_D f(x) = \int_0^\infty \int_{\Omega} \int_{\mathbb{R}^3} \nabla^x_v \left( e^{-\frac{\nu(\rho)}{\rho} |x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y) \cdot n(x)}{|x-y|^3} \right) \]
\[
\cdot \left[ f(y, \zeta') - f(x, \zeta') \right] e^{-\frac{\rho^2}{2}} \rho^2 d\zeta' dy d\rho
\]
\[- \int_0^\infty \int_{\mathbb{R}^3} \int_{\Omega} \text{div}_y \left( e^{-\frac{\nu(\rho)}{\rho} |x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y) \cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 f(x, \zeta') \right) dy d\zeta' d\rho
\]
\[+ \int_0^\infty \int_{\mathbb{R}^3} \int_{\Omega} e^{-\frac{\nu(\rho)}{\rho} |x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y) \cdot \nabla^x_v(n(x))}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 f(x, \zeta') dy d\zeta' d\rho
\]
\[=: \nabla^x D^1_f + \nabla^x D^2_f + \nabla^x D^3_f. \]
\[ \nabla^x D^2_{f, \epsilon} = - \int_0^\infty \int_{\mathbb{R}^3} \int_{\partial \Omega \setminus B(x, \epsilon)} e^{-\frac{\nu(\rho)}{\rho} |x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') (x-y) \cdot n(x) \frac{|x-y|}{|x-y|^3} \]

\[ e^{-\frac{\rho^2}{2}} \rho^2 f(x, \zeta') [v \cdot n(y)] dA(y) d\zeta' d\rho \]

\[ - \int_0^\infty \int_{\mathbb{R}^3} \int_{\partial B(x, \epsilon) \cap \Omega} e^{-\frac{\nu(\rho)}{\rho} |x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') (x-y) \cdot n(x) \frac{|x-y|}{|x-y|^3} \]

\[ e^{-\frac{\rho^2}{2}} \rho^2 f(x, \zeta') [v \cdot \frac{x-y}{|x-y|}] \frac{1}{\epsilon^2} dA(y) d\zeta' d\rho \]

\[ =: S^\epsilon + B^\epsilon. \]
We need help from geometry.

**Lemma**

*Adopting the same geometric assumption on* $\Omega$ *as stated in the main theory. Then, there exist* $\gamma_1$, $C > 0$ *such that for any* $x, y \in \partial \Omega$ *with* $y \in GB(x, r_1)$, *we have*

\[
|n(x) \cdot (x - y)| \leq C|x - y|^2, \quad (33)
\]
\[
|n(y) \cdot (x - y)| \leq C|x - y|^2, \quad (34)
\]
\[
|n(y) \cdot v| = |n(y) \cdot (v - v')| \leq C|x - y|, \quad (35)
\]

*where* $v \in T_x(\partial \Omega)$ *is a unit vector and* $v' \in T_y(\partial \Omega)$ *is the parallel transport of* $v$ *from* $T_x(\partial \Omega)$ *to* $T_y(\partial \Omega)$. 
Differentiability of $B_\psi$.

Recall

$$B_\psi(x) = 2\sqrt{\pi} \int_{\zeta \cdot n(x) > 0} \psi(p(x, \zeta))M(\zeta)e^{-\nu(|\zeta|)}\tau_-(x, \zeta)|\zeta \cdot n(x)|d\zeta.$$
Let $p(x, \zeta) = y$, $\zeta = (x - y)/l$.

\[
B_\psi(x) = 2\sqrt{\pi} \int_{\zeta \cdot n(x) > 0} \psi(p(x, \zeta)) M(\zeta) e^{-\nu(|\zeta|)} \tau - (x, \zeta) |\zeta \cdot n(x)| d\zeta
= \frac{2}{\pi} \int_0^\infty \int_{\partial \Omega} \psi(y) e^{-l^2 |x-y|^2} e^{-\nu(l|x-y|)}
\times [(x - y) \cdot n(x)] |(x - y) \cdot n(y)| l^3 dA(y) dl.
\]
Thank you!