GREEN'S FUNCTION OF COMPRESSIBLE NAVIER-STOKES AROUND A HYPERBOLIC CONTACT DISCONTINUITY

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1. Abstract

The one dimensional compressible Navier-Stokes equation modeled for an ideal gas in terms of the Lagrangian coordinate is

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = (\mu u_x/v)_x, \\ E_t + (up)_x = \Big(\mu u u_x/v_x + \kappa\theta_x/v\Big)_x, \end{cases} \tag{1}$$

where

$$\begin{cases} E = \dfrac{1}{2}u^2 + e, \\ \theta = e/(\gamma-1),\ \gamma > 1, \\ p = \theta v, \end{cases}$$

and the parameters μ and κ stand for the bulk dissipation constant and the heat conductivity. This system can be rewritten in the form

$$\vec{\boldsymbol{U}}_t + \mathsf{F}(\vec{\boldsymbol{U}})_x = (\mathsf{B}(\vec{\boldsymbol{U}})\vec{\boldsymbol{U}}_x)_x. \tag{2}$$

Let p_0 be a given positive state, any two states $(\vec{\boldsymbol{U}}_-, \vec{\boldsymbol{U}}_+)$,

$$\vec{\boldsymbol{U}}_\pm \equiv (\rho_\pm, u_\pm, E_\pm) \in \{(\rho, 0, p_0/((\gamma-1)\rho) : \ \rho > 0\},$$

can be connected by a stationary contact discontinuity for the compressible Euler equation:

$$\vec{\boldsymbol{U}}_t + \mathsf{F}(\vec{\boldsymbol{U}})_x = \vec{0}. \tag{3}$$

The stationary contact discontinuity is a two-valued vector-valued function of the form:

$$\Psi(x) = H(x)\vec{\boldsymbol{U}}_+ + (1 - H(x))\vec{\boldsymbol{U}}_-, \tag{4}$$

where $H(x)$ is the Heaviside function. The key ingredient for studying the time-asymptotic behavior of an initial value problem around a contact discontinuity for the system (**??**) is the pointwise structure of the Green's function $\mathbb{G}(x,t;z)$ in x-t domain for the linearized system around the hyperbolic contact discontinuity Ψ. The Green's function $\mathbb{G}(x,t;z)$ is defined by the solution of the initial value problem:

$$\begin{aligned} &\mathbb{G}(x,t;z) \equiv U(x,t), \\ &\begin{cases} \partial_t U + \partial_x \mathsf{F}'(\Psi)U - \partial_x \mathsf{B}(\Psi)\partial_x U = 0, \\ U(x,0) = \delta(x-z)\mathsf{I}. \end{cases} \end{aligned} \tag{5}$$

where I is a 3×3 identity matrix.

To obtain the pointwise structure of the Green's function for (**??**), one needs to find the fundamental solution $\mathbb{G}^\pm(x,t)$ for the problem

$$\partial_t U + \partial_x \mathsf{F}'(\vec{\boldsymbol{U}}_\pm)U - \partial_x \mathsf{B}(\vec{\boldsymbol{U}}_\pm)\partial_x U = \vec{0}; \tag{6}$$

and to obtain the interaction of the fundamental solutions with the contact discontinuity.

The pointwise structure of the Green's function is an instrument for developing the analysis for a quasi-linear problem toward problems with boundaries, with various different background states, etc.. Thus, the global pointwise regularity structure of the fundamental solutions in all scales in space-time domain needs to be realized. For this purpose, one applies the procedure given in [?] to expand the Fourier transform $\hat{\mathbb{G}}^{\pm}(\eta,t)$ of the fundamental solutions $\mathbb{G}_{\pm}(x,t)$ of (??) in its Fourier variable η at $\eta=\infty$ in order to remove the singularities due to the δ-functions in the initial data. It leads to the construction of the singular functions $\mathbb{G}^{\pm,*}(x,t)$, which decay exponentially in x-t domain for $x\neq 0$ and satisfy

$$\begin{cases}|\mathbb{G}^{\pm,*}(x,t)|\le O(1)e^{-(|x|+t)/C_0} \text{ for } x\neq 0,\\ |\mathbb{G}^{\pm,*}(x,0)-\mathbb{G}^{\pm,*}(x,0)|\equiv 0,\\ \left|\partial_x^k\left(\partial_t+\partial_x\mathsf{F}'(\mathsf{U}_\pm)-\partial_x\mathsf{B}(\mathsf{U}_\pm)\partial_x)\mathbb{G}^{\pm,*}\right)\right|\le O(1)e^{-(|x|+t)/C_0} \text{ for } C_0>1,\ k\le 4.\end{cases} \tag{7}$$

Then, one applies the long wave-short wave decomposition combined with a weighted energy method introduced in [?] to show that

$$|\mathbb{G}^{\pm}(x,t)-\mathbb{G}^{\pm,*}(x,t)|\le O(1)\frac{e^{-(|x|+t)/C_0}+e^{-\frac{(x-\Lambda_\pm t)^2}{C_0(t+1)}}+e^{-\frac{(x+\Lambda_\pm t)^2}{C_0(t+1)}}}{\sqrt{t+1}},$$

as well as obtain further regularity structures $\mathbb{G}^{\pm}(x,t)$, where $\Lambda_\pm$ are the sound speeds at the states $\vec{\boldsymbol{U}}_\pm$.

To study the interactions between $\mathbb{G}^{\pm}$ across the hyperbolic contact discontinuity, one uses the Laplace transform method given in [?, ?] to study the interactions. The Laplace transform of (??) in the time variable t becomes an ODE system

$$s\mathbb{L}[\mathbb{G}]+\partial_x\mathsf{F}'(\Psi)\mathbb{L}[\mathbb{G}]-\partial_x\mathsf{B}(\Psi)\partial_x\mathbb{L}[\mathbb{G}]=\delta(x-z)\mathsf{I}, \tag{8}$$

where

$$\mathbb{L}[\mathbb{G}](x,s;z)\equiv\int_0^\infty e^{-st}\mathbb{G}(x,t;z)dt.$$

The interactions across the contact discontinuity is imposed as the continuities in the x variable:

$$\begin{cases}\mathbb{L}[\mathbb{G}](x,s;z):\ \text{Continuous at } x=0,\\ (\mathsf{F}'(\Psi)-\mathsf{B}(\Psi)\partial_x)\mathbb{L}[\mathbb{G}](x,s;z):\ \text{Continuous at } x=0.\end{cases} \tag{9}$$

To realize the interaction from the continuities, one introduces the "Laplace wave numbers" $\lambda_{\pm,i}(s)$, $i=1,2,3,4$ and the Laplace wave trains $e^{\lambda_{\pm,i}x}$ to synthesize the interactions, where $\lambda_{\pm,i}(s)$ are the roots of the characteristic polynomials $p_\pm(\xi;s)$ in ξ of the ODE systems with respect to $x=\pm\infty$:

$$p_\pm(\xi;s)=det\Big(s\mathsf{I}+\xi\mathsf{F}'(\vec{\boldsymbol{U}}_\pm)-\xi^2\mathsf{B}(\vec{\boldsymbol{U}}_\pm)\Big).$$

One synthesizes $\mathbb{L}[\mathbb{G}_\pm](x,s)$ as a sum of forward Laplace wave trains and backward Laplace wave trains as follows

$$\mathbb{L}[\mathbb{G}_\pm](x,s)=H(x)\sum_{i=1}^{2}A^f_{\pm,i}(s)e^{\lambda_{\pm;i}x}+(1-H(x))\sum_{i=3}^{4}A^b_{\pm,i}(s)e^{\lambda_{\pm;i}x}, \tag{10}$$

where the matrices $A^f_\pm$ and $A^b_\pm$ are functions of s only; and for $Re(s)>0$

$$\begin{cases}Re(\lambda_{\pm,i}(s))<0 \text{ for } i=1,2,\\ Re(\lambda_{\pm,i}(s))>0 \text{ for } i=3,4.\end{cases}$$

Then, the Green's function $\mathbb{L}[\mathbb{G}](x,s;z)$ can be expressed

$$\mathbb{L}[\mathbb{G}](x,s;z) = \tag{11}$$

$$\begin{cases} H(x)\left(\mathbb{L}[\mathbb{G}_+](x,s;z)+\sum\limits_{\substack{1\le i\le 2\\3\le j\le 4}} R_{+;j,i}e^{\lambda_{+,j}z+\lambda_{+,j}x}\right)+(1-H(x))\sum\limits_{3\le i,j\le 4} T_{+;j,i}e^{\lambda_{+,j}z+\lambda_{-,j}x} & \text{for } z>0, \\ H(x)\sum\limits_{1\le i,j\le 2} T_{-;j,i}e^{\lambda_{-,j}z+\lambda_{+,j}x}+(1-H(x))\left(\mathbb{L}[\mathbb{G}_-](x,s;z)+\sum\limits_{\substack{1\le j\le 2\\3\le i\le 4}} R_{-;j,i}e^{\lambda_{-,j}z+\lambda_{-,j}x}\right) & \text{for } z<0. \end{cases}$$

Here, the reflection matrices $R_{\pm;i,j}$ and the transmission matrices $T_{\pm;i,j}$ are functions of s only, and they are obtained through (**??**).

For each individual Laplace wave train $e^{\lambda_{\pm;i}x}$, in the region $t\in(0,1)$ or in the domain $\{(x,t)|t\ge 1, |x|>2\max\{\Lambda_-,\Lambda_+\}$ one needs to use the pointwise structure of $\mathbb{G}_\pm(x,t)$ and matrices $A^f_{\pm,j}$, $A^b_{\pm,i}$ together to construct the pointwise structure of $\mathbb{L}^{-1}[e^{\lambda_{\pm;i}x}](t)$. In the domain $\mathscr{D}\equiv\{|x|\le 2t\max\{\Lambda_-,\Lambda_+\}\}\cap\{t\ge 1\}$, one uses the notion "Laplace-Fourier path" introduced in [**?**] and the complex analysis method given in [**?**] to obtain the pointwise structure of the Laplace wave $e^{\lambda_{\pm,i}x}$.

Here, the Laplace-Fourier path requires analyticity of $e^{\lambda_{\pm,i}(s)}$ around $Re(s)=0$ so that through the Cauchy integral formula the Broamwich's integral for inversing the Laplace transform becomes the inverse Fourier transform of a known function. Thus, one obtains the global pointwise structure of the Laplace wave train in all space-time scales. Finally, the global pointwise structure of $\mathbb{G}(x,t;z)$ in space-time domain is obtained.

References

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