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Some Combinatorics of the Hypergeometric Series[†]

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Symmetry formulas for the classical hypergeometric series ${}_2F_1$ are proved combinatorially. The idea of the proofs is to find weighted combinatorial structures which form models for each side of the formula and to show how to go from the first to the second model by a 'weak isomorphism' (i.e. a sequence of isomorphisms, regroupings and degroupings of structures). This is then applied to the four ${}_2F_1$-families (Meixner, Krawtchouk, Meixner–Pollaczek and Jacobi) of hypergeometric orthogonal polynomials. We give three 'weakly isomorphic' models for each family and prove in a completely combinatorial way the 3-terms recurrences for these polynomials.

1. Introduction

In this paper we describe several combinatorial models for ${}_2F_1$-hypergeometric polynomials [1] (Meixner, Krawtchouk, Meixner–Pollaczek and Jacobi polynomials) using T-structures ('pairs of complementary permutations') and M-structures ('Meixner endofunctions' [8]).

These models are 'combinatorially equivalent' because the symmetry identities for the classical hypergeometric series ${}_2F_1$, which transforms one expression for the polynomial into another one, can be proved combinatorially.

We use the models to prove the three terms recurrence formulas for these polynomials in a purely combinatorial way.

NOTATION. $[n] = \{1, 2, \ldots, n\}$ and $[0] = \phi$; $(a)_n$ denotes the rising factorial $a(a+1)\ldots(a+n-1)$; $(a)_0 = 1$; the classical hypergeometric series is

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. $$

2. Preliminaries

Let R be a ring. If (X, w) is a weighted finite set (with $w\colon X \to R$ the weight function) then the total weight of X is

$$ |X| = \sum \{w(x) \mid x \in X\} \in R $$

(sometimes we will write $|X|_w$ to be more precise).

Throughout the text we will sometimes use the language of species [11, 12, 13, 22] but a knowledge of species is not a prerequisite. Just recall that a 'weighted 1-species' T, with weights in the ring R, associates to any finite set U a weighted finite set $T[U]$ of so-called T-structures on U (the weight of $t \in T[U]$ is denoted $w(t) \in R$) and to any bijection $f\colon U \to V$ a bijection $T[f]\colon T[U] \to T[V]$. There is an (exponential) generating series for T:

$$ T(x) = \sum_{n \geq 0} |T[n]| \frac{x^n}{n!}. $$

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Weighted species can be added, multiplied, substituted and derived in such a way that these operations correspond to the same operation between generating series. Two weighted species are equipotent if they have the same generating series: they are isomorphic (a much stronger condition) if they are 'combinatorially the same' (see [11, 13] for the precise definition).

By a 'combinatorial model' for a family, $(p_n)_{n \geq 0}$, of polynomials we mean a weighted 1-species T such that $|T[U]| = p_n$ (or $\alpha_n p_n$ for some constants α_n) if $|U| = n$. Two such models are said to be 'weakly isomorphic' (stronger than equipotent but weaker than isomorphic) if one can pass the first to the second structures by 'bijections' and 'regroupings' of structures, i.e. if there is a sequence of isomorphisms and epimorphisms leading from the first weighted species to the second.

3. T-and M-Structures

Definition 1. Given a finite set U, let $S[U]$ be the set of permutations of U.

Lemma 1. *If for* $\sigma \in S[U]$ *we set* $w(\sigma) = a^{\text{cyc}(\sigma)}$, *where* cyc($\sigma$) *is the number of cycles of* σ, *then we have* $|S[U]| = (a)_n$ *if* $|U| = n$.

Proof. See [7, 10, 15, 19].

Definition 2. For two finite disjoint sets A and B, let the set of *Laguerre configurations* on (A, B) be:

$$\mathbf{L}[A, B] = \{f: A \to A + B \mid f \text{ injective}\}.$$

Lemma 2. *If for* $f \in \mathbf{L}[A, B]$ *we set* $w(f) = \alpha^{\text{cyc}(f)}$, *where* cyc($f$) *is the number of cycles of* f, *then we have*

$$|\mathbf{L}[A, B]| = (\alpha + j)_i \quad \text{if } |A| = i \text{ and } |B| = j.$$

Proof. See [7, 10, 15, 19].

Definition 3. Let $\mathbf{T}[A, B] = S[A] \times S[B]$ and $\mathbf{M}[A, B] = S[A] \times \mathbf{L}[B, A]$.

Definition 4. For any finite set U, let

$$T[U] = \{(A, B, \sigma, \tau) \mid A \cup B = U, A \cap B = \phi, (\sigma, \tau) \in \mathbf{T}[A, B]\}$$

with $w_1(A, B, \sigma, \tau) = r^{|A|} s^{|B|} u^{\text{cyc}(\sigma)} v^{\text{cyc}(\tau)}$ and

$$M[U] = \{(A, B, \sigma, f) \mid A \cup B = U, A \cap B = \phi, (\sigma, f) \in \mathbf{M}[A, B]\}$$

with

$$M[U] = \{(A, B, \sigma, f) \mid A \cup B = U, A \cap B = \phi, (\sigma, f) \in \mathbf{M}[A, B]\}$$

with $w_2(A, B, \sigma, f) = r^{|A|} s^{|B|} u^{\text{cyc}(\sigma)} v^{\text{cyc}(f)}$ where r, s, u and v are formal variables and weights are in the ring $\mathbb{Z}[r, s, u, v]$.

In the following figures, U is the set of points, $\boxed{A \} B}$ represents an ordered partition (A, B) of U, and σ, τ and f are represented by their graph. The weight of a structure is the product of the weights of its constituents.

Remarks. (1) In [8], M-structures were called 'Meixner endofunctions'.
(2) In the language of species, T and M are the 1-species associated to the 2-species $\mathbf{T}$ and $\mathbf{M}$ respectively.

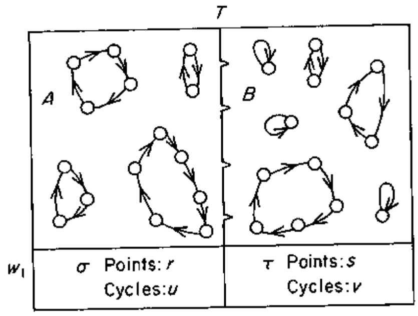

FIGURE 1.

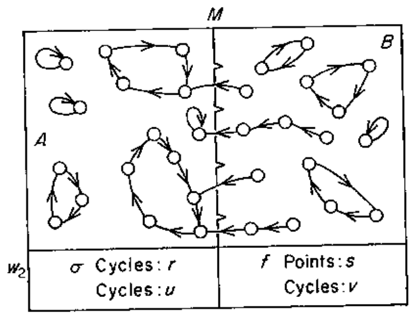

FIGURE 2.

(3) In Section 5, T and M will be used as common models for Meixner, Krawtchouk, Meixner–Pollaczek and Jacobi polynomials by choosing specific values for r, s, u and v. When this is done, we write $T(r, s; u, v)$ and $M(r, s; u, v)$ for T and M respectively.

Let $T_n = |T[n]|$ and $M_n = |M[n]|$. Later, when specific values will be given to r, s, u and v, we will write $T_n(r, s; u, v)$ and $M_n(r, s; u, v)$ respectively for T_n and M_n.

PROPOSITION 1. *We have the following* explicit expressions:

$$T_n = \sum_{i+j=n} \binom{n}{i} (u)_i (v)_j r^i s^j; \tag{3.1}$$

$$M_n = \sum_{i+j=n} \binom{n}{i} (u)_i (v + i)_j r^i s^j. \tag{3.2}$$

PROOF. Direct applications of Lemmas (3.1) and (3.2).

PROPOSITIONS 2. *We have the following* generating functions:

$$\sum_{n\geqslant 0} T_n \frac{t^n}{n!} = (1 - rt)^{-u}(1 - st)^{-v}; \tag{3.3}$$

$$\sum_{n\geqslant 0} M_n \frac{t^n}{n!} = (1 - (r + s)t)^{-u}(1 - st)^{u-v}. \tag{3.4}$$

PROOF. Recall that for any species P, an e^P-structure on U is a partition of U together with a P-structure on each of its classes, i.e. an 'assembly of P-structures'.

We have:

$$T = e^{uC(rX)} \cdot e^{vC(sX)}, \tag{3.5}$$

$$M = e^{uC(rX\,L(sX))} \cdot e^{vC(sX)}, \tag{3.6}$$

where C, L and X are respectively the species of 'cyclic permutations', 'linear orders' and 'singletons'. Since $C(t) = -\log(1-t)$ and $L(t) = (1-t)^{-1}$, (3.3) and (3.4) follow.

PROPOSITION 3. *We have the following* recurrence formulas:

$$T_{n+1} = (ru + sv + (r+s)n)T_n - rs(u + v + n - 1)nT_{n-1}; \tag{3.7}$$

$$M_{n+1} = (ru + sv + (r+2s)n)M_n - s(r+s)(v + n - 1)nM_{n-1}. \tag{3.8}$$

PROOF. Derive both sides of (3.3) (resp. (3.4)), multiply by $(1-rt)(1-st)$ (resp. $(1-(r+s)t)(1-st)$), and equate coefficients of t^{n+1}.

REMARK. A combinatorial proof of (3.7) will be given in Section 6.

4. SYMMETRY FORMULAS FOR $_2F_1$

The object of this section is to prove combinatorially the following well known symmetry formulas (in the 'terminating case' where a is $-n$):

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = (1-z)^{-a} \cdot {}_2F_1\left[\begin{matrix} a, c-b \\ c \end{matrix}; \frac{z}{z-1}\right]; \tag{4.1}$$

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \frac{(c-b)_n}{(c)_n} \cdot {}_2F_1\left[\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right]. \tag{4.2}$$

This will be applied in Section 5 to explain why $_2F_1$-hypergeometric polynomials (such as Meixner, Krawtchouk, Meixner–Polloczek and Jacobi polynomials) have several combinatorially equivalent models.

LEMMA 3. *Let* $M = M(-1, 1; a, b)$; *then*

$$M_n = (b)_n \cdot {}_2F_1\left[\begin{matrix} -n, a \\ b \end{matrix}; 1\right]. \tag{4.3}$$

PROOF. Since

$$(b)_n \cdot {}_2F_1\left[\begin{matrix} -n, a \\ b \end{matrix}; 1\right] = \sum_{i+j=n} \binom{n}{i} (a)_i (b+i)_j (-1)^i$$

this follows from (3.2).

PROPOSITION 4. *The following identity is combinatorially true*:

$$(b)_n \cdot {}_2F_1\left[\begin{matrix} -n, a \\ b \end{matrix}; 1\right] = (b-a)_n. \tag{4.4}$$

PROOF. In Fig. 3, every rectangle contains a typical structure on $U = [n]$ whose weight is described under this rectangle. The maps Φ, ψ and χ will now be described (Φ is a bijection, and ψ and χ are surjective maps which are used to 'regroup structures' in a nice way):

Φ is Foata's isomorphism (see [8, 17]) between 'Meixner endofunctions' and 'bicolored permutations' on U. It preserve weights, i.e. $w_1(\sigma, f) = w_2(\Phi(\sigma, f))$;
ψ is the forgetful epimorphism which 'forgets the coloring but separates mixed cycles (i.e. with at least one black vertex) from white cycles';
χ is the forgetful epimorphism which 'forgets the separation'.

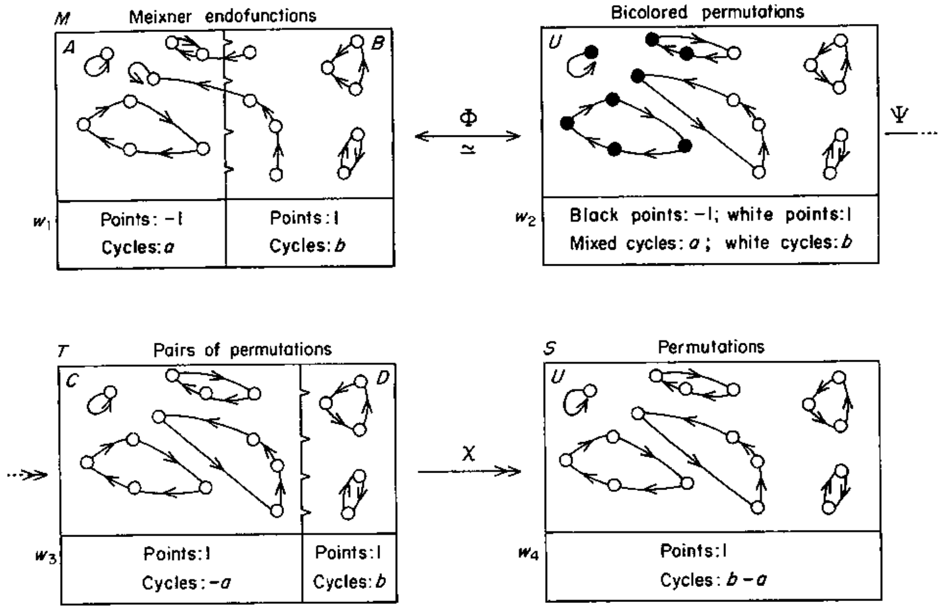

FIGURE 3.

Both epimorphisms ψ and χ (and obviously also Φ) have the property that the weight of a structure is the total weight of its inverse image. If $\sigma \in S[U]$ is a permutation (say with $\mathrm{cyc}(\sigma) = k$), then in rebuilding all the T-structures above σ by χ every one of the k cycles of σ can be put either on the left (with weight $-a$) or on the right (with weight b), so that:

$$|\chi^{-1}(\sigma)|_{w_3} = (-a + b)^k = w_4(\sigma).$$

Moreover, if c is an m-cycle then:

$$|(\chi \circ \psi)^{-1}(c)|_{w_2} = \left(\sum_{i=1}^{m} \binom{m}{i} (-1)^i\right) a + b = b - a.$$

Since, in rebuilding a bicolored cycle above c by $\chi \circ \psi$, we can make it completely white (of weight b) or mixed (of weight a) with i ($1 \leqslant i \leqslant m$) black points (each of weight -1), for $\sigma \in S[U]$, if $\mathrm{cyc}(\sigma) = k$, then

$$(b - a)^k = w_4(\sigma) = |\chi^{-1}(\sigma)|_{\omega_3} = |(\chi \circ \psi)^{-1}(\sigma)|_{w_2} = |(\chi \circ \psi \circ \Phi)^{-1}(\sigma)|_{w_1},$$

so that, using (4.3) for the last equality,

$$(b - a)_n = |S[U]|_{w_4} = |M[U]|_{w_1} = (b)_n \cdot {}_2F_1\left[\begin{matrix} -n, a \\ b \end{matrix}; 1\right].$$

REMARK. Proposition 4 may be written $M_n(-1, 1; a, b) = (b - a)_n$. It is proved by showing that the two weighted species $M(-1, 1; a, b)$ and S_{b-a} ('permutations with weight $b - a$ on each cycle') are weakly isomorphic.

COROLLARY 1. *We have:*

$$(x + y)_n = \sum_{i+j=n} \binom{n}{i} (-1)^i (-x)_i (y + i)_j = \sum_{i+j=n} \binom{n}{i} (x - i + 1)_i (y + i)_j. \tag{4.5}$$

REMARK. The binomial formula, $\Sigma_{i+j=n} \binom{n}{i} a^i b^j = (a + b)^n$, corresponds to:

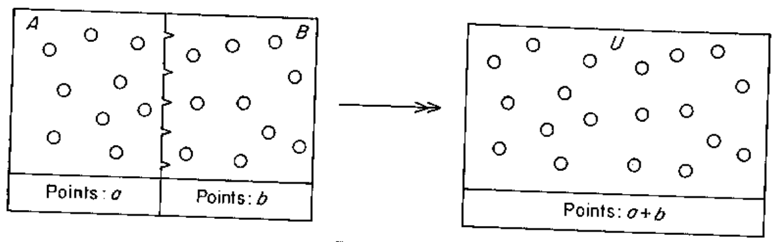

FIGURE 4.

LEMMA 4. *Given three finite mutually disjoint sets A, B and C there is a nice bijection*

$$\mathbf{L}[A \cup B, C] \approx \mathbf{L}[A, C] \times \mathbf{L}[B, A \cup C].$$

(Taking weights (with α for each cycle) this corresponds to $(\alpha + k)_{i+j} = (\alpha + k)_i (\alpha + k + i)_j$, where $|A| = i$, $|B| = j$ and $|C| = k$.)

PROOF. See [19, Lemma 3.1].

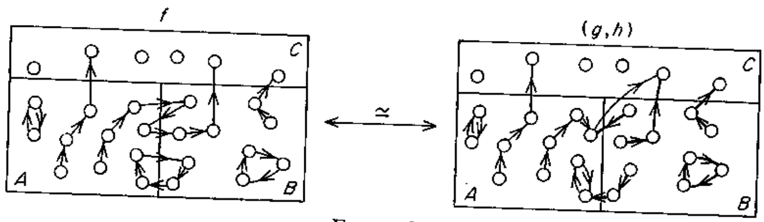

FIGURE 5.

PROPOSITION 5. We have

$$(c)_n \cdot {}_2F_1\left[\begin{matrix} -n, b \\ c \end{matrix}; z\right] = M_n(-z, 1; b, c). \tag{4.6}$$

PROOF. Both sides are $\Sigma_{i+j=n} \binom{n}{i} (b)_i (c + i)_j (-z)^i$.

THEOREM I. *The following formula is combinatorially true:*

$${}_2F_1\left[\begin{matrix} -n, b \\ c \end{matrix}; z\right] = (1 - z)^n \cdot {}_2F_1\left[\begin{matrix} -n, c - b \\ c \end{matrix}; \frac{z}{z - 1}\right]. \tag{4.7}$$

PROOF. We will prove

$$(c)_n (1 - z)^n \cdot {}_2F_1\left[\begin{matrix} -n, c - b \\ c \end{matrix}; \frac{z}{z - 1}\right] = (c)_n \cdot {}_2F_1\left[\begin{matrix} -n, b \\ c \end{matrix}; z\right]. \tag{4.8}$$

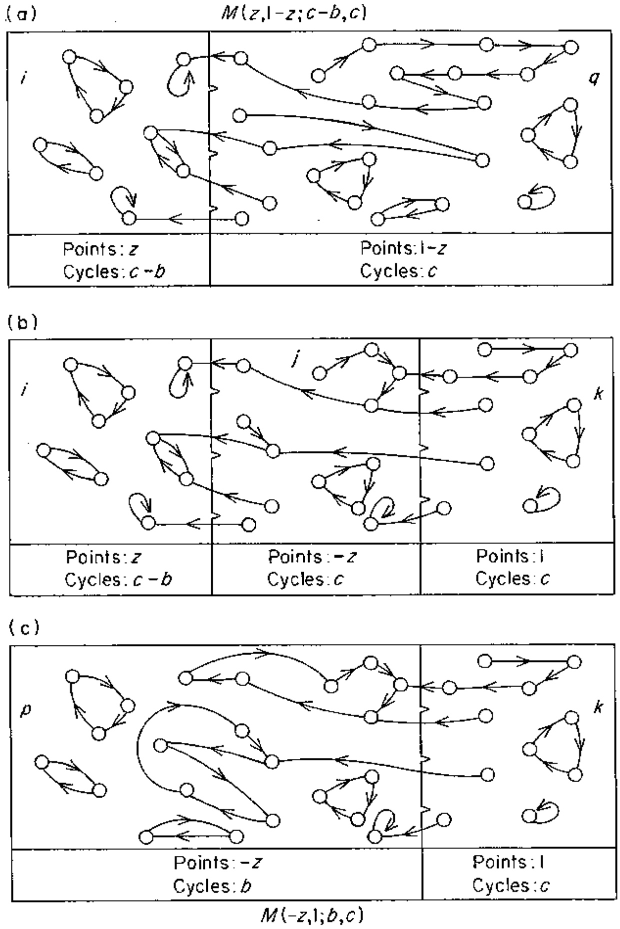

FIGURE 6.

In what follows, the algebraic expression is the total weight of those structures on U, $|U| = n$, for which a typical one is described in the corresponding rectangle in Fig. 6:

(a) $$(c)_n(1 - z)^n {}_2F_1\left[\begin{matrix} -n, c - b \\ c \end{matrix}; \frac{z}{z - 1}\right] = \sum_{i+q=n} \binom{n}{i} (c - b)_i (c + i)_q z^i (1 - z)^q$$

by the remark below Corollary 1,

$$= \sum_{i+q=n} \binom{n}{i} (c - b)_i (c + i)_q z^i \sum_{j+k=q} \binom{q}{j} (-z)^j$$

by Lemma 4,

$$= \sum_{i+q=n} \binom{n}{i} (c - b)_i z^i \sum_{j+k=q} \binom{q}{j} (-z)^j (c + i)_j (c + i + j)_k;$$

(b) $$= \sum_{i+j+k=n} \binom{n}{i, j, k} (c - b)_i (c + i)_j (c + i + j)_k z^{i+j} (-1)^j;$$

(c) $$= \sum_{p+k=n} \binom{n}{p} (-1)^p z^p (c + p)_k \sum_{i+j=p} \binom{p}{i} (c + i)_j (c - b)_i (-1)^i$$

by (4.5),

$$= \sum_{p+k=n} \binom{n}{p} (b)_p (c+p)_k (-z)^p$$

$$= (c)_n \cdot {}_2F_1\left[\begin{matrix} -n, b \\ c \end{matrix}; z\right].$$

REMARK. Theorem I (more precisely (4.8)) may be written as

$$M_n(z, 1-z; c-b, c) = M_n(-z, 1; b, c). \tag{4.9}$$

We proved it by showing that the two weighted species $M(z, 1-z; c-b, c)$ and $M(-z, 1; b, c)$ are weakly isomorphic.

THEOREM II. *The following formula is combinatorially true:*

$$(c)_n \cdot {}_2F_1\left[\begin{matrix} -n, b \\ c \end{matrix}; z\right] = (c-b)_n \cdot {}_2F_1\left[\begin{matrix} -n, b \\ -n+b-c+1 \end{matrix}; 1-z\right]. \tag{4.10}$$

PROOF. The proof is quite similar to the preceding one (see [18]).

REMARK. Theorem II may be written

$$M_n(-z, 1; b, c) = T_n(1-z, 1; b, c-b). \tag{4.11}$$

We prove it by showing that the two weighted species $M(-z, 1; b, c)$ and $T(1-z, 1; b, c-b)$ are weakly isomorphic.

Note that Theorem II also follows from Theorem I and the following proposition. More precisely, (4.10) is obtained by applying (4.7) then (4.12) and then (4.7) again.

PROPOSITION 6. *We have*

$$(b)_n \cdot {}_2F_1\left[\begin{matrix} -n, a \\ -n-b+1 \end{matrix}; z\right] = (a)_n z^n \cdot {}_2F_1\left[\begin{matrix} -n, b \\ -n-a+1 \end{matrix}; 1/z\right]. \tag{4.12}$$

PROOF. The left-hand side is

$$T_n(z, 1; a, b) = \sum_{i+j=n} \binom{n}{i} (a)_i (b)_j z^i,$$

while the right-hand side is

$$T_n(1, z; b, a) = \sum_{i+j=n} \binom{n}{i} (b)_i (a)_j z^j.$$

5. APPLICATION TO ${}_2F_1$ HYPERGEOMETRIC POLYNOMIALS

In this section we will describe several combinatorial models for Meixner, Krawtchouk and Meixner–Pollaczek polynomials using T- and M-structures. Recall that these polynomials are defined as follows [1, 3]:

DEFINITION 5. *Meixner polynomials* are

$$m_n(x;\beta,c) = (\beta)_n \cdot {}_2F_1\left[\begin{matrix} -n,\ -x \\ \beta \end{matrix};\ 1 - 1/c\right]; \tag{5.1}$$

Krawtchouk polynomials are $(0 \leqslant n \leqslant N,\ p + q = 1)$

$$K_n(x;p,N) = {}_2F_1\left[\begin{matrix} -n,\ -x \\ -N \end{matrix};\ 1/p\right]; \tag{5.2}$$

Meixner–Pollaczek polynomials are

$$P_n^a(x;\varphi) = \mathrm{e}^{in\varphi}\,\frac{(2a)_n}{n!}\cdot {}_2F_1\left[\begin{matrix} -n,\ a + \mathrm{i}x \\ 2a \end{matrix};\ 1 - \mathrm{e}^{-2i\varphi}\right]. \tag{5.3}$$

PROPOSITION 7. *We have*

(*a*)
$$\begin{aligned} m_n(x;\beta,c) &= M_n(1/c - 1, 1; -x, \beta) \\ &= M_n(1 - 1/c, 1/c; \beta + x, \beta) \\ &= T_n(1/c, 1; -x, \beta + x); \end{aligned}$$

(*b*)
$$\begin{aligned} (-N)_n K_n(x;p,N) &= M_n(-1/p, 1; -x, -N) \\ &= M_n(1/p, 1 - 1/p; x - N, -N) \\ &= T_n(1 - 1/p, 1; -x, x - N); \end{aligned}$$

(*c*)
$$\begin{aligned} \mathrm{e}^{-in\varphi} n!\,P_n^a(x;\varphi) &= M_n(\mathrm{e}^{-2i\varphi} - 1, 1; a + \mathrm{i}x, 2a) \\ &= M_n(1 - \mathrm{e}^{-2i\varphi}, \mathrm{e}^{-2i\varphi}; a - \mathrm{i}x, 2a) \\ &= T_n(\mathrm{e}^{-2i\varphi}, 1; a + \mathrm{i}x, a - \mathrm{i}x). \end{aligned}$$

PROOF. In each case, the first equality comes from Proposition 5 and, the second and third equalities by applying (4.9) and (4.11) respectively to it.

PROPOSITION 8. *Writing* $m_n = m_n(x;\beta,c)$, $K_n = K_n(x;p,N)$ *and* $P_n = P_n^a(x;\varphi)$, *we have the following recurrences*:

$$cm_{n+1} = ((c - 1)x + (c + 1)n + c\beta)m_n - n(n + \beta - 1)m_{n-1}; \tag{5.4}$$

$$(N - n)p(K_{n+1} - K_n) = nq(K_n - K_{n-1}) - xK_n; \tag{5.5}$$

$$(n + 1)P_{n+1} = 2((n + a)\cos\varphi + x\sin\varphi)P_n - (n + 2a - 1)P_{n-1}. \tag{5.6}$$

PROOF. Apply (3.7) (or 3.8) to Proposition 7.

REMARK. Since (3.7) will be proved combinatorially in Section 6, we have combinatorial proofs of these three recurrence formulas.

DEFINITION 6. *Jacobi polynomials* are defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(1 + \alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix} -n,\ 1 + \alpha + \beta + n \\ 1 + \alpha \end{matrix};\ \frac{1 - x}{2}\right] \\ &= \frac{1}{n!}\sum_{i+j=n}\binom{n}{i}(1 + \alpha + i)_j(1 + \alpha + \beta + n)_i\left(\frac{x - 1}{2}\right)^i. \end{aligned} \tag{5.7}$$

PROPOSITION 9. We have

$$n!P_n^{(\alpha,\beta)}(x) = M_n\left(\frac{x-1}{2}, 1; 1+\alpha+\beta+n, 1+\alpha\right)$$
$$= M_n\left(\frac{1-x}{2}, \frac{1+x}{2}; -\beta-n, 1+\alpha\right)$$
$$= T_n\left(\frac{1+x}{2}, 1; 1+\alpha+\beta+n, -\beta-n\right). \tag{5.8}$$

PROOF. Same proof as Proposition 7.

REMARK. These are 'ugly models' since the weight of some constituents in the structures depend on n. But they are combinatorially equivalent to the following 'nice model' due to Foata and Leroux [9]. Recall that for two disjoint finite sets A and B, we set:

$$\mathbf{P}[A, B] = \mathbf{L}[A, B] \times \mathbf{L}[B, A].$$

Moreover, for a finite set U,

$$P[U] = \{(A, B, f, g) \mid A \cup B = U, A \cap B = \phi, (f, g) \in \mathbf{P}[A, B]\}$$

and

$$w(A, B, f, g) = \left(\frac{x-1}{2}\right)^{|A|}\left(\frac{x+1}{2}\right)^{|B|}(1+\beta)^{\mathrm{cyc}(f)}(1+\alpha)^{\mathrm{cyc}(g)}.$$

PROPOSITION 10. *We have* $|P[n]| = n!P_n^{(\alpha,\beta)}(x)$.

PROOF. (see [9, 19, 16]). By (5.8) we have:

$$n!P_n^{(\alpha,\beta)}(x) = \sum_{i+j=n}\binom{n}{i}(-\beta-n)_i(1+\alpha+i)_j\left(\frac{1-x}{2}\right)^i\left(\frac{1+x}{2}\right)^j. \tag{5.9}$$

But since $(-\beta-n)_i = (-1)^i(\beta+1+n-i)_i$, we have:

$$n!P_n^{(\alpha,\beta)}(x) = \sum_{i+j=n}\binom{n}{i}(1+\beta+j)_i(1+\alpha+i)_j\left(\frac{x-1}{2}\right)^i\left(\frac{x+1}{2}\right)^j$$
$$= |P[n]| \tag{5.10}$$

by Lemma (3.2).

REMARK. As noted by F. Bergeron [2], we also have (by (5.10) and (3.1));

$$(-1)^n P_n^{(\alpha,\beta)}(x) = T_n((x-1)/2, (x+1)/2; -n-\beta, -n-\alpha). \tag{5.11}$$

PROPOSITION 11. *The following identity is combinatorially true*:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{5.12}$$

PROOF. Apply Theorem II to (5.7) or use (5.10) or (5.11).

6. COMBINATORIAL PROOFS OF THE RECURRENCE FORMULAS

The object of this section is to prove combinatorially the recurrence formula (3.7) (which implies the recurrence formulas for Meixner, Krawtchouk and Meixner–Pollaczek polynomials (Proposition 8)) and the three-terms recurrence for Jacobi polynomials.

THEOREM III. *The following recurrence formula is combinatorially true:*

$$T_{n+1} = (ru + sv + n(r + s))T_n - rs(u + v + n - 1)nT_{n-1}$$

where $T_n = |T(r, s; u, v)[n]|$.

PROOF. Let $L_n = |\{(A, B, \sigma, \tau) \in T[n]$ such that $1 \in A\}|$ and $R_n = |\{A, B, \sigma, \tau) \in T[n]$ such that $1 \in B\}|$. We have:

$$T_n = L_n + R_n \text{ for all } n. \tag{1}$$

Now look at what happens to the point $n + 1$ in a T-structure on $[n + 1]$:
1^0 case: $n + 1$ 'is alone on the left', i.e. $n + 1 \in A$ and $\sigma(n + 1) = n + 1$.

$$|\{(A, B, \sigma, \tau) \in T[n + 1] | n + 1 \in A, \sigma(n + 1) = n + 1\}| = ruT_n.$$

2^0 case: $n + 1$ 'is alone on the right', i.e. $n + 1 \in B$ and $\tau(n + 1) = n + 1$.

$$|\{(A, B, \sigma, \tau) \in T[n + 1] | n + 1 \in B, \tau(n + 1) = n + 1\}| = svT_n.$$

3^0 case: $n + 1$ 'is not alone', i.e. $\exists i \in [n]$ such that $\sigma(i) = n + 1$ or $\tau(i) = n + 1$.

$$|\{(A, B, \sigma, \tau) \in T[n + 1] | n + 1 \text{ is not alone}\}| = n(rL_n + sR_n).$$

Since by replacing $i \rightarrowtail n + 1$ by a single point $\underline{i}$ (of weight r^2 if it is on the left or s^2 otherwise) we obtain a T-structure on the set $\{1, 2, \ldots, i - 1, \underline{i}, i + 1, \ldots, n\}$. We have:

$$T_{n+1} = (ru + sv)T_n + n(rL_n + sR_n) \text{ for all } n. \tag{2}$$

In order to obtain a third equation, we look at what happens to the point $n + 2$ in a T-structure on $[n + 2]$:
1^0 case: $n + 2$ is alone. As before,

$$|\{(A, B, \sigma, \tau) \in T[n + 2] | n + 2 \text{ is alone}\}| = (ru + sv)T_{n+1}.$$

2^0 case: $n + 2$ is not alone but $\underline{i}$ is.

More precisely, $\exists i \in [n + 1]$ such that $\sigma(i) = n + 2$ or $\tau(i) = n + 2$, and in the T-structure on $\{1, 2, \ldots, i - 1, \underline{i}, i + 1, \ldots, n + 1\}$, obtained by replacing the arrow $i \rightarrowtail n + 2$ by a single point $\underline{i}$, this point $\underline{i}$ is alone. The total weight of these T-structures on $[n + 2]$ is $(n + 1)(r^2u + s^2v)T_n$.

3^0 case: $n + 2$ and $\underline{i}$ are not alone. More precisely, $\exists i \in [n + 1]$ such that $\sigma(i) = n + 2$ or $\tau(i) = n + 2$, and in the T-structure on $\{1, 2, \ldots, i - 1, \underline{i}, i + 1, \ldots, n + 1\}$, there is $j \in \{1, 2, \ldots, i - 1, i + 1, \ldots, n + 1\}$ with $\underline{i}$ as image. There is $n + 1$ choice for i, n choice for j, and if we replace $j \rightarrowtail i \rightarrowtail n + 2$ by a single point, say $\underline{ji}$, we obtain a T-structure on n points where $\underline{ji}$ is either on the left with weight r^3 (instead of r) or on the right with weight s^3 (instead of s).

The total weight of these T-structures on $[n + 2]$ is:

$$T_{n+2} = (ru + sv)T_{n+1} + (n + 1)(r^2u + s^2v)T_n + (n + 1)n(r^2L_n + s^2R_n). \tag{3}$$

From equations (1) and (2) we obtain:

$$n(r - s)L_n = -(ru + s(v + n))T_n + T_{n+1},$$

$$n(r - s)R_n = (r(u + n) + sv)T_n - T_{n+1}.$$

Substituting these expressions for L_n and R_n in (3), we obtain:

$$T_{n+2} = (ru + sv + (n + 1)(r + s))T_{n+1} - rs(u + v + n)(n + 1)T_n.$$

REMARK. One can proceed differently as was first done in [2] for Meixner polynomials and in [17] for formula (3.7). By 'pointing' a T-structure (resp. an M-structure) we prove combinatorially the following:

$$T' = (ruL(rX) + svL(sX)) \cdot T, \tag{6.1}$$

$$M' = \{uL(rX \cdot L(sX)) \cdot (rL(sX) + rsX \cdot L(sX) \cdot L(sX)) + svL(sX)\} \cdot M. \tag{6.2}$$

Note that (6.1) and (6.2) follow from (3.5) and (3.6) respectively by using the rules of the differential calculus of species and the facts that $C' = L$ and $L' = L \cdot L$. Taking generating series we can then proceed analytically as in the proof of Proposition 3, Section 3. The present proof is at the combinatorial structures level rather than at the generating series level.

THEOREM IV. *The three-terms recurrence for Jacobi polynomials is:*

$$\begin{aligned} &2(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x) \\ &= (\alpha+\beta+2n+1)\{(\alpha+\beta+2n+2)(\alpha+\beta+2n)x+\alpha^2-\beta^2\}P_n^{(\alpha,\beta)}(x) \\ &\quad -2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x). \end{aligned} \tag{6.3}$$

Before we prove the theorem, we need two lemmas:

LEMMA 5. *We have*

$$m_n(x;\beta,c) = n!P_n^{(\beta-1,-x-\beta-n)}\left(\frac{2}{c}-1\right). \tag{6.4}$$

PROOF. By Propositions 7 and 9, both sides are $M_n(1/c-1, 1; -x, \beta)$.

LEMMA 6. *The following formula is combinatorially true:*

$$(1+\alpha+\beta+2n)P_n^{(\alpha,\beta)}(x) = (1+\alpha+\beta+n)P_n^{(\alpha,\beta+1)}(x) + n(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x). \tag{6.5}$$

PROOF. See [19, formula 4.15].

PROOF OF (6.3). In the three-terms recurrence (5.4), for Meixner polynomials, apply (6.4) to obtain:

$$\begin{aligned} c(n+1)P_{n+1}^{(\beta-1,-x-\beta-n-1)}\left(\frac{2}{c}-1\right) &= (c\beta+n(c+1)+x(c-1))P_n^{(\beta-1,-x-\beta-n)} \\ &\times\left(\frac{2}{c}-1\right) - (\beta+n-1)P_{n-1}^{(\beta-1,-x-\beta-n-1)}\left(\frac{2}{c}-1\right). \end{aligned} \tag{6.6}$$

Applications of (6.5) to the two right-side terms of (6.6) leads to (6.3) after a painful calculation.

REMARK. We are told that V. Strehl found (in 1985) a beautiful combinatorial proof of (6.3) which he mentioned briefly in his talk at the 'Séminaire lotharingien de combinatoire' of Bologne (1985).

7. CONCLUSION

Most polynomials in R. Askey's chart [1, 14] of hypergeometric orthogonal polynomials now have several weakly isomorphic models. Note that every one of these models has a natural q-analogue [4] which will eventually be computed explicitly.

Using these models, the three-terms recurrence formulas for these families of polynomials (see [15] for Hermite, Charlier and Laguerre polynomials) and several other formulas can be proved in a purely combinatorial way.

The combinatorics of the hypergeometric series ${}_2F_1$, which was considered in Section 4, will be extended to ${}_1F_1$, ${}_2F_0$, ${}_3F_2$ in [18]. (See also Foata [5] for a combinatorial proof of the Pfaff–Saalschütz formula involving ${}_3F_2$.) This will also be applied to combinatorial models for Laguerre, Charlier, Hahn and other polynomials.

References

1. R. Askey and J. A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Am. Math. Soc.*, **54** (319) (1985).
2. F. Bergeron, Modèles combinatoires de familles de polynômes orthgonaux classiques; une approche unifiée, *Europ. J. Comb.* (to appear).
3. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
4. H. Décoste, Séries indicatrices pondérées et q-analogues en théorie des espèces, Ph.D. Thesis (Université de Montréal) (to appear).
5. D. Foata, Une démonstration combinatoire de l'identité de Pfaff–Saalschütz, *C.R. Acad. Sci. Paris*, **297** (1983), 221–224.
6. D. Foata, A combinatorial proof of the Mehler formula, *J. Comb. Theory, Ser. A.*, **24** (1978), 367–376.
7. D. Foata, Combinatoire de identités sur les polynômes orthogonaux, *Proc. Int. Congr. Mathematicians*, Warsaw, Poland, 1983.
8. D. Foata and J. Labelle, Modèles combinatoires pour les polynômes de Meixner, *Europ. J. Comb.*, **4** (1983), 305–311.
9. D. Foata and P. Leroux, Polynômes de Jacobi, interprétation combinatoire et fonction génératrice, *Proc. Am. Math. Soc.*, **87** (1983), 47–53.
10. D. Foata and V. Strehl, Combinatorics of the Laguerre polynomials, *Proc. Waterloo Silver Jubilee Conf., 1982*, in *Enumeration and Design*, ed. D. M. Jackson and S. A. Vanstone, Academic Press, Toronto, 1984, pp. 123–140.
11. A. Joyal, Une théorie combinatoire des séries formelles, *Adv. Math.*, **42** (1981), 1–82.
12. G. Labelle, Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange, *Adv. Math.*, **42** (1981), 217–247.
13. J. Labelle, Applications diverses de la théorie des espèces de structures, *Ann. Sci. Math. Québec*, **7** (1983), 59–94.
14. J. Labelle, Tableau d'Askey, in *Proc. Symposium Bar-le-Duc 1984, L. N. Math.*, **1171**, pp. XXXVI and XXXVII, Springer-Verlag, Berlin.
15. J. Labelle and Y. N. Yeh, The combinatorics of Laguerre, Charlier and Hermite polynomials, Studies in Applied Math. (to appear), Rapport de recherche de Dépt. de math. et d'info. UQAM, no. 21.
16. J. Labelle and Y. N. Yeh, A combinatorial model for Hahn polynomials, in *Séminaire Lotharingien de combinatoire XVI, Liebfraunberg (France), 1987, Actes du séminaire*, ed. D. Foata.
17. J. Labelle and Y. N. Yeh, Combinatorial proofs of some limit formulas involving orthogonal polynomials, *Discr. Math.* (to appear), Rapport de recherche du Dépt. de math. et d'info. UQAM, no. 25.
18. J. Labelle and Y. N. Yeh, Combinatorial proofs of symmetry formulas for the generalized hypergeometric series, *J. Math. Anal. Appl.* (to appear), Rapport de recherche de Dépt. de math. et d'info. UQAM, no. 41.
19. P. Leroux and V. Strehl, Jacobi polynomials: combinatorics of the basic identities, *Discr. Math.*, **57** (1985), 167–187.
20. V. Strehl, Polynômes d'Hermite généralisés et identités de Szegö—une version combinatoire, in *Proc. Symposium Bar-le-Duc 1984, L. N. Math.*, **1171**, pp. 129–138, Springer-Verlag, Berlin.
21. G. Szego, *Orthogonal Polynomials*, Am. Math. Soc. Colloquium Publications, vol. 23, Providence, 1975 (4th edn).
22. Y. N. Yeh, The calculus of virtual species and K-species, in *Proc. Coll. Combin. Enumérative, Montréal 1985*, ed. G. Labelle and P. Leroux, *L. N. Math.*, **1234**, pp. 129–138, Springer-Verlag, Berlin.

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