GREEN'S FUNCTION OF COMPRESSIBLE NAVIER-STOKES AROUND A HYPERBOLIC CONTACT DISCONTINUITY

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1. Abstract

The one dimensional compressible Navier-Stokes equation modeled for an ideal gas in terms of the Lagrangian coordinate is

\[
\begin{align*}
  v_t - u_x &= 0, \\
  u_t + p_x &= (\mu u_x / v)_x, \\
  E_t + (up)_x &= \left( \frac{\mu uu_x}{v} + \kappa \theta x / v \right)_x,
\end{align*}
\]

where

\[
\begin{align*}
  E &= \frac{1}{2} u^2 + e, \\
  \theta &= e / (\gamma - 1), \quad \gamma > 1, \\
  p &= \theta v,
\end{align*}
\]

and the parameters \( \mu \) and \( \kappa \) stand for the bulk dissipation constant and the heat conductivity. This system can be rewritten in the form

\[
\vec{U}_t + F(\vec{U})_x = (B(\vec{U})\vec{U}_x)_x.
\]

Let \( p_0 \) be a given positive state, any two states \((\vec{U}^-_+, \vec{U}^+_+)\) can be connected by a stationary contact discontinuity for the compressible Euler equation:

\[
\vec{U}_t + F(\vec{U})_x = \vec{0}.
\]

The stationary contact discontinuity is a two-valued vector-valued function of the form:

\[
\Psi(x) = H(x)\vec{U}^+_+ + (1 - H(x))\vec{U}^-_-, 
\]

where \( H(x) \) is the Heaviside function. The key ingredient for studying the time-asymptotic behavior of an initial value problem around a contact discontinuity for the system (??) is the pointwise structure of the Green’s function \( G(x, t; z) \) in \( x-t \) domain for the linearized system around the hyperbolic contact discontinuity \( \Psi \). The Green’s function \( G(x, t; z) \) is defined by the solution of the initial value problem:

\[
\begin{align*}
  \partial_t U + \partial_x F'(\Psi)U - \partial_x B(\Psi) \partial_x U &= 0, \\
  U(x, 0) &= \delta(x - z)I,
\end{align*}
\]

where \( I \) is a \( 3 \times 3 \) identity matrix.

To obtain the pointwise structure of the Green’s function for (??), one needs to find the fundamental solution \( G^\pm(x, t) \) for the problem

\[
\partial_t U + \partial_x F'(\vec{U}_\pm)U_1 - \partial_x B(\vec{U}_\pm) \partial_x U_1 = \vec{0};
\]
and to obtain the interaction of the fundamental solutions with the contact discontinuity.

The pointwise structure of the Green’s function is an instrument for developing the analysis for a quasi-linear problem toward problems with boundaries, with various different background states, etc. Thus, the global pointwise regularity structure of the fundamental solutions in all scales in space-time domain needs to be realized. For this purpose, one applies the procedure given in [7] to expand the Fourier transform $\hat{G}^\pm(\eta, t)$ of the fundamental solutions $G_\pm(x, t)$ of (7) in its Fourier variable $\eta$ at $\eta = \infty$ in order to remove the singularities due to the $\delta$-functions in the initial data. It leads to the construction of the singular functions $G^{\pm, *}(x, t)$, which decay exponentially in $x$-t domain for $x \neq 0$ and satisfy

$$\begin{align*}
|G^{\pm, *}(x, t)| &\leq O(1)e^{-(|x|+t)/C_0} \text{ for } x \neq 0, \\
|G^{\pm, *}(x, 0) - G^{\pm, *}(x, 0)| &\equiv 0, \\
|\partial_x (\partial_t + \partial_x F(U_\pm) - \partial_x B(U_\pm) \partial_x)G^{\pm, *}| &\leq O(1)e^{-(|x|+t)/C_0} \text{ for } C_0 > 1, \ k \leq 4.
\end{align*}$$

Then, one applies the long wave-short wave decomposition combined with a weighted energy method introduced in [7] to show that

$$|G^{\pm}(x, t) - G^{\pm, *}(x, t)| \leq O(1)\frac{e^{-(|x|+t)/C_0} + e^{-(x+A_t)/C_0 + t} + e^{-(x-A_t)/C_0 + t}}{\sqrt{1+t}}$$

as well as obtain further regularity structures $G^{\pm}(x, t)$, where $A_\pm$ are the sound speeds at the states $U_\pm$.

To study the interactions between $G^\pm$ across the hyperbolic contact discontinuity, one uses the Laplace transform method given in [7, 8] to study the interactions. The Laplace transform of (7) in the time variable $t$ becomes an ODE system

$$sL[G] + \partial_x F(\Psi)L[G] - \partial_x B(\Psi)\partial_x L[G] = \delta(x-z)l,$$

where

$$L[G](x, s; z) = \int_0^\infty e^{-st}G(x, t; z)dt.$$

The interactions across the contact discontinuity is imposed as the continuities in the $x$ variable:

$$\begin{align*}
L[G](x, s; z) : \text{ Continuous at } x = 0, \\
(F(\Psi) - B(\Psi)\partial_x)L[G](x, s; z) : \text{ Continuous at } x = 0.
\end{align*}$$

To realize the interaction from the continuities, one introduces the “Laplace wave numbers” $\lambda_{\pm,i}(s)$, $i = 1, 2, 3, 4$ and the Laplace wave trains $e^{\lambda_{\pm, i}x}$ to synthesize the interactions, where $\lambda_{\pm, i}(s)$ are the roots of the characteristic polynomials $p_\pm(\xi; s)$ in $\xi$ of the ODE systems with respect to $x = \pm \infty$:

$$p_\pm(\xi; s) = det\left(s \xi F(\bar{U}_\pm) + \xi^2B(\bar{U}_\pm) \right).$$

One synthesizes $L[G_\pm](x, s)$ as a sum of forward Laplace wave trains and backward Laplace wave trains as follows

$$L[G_\pm](x, s) = H(x)\sum_{i=1}^2 A^f_{\pm,i}(s)e^{\lambda_{\pm,i}x} + (1 + H(x))\sum_{i=3}^4 A^b_{\pm,i}(s)e^{\lambda_{\pm,i}x},$$

where the matrices $A^f_{\pm}$ and $A^b_{\pm}$ are functions of $s$ only; and for $Re(s) > 0$

$$\begin{align*}
Re(\lambda_{\pm,i}(s)) < 0 \text{ for } i = 1, 2, \\
Re(\lambda_{\pm,i}(s)) > 0 \text{ for } i = 3, 4.
\end{align*}$$
Then, the Green’s function $L[G](x, s; z)$ can be expressed

$$L[G](x, s; z) = egin{cases} H(x) \left( L[G_+](x, s; z) + \sum_{1 \leq i,j \leq 2} R_{+,ij} e^{\lambda_{+,j} z + \lambda_{+,j} x} \right) + (1 - H(x)) \sum_{3 \leq i,j \leq 4} T_{+,ij} e^{\lambda_{+,j} z + \lambda_{+,j} x} & \text{for } z > 0, \\ H(x) \sum_{1 \leq i,j \leq 2} T_{-,ij} e^{\lambda_{-,j} z + \lambda_{+,j} x} + (1 - H(x)) \left( L[G_-](x, s; z) + \sum_{3 \leq i,j \leq 4} R_{-,ij} e^{\lambda_{-,j} z + \lambda_{-,j} x} \right) & \text{for } z < 0. \end{cases}$$

Here, the reflection matrices $R_{\pm,ij}$ and the transmission matrices $T_{\pm,ij}$ are functions of $s$ only, and they are obtained through (??).

For each individual Laplace wave train $e^{\lambda_{\pm,i} x}$, in the region $t \in (0, 1)$ or in the domain $\{(x, t)|t \geq 1, |x| > 2 \max\{\Lambda_-, \Lambda_+\}\}$ one needs to use the pointwise structure of $G_{\pm}(x, t)$ and matrices $A_{\pm,ij}^f$, $A_{\pm,ij}^b$ together to construct the pointwise structure of $L^{-1}[e^{\lambda_{\pm,i} x}](t)$. In the domain $\mathcal{D} \equiv \{|x| \leq 2t \max\{\Lambda_-, \Lambda_+\}\} \cap \{t \geq 1\}$, one uses the notion “Laplace-Fourier path” introduced in [?] and the complex analysis method given in [?] to obtain the pointwise structure of the Laplace wave $e^{\lambda_{\pm,i} x}$.

Here, the Laplace-Fourier path requires analyticity of $e^{\lambda_{\pm,i}(s)}$ around $Re(s) = 0$ so that through the Cauchy integral formula the Broamwich’s integral for in versing the Laplace transform becomes the inverse Fourier transform of a known function. Thus, one obtains the global pointwise structure of the Laplace wave train in all space-time scales. Finally, the global pointwise structure of $G(x, t; z)$ in space-time domain is obtained.

**References**


[4] Liu, T.-P.; Yu, S.-H.,

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