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The Relation between Burnside Rings and Combinatorial Species*

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We describe the close relationship between permutation groups and combinatorial species (introduced by A. Joyal, *Adv. in Math.* **42**, 1981, 1–82). There is a bijection Φ between the set of transitive actions (up to isomorphism) of S_n on finite sets and the set of "molecular" species of degree n (up to isomorphism). This bijection extends to a ring isomorphism between $B(S_n)$ (the Burnside ring of the symmetric group) and the ring $\mathscr{V}\mathscr{S}_n$ (of virtual species of degree n). Since permutation groups are well known (and often studied using computers) this helps in finding examples and counterexamples in species. The cycle index series of a molecular species, which is hard to compute directly, is proved to be simply the (Pólya) cycle polynomial of the corresponding permutation group. Conversely, several operations which are hard to define in $\prod_n B(S_n)$ have a natural description in terms of species. Both situations are extended to coefficients in λ-rings and binomial rings in the last section.

INTRODUCTION

In this paper, we want to highlight the close relationship between permutation groups and combinatorial species introduced by A. Joyal in [4].

In Sections 1 and 2, the main properties of species (operations, cycle index series, molecular and atomic species, ring of virtual species) and of the Burnside ring (of the symmetric group S_n) are reviewed.

There is a bijection Φ (with inverse Ψ) between the set of isomorphism classes of transitive S_n-sets (or the set of conjugacy classes of subgroups of S_n) and the set $\mathscr{M}_n$ of isomorphism classes of "molecular" species of degree n. Properties of these bijections are considered in Section 3. Since permutations groups are studied a lot in the literature (sometimes using computers), this can be applied to answer some questions about species.

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In Section 4, various extensions of the coefficients (from $\mathbb{N}$ to a ring A) are considered. The main problem here is to define substitution of A-species. This was done in [17] when A is a "binomial ring." A. Joyal solved the problem when A is $\mathbb{Z}$ or any "splitting λ-ring" (Talk at the "Colloque de combinatoire énumérative," UQAM, May 1985). Both cases can be transfered to Burnside rings using bijection Φ.

1. Combinatorial Species

Let $\mathbf{B}$ be the category (in fact, a groupoid) of finite sets and bijections. A *species* T is a functor from $\mathbf{B}$ to $\mathbf{B}$. Given a finite set U, $T[U]$ is called the set of *T-structures* (or structures of species T) on U; given a bijection $u: U \to V$, $T[u]: T[U] \to T[V]$ (which is a bijection) is called the *transport of T-structures along u.* Note that $T[U]$ is a $\mathbf{S}_U$-set (where $\mathbf{S}_U$ denotes the group of permutations of U) with the action defined by

$$\sigma \cdot t = T[\sigma](t) \qquad \text{for} \quad \sigma \in \mathbf{S}_U \qquad \text{and} \qquad t \in T[U].$$

There are three formal series associated to T:

1. the *exponential generating series*

$$\text{Card } T = T(x) = \sum_{n \geqslant 0} |T[n]| x^n/n!, \qquad \text{where} \quad [n] = \{1, 2, ..., n\}$$

2. the *type-generating series*

$$\tilde{T}(x) = \sum_{n \geqslant 0} |T[n]/\mathbf{S}_n|\, x^n, \qquad \text{where} \quad \mathbf{S}_n = \mathbf{S}_{[n]}$$

3. the *cycle index series*

$$Z_T(x_1, x_2, ...) = \sum_{n \geqslant 0} \sum_{\Sigma i\lambda_i = n} \text{Fix}_T(\sigma) \frac{x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}}{1^{\lambda_1}\lambda_1!\, 2^{\lambda_2}\lambda_2! \cdots n^{\lambda_n}\lambda_n!},$$

where $\sigma \in \mathbf{S}_n$ is any permutation of type $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\text{Fix}_T(\sigma) = |\{t \in T[n]| \mid \sigma \cdot t = t\}|$.

As usual let 0 be the *empty species* and 1, X, E, $E^{\pm}$, C, S, and L be the species of *empty set*, *singletons*, *sets*, *oriented sets*, *cyclic permutations*, *permutations*, and *linear orders*, respectively.

EXAMPLES.

$$X(x)=x \qquad E(x)=e^x \qquad L(x)=1/(1-x)$$

$$\tilde{X}(x)=x \qquad \tilde{E}(x)=1/(1-x) \qquad \tilde{L}(x)=1/(1-x)$$

$$Z_X(x_1,\ldots)=x_1 \qquad Z_E(x_1,\ldots)=\prod_{n\geqslant 1} e^{x_n/n} \qquad Z_L(x_1,\ldots)=1/(1-x_1)$$

$$C(x)=-\ln(1-x) \qquad S(x)=1/(1-x)$$

$$\tilde{C}(x)=x/(1-x) \qquad \tilde{S}(x)=\prod_{n\geqslant 1}(1-x^n)^{-1}=\sum_{n\geqslant 0} p_n x^n$$

$$Z_C(x_1,\ldots)=\sum_{n\geqslant 1}\frac{\varphi(n)}{n}\ln\left(\frac{1}{1-x_n}\right) \qquad Z_S(x_1,\ldots)=\prod_{n\geqslant 1}(1-x_n)^{-1}.$$

We assume the reader is familiar with the following operations on species; the sum, $T+M$; the product, $T\cdot M$; the cartesian product, $T\times M$; the substitution, $T\circ M$ (if $M[\varnothing]=\varnothing$), and the derivative, T'.

Two species, T and M, are said to be *equipotent*

$$\text{if} \quad T(x)=M(x), \qquad \text{i.e.,} \quad \forall U, \quad |T[U]|=|M[U]|;$$

they are said to be *isomorphic* (we write $T=M$) if there is a natural equivalence between the two functors T and M. If $T=M$ then

$$T(x)=M(x), \qquad \tilde{T}(x)=\tilde{M}(x), \qquad \text{and} \qquad Z_T=Z_M,$$

but the converse is not true.

Recall the following results from [4] (see also [7, 11, 18]).

THEOREM I. *We have*

$$(T+M)(x)=T(x)+M(x), \qquad (T+M)^{\sim}(x)=\tilde{T}(x)+\tilde{M}(x),$$

$$(T\cdot M)(x)=T(x)\cdot M(x), \qquad (T\cdot M)^{\sim}(x)=\tilde{T}(x)\cdot\tilde{M}(x),$$

$$\frac{d}{dx}T(x)=(T')(x), \qquad \frac{d}{dx}\tilde{T}(x)=(T')^{\sim}(x),$$

$$(T\circ M)(x)=T(M(x)), \qquad (T\circ M)^{\sim}(x)=Z_T(\tilde{M}(x),\tilde{M}(x^2),\tilde{M}(x^3),\ldots),$$

$$Z_{T+M}=Z_T+Z_M, \quad Z_{T\cdot M}=Z_T\cdot Z_M, \quad Z_{T'}=\frac{\partial}{\partial x_1}Z_T, \quad Z_{T\circ M}=Z_T(Z_M)$$

(*where* $Z_T(Z_M)$ *is the plethym substitution of* Z_M *into* Z_T).

THEOREM II. *We have* $Z_M(x,0,0,\ldots)=M(x)$ *and* $Z_M(x,x^2,x^3,\ldots)=\tilde{M}(x)$.

DEFINITION 1.1. Given a species T, let T_n be defined by $T_n[U]$ is $T[U]$ if $|U|=n$ and $T_n[U]=\varnothing$ if not.

We always have $T=\sum_{n\geqslant 0} T_n$; this is the *canonical decomposition* of T. When this sum is finite, we say that T is *polynomial.* We say that T is a species of *degree n* (or concentrated on the cardinality n) if $T=T_n$.

DEFINITION 1.2. T is *molecular* if $T=M+K \Rightarrow M=0$ *or* $K=0$. T is *atomic* if T is molecular and $T=M\cdot K \Rightarrow M=1$ or $K=1$.

Remark. If T is molecular than $T=T_n$ for some n and $T[n]$ is a transitive $\mathbf{S}_n$-set (i.e., all the T-structures are isomorphic).

From now on we work up to isomorphism and identify a species with its isomorphism class.

Let $\mathscr{M}_n$ (resp. $\mathscr{A}_n$) be the set of molecular (resp. atomic) species of degree n and $\mathscr{M}=\bigcup_{n\geqslant 0}\mathscr{M}_n$ and $\mathscr{A}=\bigcup_{n\geqslant 0}\mathscr{A}_n$.

The sets $\mathscr{M}_n$ and $\mathscr{A}_n$ have been studied extensively (for $n\leqslant 5$) in [8, 12]. We have

$$\begin{aligned}
&\mathscr{M}_0=\{1\},\ \mathscr{A}_0=\varnothing;\\
&\mathscr{M}_1=\{X\}=\mathscr{A}_1;\ \mathscr{M}_2=\{E_2, X^2\},\ \mathscr{A}_2=\{E_2\};\\
&\mathscr{M}_3=\{E_2, C_3, X\cdot E_2, X^3\},\ \mathscr{A}_3=\{E_3, C_3\};\\
&\mathscr{M}_4=\{E_4, E_4^{\pm}, E_2\circ E_2, X\cdot E_3, E_2\cdot E_2, P_4^{\mathrm{bic}}, C_4, X\cdot C_3, X^2\cdot E_2,\\
&\qquad E_2\circ X^2, X^4\};\\
&\mathscr{A}_4=\{E_4, E_4^{\pm}, E_2\circ E_2, P_4^{\mathrm{bic}}, C_4, E_2\circ X^2\}\\
&\mathscr{M}_5=\{E_5, E_5^{\pm}, X\cdot E_4, P_5/\mathbb{Z}_2, X\cdot E_4^{\pm}, E_2\cdot E_3, P_5, X\cdot(E_2\circ E_2),\\
&\qquad X^2\cdot E_3, E_2\cdot C_3, (E_2^{\pm}\cdot E_3^{\pm})/\mathbb{Z}_2, C_5, X\cdot E_2\cdot E_2, X\cdot P_4^{\mathrm{bic}}, X\cdot C_4,\\
&\qquad X^2\cdot C_3, X^3\cdot E_2, X\cdot(E_2\circ X^2), X^5\}\\
&\mathscr{A}_5=\{E_5, E_5^{\pm}, P_5/\mathbb{Z}_2, P_5, (E_2^{\pm}\cdot E_3^{\pm})/\mathbb{Z}_2, C_5\}.
\end{aligned}$$

Let $\mathscr{S}_n$ be the semi-group of degree n species (with cartesian product, it is in fact a semi-ring), then $\mathscr{P}\mathscr{S}=\bigoplus_{n\geqslant 0}\mathscr{S}_n$ and $\mathscr{S}=\prod_{n\geqslant 0}\mathscr{S}_n$ are the semi-groups of polynomial species and species, respectively. Let $\mathscr{V}\mathscr{S}_n$, $\mathscr{V}\mathscr{P}\mathscr{S}=\bigoplus_{n\geqslant 0}\mathscr{V}\mathscr{S}_n$ and $\mathscr{V}\mathscr{S}=\prod_{n\geqslant 0}\mathscr{V}\mathscr{S}_n$ be the group completions of these semi-groups. By definition, an element $T-R=\mathscr{V}\mathscr{S}$ is called a *virtual* species and $T_1-R_1=T_2-R_2$ iff $T_1+R_2=T_2+R_1$. It is easy to see [8, 12, 18] that $\mathscr{V}\mathscr{S}_n$ is the free abelian group generated by $\mathscr{M}_n$. Any element T of $\mathscr{S}_n$ (resp. $\mathscr{V}\mathscr{S}_n$) can the written uniquely as

$$T=\sum_{M\in\mathscr{M}_n} t_M M, \qquad \text{where} \quad t_M\in\mathbb{N}\ (\text{resp. } \mathbb{Z}).$$

This is the *molecular decomposition* of T.

Moreover, Yeh [17] proved that any $M \in \mathcal{M}$ can be written uniquely (up to permutation of the factors) as

$$M = \prod_{A \in \mathcal{A}} A^{\gamma_A(M)}, \qquad \text{where} \quad \gamma_A(M) \in \mathbb{N} \qquad \text{and} \qquad \sum_{A \in \mathcal{A}} \gamma_A(M) < \infty.$$

This is the *atomic decomposition* of M.

We have (considering only the additive structure involved)

$$\mathscr{P}\mathscr{S} = \bigoplus_n \mathscr{S}_n = \bigoplus_n \mathbb{N}^{(\mathcal{M}_n)} = \mathbb{N}^{(\mathcal{M})} \qquad \mathscr{S} = \prod_n \mathscr{S}_n = \prod_n \mathbb{N}^{(\mathcal{M}_n)} = \mathbb{N}^{\mathcal{M}}$$

$$\mathscr{V}\mathscr{P}\mathscr{S} = \bigoplus_n \mathscr{V}\mathscr{S}_n = \bigoplus_n \mathbb{Z}^{(\mathcal{M}_n)} = \mathbb{Z}^{(\mathcal{M})} \qquad \mathscr{V}\mathscr{S} = \prod_n \mathscr{S}_n = \prod_n \mathbb{Z}^{(\mathcal{M}_n)} = \mathbb{Z}^{\mathcal{M}},$$

where $\mathbb{N}^{(X)}$ (resp. $\mathbb{Z}^{(X)}$) is the free abelian semi-group (resp. group) generated by the set X.

With the cartesian product, $\mathscr{S}_n$ is a ring; with the usual product $\cdot$ of species, $\mathscr{V}\mathscr{P}\mathscr{S}$ (resp. $\mathscr{V}\mathscr{S}$) is a polynomial (resp. power series) ring in an infinite number of variables, the atomic species

$$\mathscr{V}\mathscr{P}\mathscr{S} = \mathbb{Z}[\mathcal{A}] \qquad \mathscr{V}\mathscr{S} = \mathbb{Z}[[\mathcal{A}]].$$

Both $\mathscr{V}\mathscr{P}\mathscr{S}$ and $\mathscr{V}\mathscr{S}$ are *UFD*, as proved by Yeh [17].

DEFINITION 1.3. Let G be a finite group and T a species, we say that G *acts naturally on* T if $\forall U$ we have an action $G \times T[U] \to T[U]$ such that $\forall f\colon U \to V$ (bijection) the following diagram commutes

$$\begin{array}{ccc} G \times T[U] & \longrightarrow & T[U] \\ \Big\downarrow {\scriptstyle 1_G \times T[f]} & & \Big\downarrow {\scriptstyle T[f]} \\ G \times T[V] & \longrightarrow & T[V] \end{array}$$

The species T/G is then defined by $T/G[U] = T(U)/G$ (with obvious transport of structures). See [12] for more details on the following three examples and others.

EXAMPLES.

(1) $\mathbb{Z}_2$ acts on C_n $(n \geqslant 3)$ by $1 \cdot \sigma = \sigma^{-1}$; $C_n/\mathbb{Z}_2 = P_n$ is the species of n-gons.

(2) $\mathbb{Z}_2$ acts on P_5 by "complementation of the edges"; $P_5/\mathbb{Z}_2$ is a nice atomic species whose derivative is C_4.

(3) $\mathbb{Z}_2$ acts on $E_2^{\pm} \cdot E_3^{\pm}$ by "switching both orientations"; $(E_2^{\pm} \cdot E_3^{\pm})/\mathbb{Z}_2$ is an atomic species whose derivative $x \cdot C_3 + E_2 \circ L_2$, is not molecular.

Let H be a subgroup of $\mathbf{S}_n$. There is a natural action of H on the species X^n defined by: $\sigma \cdot (\mu_1, \mu_2, ..., \mu_n) = (\mu_{\sigma^{-1}(1)}, \mu_{\sigma^{-1}(2)}, ..., \mu_{\sigma^{-1}(n)})$, where $\sigma \in \mathbf{H}_n$, $\sigma: [n] \to [n]$ and $(\mu_1, ..., \mu_n) \in X^n[U] = L_n[U]$, $|U| = n$.

It is not hard to see that if M_1 and M_2 are molecular then $M_1 \cdot M_2$ and $M_1 \circ M_2$ will also be. For a cartesian product and derivative, things are more complicated since $M_1 \times M_2$ and M' are not, in general, molecular. In fact, we have the following formulas [17, 18].

THEOREM III. *Let $H \subseteq \mathbf{S}_n$ and $K \subseteq \mathbf{S}_m$ be two permutation groups, then*

$$(X^n/H) \cdot (X^m/K) = X^{n+m}/H * K, \quad \textit{where} \quad H * K \subseteq \mathbf{S}_{n+m} \tag{1}$$

$$(X^n/H) \circ (X^m/K) = X^{mn}/K \wr H \quad \textit{where} \quad K \wr H \subseteq \mathbf{S}_{mn} \text{ is the wreath product} \tag{2}$$

$$\frac{X^n}{H} \times \frac{X^m}{K} = \begin{cases} \sum_L \dfrac{|L| \cdot |\{g \in \mathbf{S}_n \mid gHg^{-1} \cap K = L\}|}{|H| \cdot |K|} \dfrac{X^n}{L} & \textit{if} \quad n = m \\ 0 & \textit{if} \quad n \neq m \end{cases} \tag{3}$$

$$\left(\frac{X^n}{H}\right)' = \sum_e \frac{X^{n-1}}{H \cap ([n] - \{e\})!} \tag{4}$$

where e runs through a set of representatives of the orbits of the action of H on $[n]$ and $A!$ means $\mathbf{S}_A$.

We now show that Φ, which sends H (up to conjugacy) to X^n/H (up to isomorphism), also "preserves cycle polynomials." Let $\lambda = (\lambda_1, \lambda_2, ...)$ be a partition of n. We denote $\mathbf{S}_\lambda \subseteq \mathbf{S}_n$ the set of permutations of type λ. It is well known that $|\mathbf{S}_\lambda| = n!/1^{\lambda_1}\lambda_1!\, 2^{\lambda_2}\lambda_2! \ldots$ Consider the action of $\mathbf{S}_n$ on $\mathbf{S}_n/H$, where H is an arbitrary subgroup of $\mathbf{S}_n$.

LEMMA 1.1. *For any $\sigma \in \mathbf{S}_\lambda$, we have* $\mathrm{Fix}(\sigma) \cdot |H| \cdot |\mathbf{S}_\lambda| = n! \cdot |\mathbf{S}_\lambda \cap H|$.

Proof. Since $\mathrm{Fix}(\sigma) = |\{\tau H \mid \tau \in \mathbf{S}_n, \tau^{-1}\sigma\tau \in H\}$ we have

$$\begin{aligned} \mathrm{Fix}(\sigma) \cdot |H| &= |\{\tau \in \mathbf{S}_n \mid \tau^{-1}\sigma\tau \in H\}| = \sum_{h \in H} |\{\tau = \mathbf{S}_n \mid \tau^{-1}\sigma\tau = h\}| \\ &= \sum_{h \in H \cap \mathbf{S}_\lambda} |\{\tau \in \mathbf{S}_n \mid \tau^{-1}\sigma\tau = h\}| \\ &= (1^{\lambda_1}\lambda_1!\, 2^{\lambda_2}\lambda_2! \cdots n^{\lambda_n}\lambda_n!) \cdot |\mathbf{S}_\lambda \cap H| \\ &= n! \cdot |\mathbf{S}_\lambda|^{-1} \cdot |\mathbf{S}_\lambda \cap H|. \end{aligned}$$

DEFINITION 1.4. Given $H \subseteq \mathbf{S}_n$, the cycle polynomial of H is

$$Z_H = \frac{1}{|H|} \sum_{\lambda} |H \cap \mathbf{S}_\lambda| \, x^\lambda,$$

where $x^\lambda = x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \cdots$

PROPOSITION 1.1. *We have* $Z_H = Z_{X^n/H}$.

Proof.

$$Z_{X^n/H} \equiv \sum_\lambda \operatorname{Fix}(\sigma) \frac{x_1^{\lambda_1} x_2^{\lambda_2} \cdots}{1^{\lambda_1} \lambda_1! \, 2^{\lambda_2} \lambda_2! \cdots} = \sum_{\lambda} \frac{\operatorname{Fix}(\sigma)}{n!} \cdot |\mathbf{S}_\lambda| \cdot x_1^{\lambda_1} x_2^{\lambda_2} \cdots$$

$$= \frac{1}{|H|} \sum_{\lambda} |\mathbf{S}_\lambda \cap H| \, x^\lambda = Z_H.$$

2. THE BURNSIDE RING

Let G be a finite group; the set of isomorphism classes of G-sets forms a semi-ring ($+$ and $\times$ being induced by disjoint union and cartesian product, respectively) whose completion is called the *Burnside ring* $\mathrm{B}(G)$ of G.

It is well known [3] that, as an abelian group, $\mathrm{B}(G)$ is freely generated by the transitive G-sets. If H is a subgroup of G then G/H is a transitive G-set $(g'(gH) = (g'g)H)$ uniquely defined, up to isomorphism, by the conjugacy class $[H]$ of H. Moreover any transitive G-set is of that sort. In other words, any G-set is isomorphic to

$$\sum_{[H] \in P_G} a_H G/H$$

with a's in $\mathbb{N}$ (P_G being the poset of conjugacy classes of subgroups of G).

An arbitrary element of $\mathrm{B}(G)$ (called a *virtual* G-set or a *virtual permutation representation* of G) can always be written uniquely as

$$\sum_{[H] \in P_G} a_H G/H \qquad \text{with} \quad a_H \in \mathbb{Z}.$$

Multiplication in $\mathrm{B}(G)$ is usually computed using the *table of marks* of Burnside [1] (also called the *supercharacter table* of G [16]).

More specifically, for $[H]$, $[K] \in P_G$ let

$$\phi_{G/H}(K) = |\{x \in G/H \mid \forall k \in K, kx = x\}|$$

be the *mark* of K *on the representation* G/H.

This defines a triangular $|P_G| \times |P_G|$ matrix (the *matrix of marks* or the *supercharacter table* of G) whose inverse is called the *Burnside matrix* of G. We have

PROPOSITION 2.1. *Let*

$$G/H \times G/K = \sum_{[J] \in P_G} a_J G/J.$$

Then the column vector $(a_J)_{[J] \in P_G}$ *is computed by multiplying the Burnside matrix of* G *by the column vector* $(\phi_{G/H}(J) \cdot \phi_{G/K}(J))_{[J] \in P_G}$.

Proof. Compute the marks of both sides and solve the system of linear equations in the a's so obtained.

3. THE FUNDAMENTAL BIJECTION

We now return to the bijection Φ between $P_n = \{[H] \mid H \subseteq \mathbf{S}_n\}$, the set of conjugacy classes of subgroups of the symmetric group $\mathbf{S}_n$, and $\mathscr{M}_n$, the set of molecular species of degree n:

$$P_n \overset{\Phi}{\leftrightarrow} \mathscr{M}_n$$
$$[H] \rightarrow [X^n/H].$$

Properties of Φ were studied intensively by Yeh in is Ph.D. thesis (see Theorem III).

The inverse Ψ of Φ is obtained in the following way: let $[M] \in \mathscr{M}_n$ and $s \in M[n]$ be an arbitrary M-structure on $[n]$; let $H \subseteq \mathbf{S}_n$ be the stabiliser of s under the action of $\mathbf{S}_n$ on $M[n]$ induced by transport of structures (i.e., $H = \{\sigma \in \mathbf{S}_n \mid M[\sigma](s) = s\}$). Then $[H] \in P_n$ is the image of $[M]$ by Ψ.

For example, Fig. 1 shows the case $n = 4$, where

$$I^4 = \{\mathrm{id}\}, \qquad C_2' = \{\mathrm{id}, (12)(34)\}, \qquad C_2 * I^2 = \{\mathrm{id}, (12)\},$$
$$C_3 * I = \{\mathrm{id}, (123), (132)\},$$
$$K = \{\mathrm{id}, (12), (34), (12)(34)\}, \qquad K' = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\},$$
$$C_4 = \{\mathrm{id}, (1234), (13)(24), (1432)\}, \qquad \mathbf{S}_3 * I = \{\sigma \in \mathbf{S}_4 \mid \sigma(4) = 4\},$$
$$D_4 = \{\mathrm{id}, (13), (24), (13)(24), (1234), (1432), (14)(23), (12)(34)\},$$

and A_4 is the alternating group.

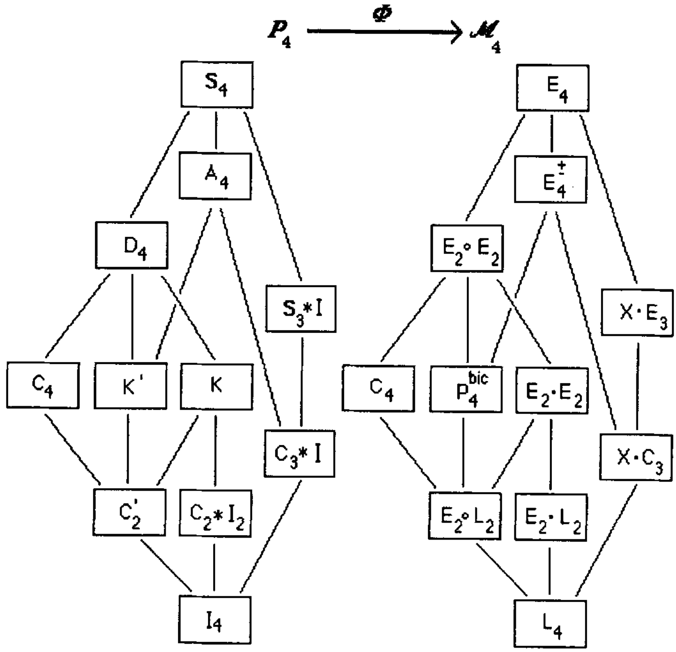

FIGURE 1

Since P_n and $\mathcal{M}_n$ are sets of generators for the free abelian groups $B(S_n)$ and $\mathcal{V}\mathcal{S}_n$, respectively, we have a group isomorphism

$$B(S_n) \overset{\Phi}{\leftrightarrow} \mathcal{V}\mathcal{S}_n$$

and, more generally,

$$PB(S) = \bigoplus_{n \geqslant 0} B(S_n) \overset{\Phi}{\leftrightarrow} \bigoplus_{n \geqslant 0} \mathcal{V}\mathcal{S}_n = \mathcal{P}\mathcal{V}\mathcal{S}$$

$$B(S) = \prod_{n \geqslant 0} B(S_n) \overset{\Phi}{\leftrightarrow} \prod_{n \geqslant 0} \mathcal{V}\mathcal{S}_n = \mathcal{V}\mathcal{S};$$

all the operations $+$, $\times$, $\cdot$, $\circ$, $'$ in $\mathcal{P}\mathcal{V}\mathcal{S}$ and $\mathcal{V}\mathcal{S}$ can then be transferred to PB(S) and B(S) so that Φ preserves all the structures involving these operations.

For example, the fact that the rings $\mathcal{P}\mathcal{V}\mathcal{S}$ and $\mathcal{V}\mathcal{S}$ are UFD was proved [17] by showing that PB(S) and B(S) are. The cartesian product of two molecular species of degree $\leqslant 7$ can be computed using the table of marks of S_7 (a triangular 96 by 96 matrix(!) given by Wensley [16]) with Proposition 2.1 or directly in $\mathcal{V}\mathcal{S}_n$, $n \leqslant 4$, as in [18, p. 365].

Let SCF(G) denote the ring of *supercentral* functions of G: i.e., SCF(G) $= \{f: P_G \to \mathbb{C}\}$. It is well known that the map: $\mathrm{B}(G) \to \mathrm{SCF}(G)$ (G-set X goes to the map φ_X), where $\varphi_X[H] =$ mark of H on $X = |\{x \in X \mid \forall h \in H, hx = x\}$, is a ring monomorphism (see [6]). In terms of species this corresponds to

$$\mathscr{V}\mathscr{S}_n \cong_\Phi \mathrm{B}(\mathrm{S}_n) \to \mathrm{SCF}(\mathrm{S}_n)$$

defined by sending T to the super central function, $P_n \to \mathbb{C}$, defined by H goes to $t_{X^n/H}$, where

$$T = \sum_{M \in \mathscr{M}_n} t_M M.$$

PROPOSITION 3.1. *H is transitive subgroup of S_n if and only if the corresponding molecular species, X^n/H, has a molecular derivative.*

Proof. Use (4) of Theorem III.

Those species are always atomic (the converse is not true (see [12]), since if M is non-atomic then $M = S \cdot T$ with $\deg S > 0$ and $\deg T > 0$ and $M' = S' \cdot T + S \cdot T'$ is non-molecular.

If H is also primitive (see [14] for definition) then X^n/H will certainly not be a non-trivial substitution. But again the converse is not true since a species as simple as C_4 (cycles of length 4) is not a substitution but has an imprimitive "stabiliser."

PROPOSITION 3.2. *Let $H \subseteq S_n$, $H \not\subseteq A_n$; then $X^n/(H \cap A_n)$ is $(X^n/H) \times (X^n/A_n) = (X^n/H) \times E_n^{\pm}$.*

Proof (This is a very special case of (3) Theorem III). Take $(s, t) \in (X^n/H) \times (X^n/A_n)[n]$, then $\mathrm{Stab}(s, t) = \{\sigma \in \mathrm{Stab}(s) \mid \sigma \in A_n\} = \mathrm{Stab}(s) \cap A_n = H \cap A_n$.

Remark. If $H \subseteq A_n$ then $(X^n/H) \times (X^n/A_n)$ is not molecular.

Since there are 4 molecular species of degree 3, but only 3 partitions of 3, one can easily find two (non-molecular) species of degree 3 having the same cycle index series but which are not isomorphic, namely, $L_3 + 2E_3$ and $C_3 + 2X \cdot E_2$, with

$$Z = \tfrac{4}{3}x_1^3 + x_1 x_2 + \tfrac{2}{3}x_3.$$

Looking at tables of subgroups of S_n (see [16]) one can find two *molecular* non-isomorphic species having the same cycle index series, namely, X^6/Q_1 and X^6/Q_2, where

$$Q_1 = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \subset \mathrm{S}_6$$

and

$$Q_2 = \{\mathrm{id}, (12)(34), (12)(56), (34)(56)\} \subset \mathbf{S}_6.$$

It is not difficult to see that X^6/Q_1 is isomorphic to $X^2 \cdot P_4^{\mathrm{bic}}$ and X^6/Q_2 to $(L_3 \circ E_2) \times E_6^{\pm}$. Figure 2 describes a $X^2 \cdot P_4^{\mathrm{bic}}$-structure and a $(L_3 \circ E_2) \times E_6^{\pm}$-structure on [6] whose stabilisers are respectively Q_1 and Q_2.

Moreover, there are two transitive subgroups of degree 8, denoted T_{10} and T_{11} in [2], with same "cycle type." The two atomic species X^8/T_{10} and X^8/T_{11} (which have molecular derivatives) are non-isomorphic and have the same cycle index series:

$$\tfrac{1}{16}(x_1^8 + 2x_1^4x_2^2 + 5x_2^4 + 8x_4^2).$$

Here are the first few terms of some interesting sequences:

(1) the number of subgroups of $\mathbf{S}_n$ (see [16]):

1, 2, 6, 30, 156, 1455, 11300, ...

(2) the number of conjugacy classes of subgroups of $\mathbf{S}_n$ (i.e., $|P_n|$) (see [13]) which we know is also the number of molecular species of degree n (i.e., $|\mathscr{M}_n|$):

1, 2, 4, 11, 19, 56, 96, 296, 554, 1593, 3093, ...

(3) the number of atomic species of deree n (i.e., $|\mathscr{A}_n|$):

1, 1, 2, 6, 6, 27, 20, 130, 124, 598, 640, ...

This is computed using the following theorem [12, p. 53]: Let $m_n = |\mathscr{M}_n|$ and $a_n = |\mathscr{A}_n|$, then

$$1 + \sum_{n \geqslant 1} m_n x^n = \prod_{k \geqslant 1} \left(\frac{1}{1 - x^k}\right)^{a_k}$$

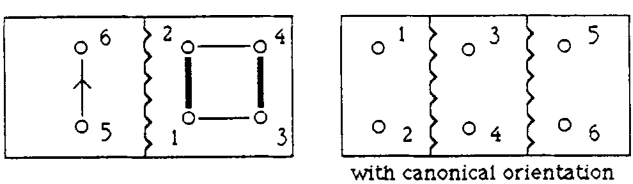

FIGURE 2

(4) the number of conjugacy classes of transitive subgroups of $\mathbf{S}_n$ (see [2]) which we know is the number of isomorphic classes of molecular species with a molecular derivative:

$$1, 1, 2, 5, 5, 16, 7, 50, 34, 45, 8, \ldots$$

(5) the number of (isomorphism classes of) atomic species of degree n which are not non-trivial substitution (see tables A, B, and C of [12]):

$$1, 1, 2, 4, 6, 19, 20, 107, 116, 567, 640, \ldots$$

(6) the number of conjugacy classes of primitive subgroups of $\mathbf{S}_n$ (see [15]):

$$1, 1, 2, 2, 5, 4, 7, 6, 11, 9, 8, \ldots.$$

4. $\mathbb{K}$-Species and λ-Species

Definition 4.1. $(\mathbb{K}, 0, 1, +, \cdot)$ is a *half-ring* if $(\mathbb{K}, +)$ and $(\mathbb{K}, \cdot)$ are commutative monoids and the two "distributive" laws: (1) $(a+b)c = ac + bc$; (2) $0c = 0$ hold in $\mathbb{K}$.

Definition 4.2 ([17]). A half-ring $\mathbb{K}$ is called a *binomial half-ring* if

(a) there exists a $\mathbb{Q}$-algebra $\mathbf{L}$ containing $\mathbb{K}$, and

(b) for every $a \in \mathbb{K}$ and $i \in \mathbb{N}$, $\binom{a}{i} = a(a-1)(a-2)\cdots(a-i+1)/i! \in \mathbb{K}$.

For example, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}[i]$, and $\mathbb{N} + \mathbb{Q}e$ $(e^2 = 0)$ are all binomial half-rings, but $\mathbf{F}_p$, p prime, and $\mathbb{Z}[i]$ are not binomial half-rings.

Definition 4.3 ([17]). Let $\mathbb{K}$ be a binomial half-ring. A $\mathbb{K}$-*species* is an element S of $\mathbb{K}\langle \mathscr{M} \rangle$, i.e., a formal linear combination of the molecular species with coefficients in $\mathbb{K}$.

The concepts of species (resp. virtual species) and $\mathbb{N}$-species (resp. $\mathbb{Z}$-species) coincide. Let $\mathbb{K}$ be a binomial half-ring. We can extend the operations $+$, $\cdot$, $\times$, and $'$ to the set

$$\mathbb{K}\langle \mathscr{M} \rangle = \left\{ \sum_{T \in \mathscr{M}} a_T T \mid a_T \in \mathbb{K} \right\}$$

as

$$\Big(\sum_{T\in\mathcal{M}} a_T T\Big)+\Big(\sum_{T\in\mathcal{M}} b_T T\Big)=\sum_{T\in\mathcal{M}}(a_T+b_T)T \tag{4.1}$$

$$\Big(\sum_{T\in\mathcal{M}} A_T T\Big)\cdot\Big(\sum_{S\in\mathcal{M}} b_S S\Big)=\sum_{T,S\in\mathcal{M}}(a_T\cdot b_S)(T\cdot S) \tag{4.2}$$

$$\Big(\sum_{T\in\mathcal{M}} a_T T\Big)\times\Big(\sum_{S\in\mathcal{M}} b_S S\Big)=\sum_{T,S\in\mathcal{M}}(a_T+b_S)(T\times S) \tag{4.3}$$

$$\Big(\sum_{T\in\mathcal{M}} a_T T\Big)'=\sum_{T\in\mathcal{M}} a_T T'. \tag{4.4}$$

Of course, the terms must be collected on the right sides of (4.2), (4.3), (4.4). It is possible to do so because: given a molecular species M, there are only finitely many pairs of molecular species (S, T) such that $M = S\cdot T$; finitely many pairs of molecular species (U, V) such that M is a subspecies of $U\times V$; and finitely many molecular species W such that M is a subspecies of W'.

LEMMA 4.1. *For all* $n\in\mathbb{N}$,

$$e^{nX}=(e^X)^n=\sum_{k\geqslant 0}\ \sum_{r_1+2r_2+\cdots+kr_k=k}\binom{n}{r_1, r_2, ..., r_k}(X/1!)^{r_1}(X^2/2!)^{r_2}\cdots(X^k/k!)^{r_k}$$

$$=\sum_{M\in\mathcal{M}} g_M(n)M, \tag{4.5}$$

where all r_i *are non-negative integers and all* $g_M(p)\in\mathbb{N}[\binom{p}{i}]$.

PROPOSITION 4.1. *Let* S *be a species and* $n\in\mathbb{N}$, *then*

$$S\times e^{nX}=S\circ(nX).$$

PROPOSITION 4.2. *Let* S *be a species and* $n\in\mathbb{N}$, *then*

$$S(nX)=\sum_{M\in\mathcal{M}} f_M(n)M \qquad \text{for some } f_M(p)\in\mathbb{N}[\tbinom{p}{i}].$$

Now, we can extend this proposition to $\mathbb{K}$-species.

DEFINITION 4.4. Let $\mathbb{K}$ be a binomial half-ring, $a\in\mathbb{K}$, and S be a $\mathbb{K}$-species. Then

$$S(aX)=\sum_{M\in\mathcal{M}} f_M(a)M \qquad \text{with } f_M(p) \text{ defined in Proposition 4.2.}$$

Tables for $S(-X)$ and $S(nX)$ for molecular species of small degree can be found in [18].

PROPOSITION 4.3. *Let T be a species, then*

$$T\circ\left(\sum_{A\in\mathcal{M}^*} n_A A\right)=\sum_{B\in\mathcal{M}} f_B((n_A)_{A\in\mathcal{M}})\,B$$

where $\mathcal{M}^*=\mathcal{M}-\{1\}$ and

$$f_B((p_A)_{A\in\mathcal{M}})\in\mathbb{N}\left[\binom{p_A}{j}\right]_{A\in\mathcal{M}}. \tag{4.6}$$

DEFINITION 4.5 ([17]). Let $\mathbb{N}$ be a binomial half-ring and S, T be two $\mathbb{K}$-species with $T=\sum_{A\in\mathcal{M}^*} n_A A$ for $n_A\in\mathbb{K}$. The substitution, $S\circ T$, of T in S is defined by

$$\sum_{B\in\mathcal{M}} f_B((n_A)_{A\in\mathcal{M}})B \qquad \text{with} \quad f_B((p_A)_{A\in\mathcal{M}}) \text{ given in Proposition 4.3.}$$

THEOREM IV. *Let S, T, and U be $\mathbb{K}$-species with $T_0=U_0=0$, then*

$$(S\circ T)\circ U=S\circ(T\circ U). \tag{4.7}$$

Proof. Let $S=\sum_{A\in\mathcal{M}} s_A A$, $T=\sum_{B\in\mathcal{M}^*} t_B B$ *and* $U=\sum_{C\in\mathcal{M}^*} u_C C$. We have

$$(S\circ T)\circ U=\sum_{M\in\mathcal{M}} f_M((s_A,t_B,u_C)_{A,B,C\in\mathcal{M}})M,$$

$$S\circ(T\circ U)=\sum_{M\in\mathcal{M}} g_M((s_A,t_B,u_C)_{A,B,C\in\mathcal{M}})M,$$

where

$$f_M((p_A,q_B,r_C)_{A,B,C\in\mathcal{M}}),\, g_M((p_A,q_B,r_C)_{A,B,C\in\mathcal{M}})\in\mathbb{N}\left[\tbinom{p_A}{i},\tbinom{q_B}{j},\tbinom{r_C}{k}\right]_{A,B,C\in\mathcal{M}}.$$

By associativity of substitution for real species, f_M and g_M agree when natural numbers are substituted for p_A, q_B, and r_C, and hence they agree when arbitrary elements of $\mathbb{K}$ are substituted.

Similar arguments prove all the identities involving $+$, $\cdot$, $\times$, $\circ$, $'$, 0, and 1.

DEFINITION 4.6. A pre-λ-ring is a commutative ring R with identity 1, with a series of operations λ^i: $R\to R$, $i=0, 1, \ldots$, satisfying for all $x, y\in R$,

(1) $\lambda^0(x)=1$

(2) $\lambda^1(x)=x$

(3) $\lambda^n(x+y)=\sum\lambda^i(x)\lambda^{n-i}(y)$.

There is an equivalent definition. For $x \in R$; consider the formal power series

$$\lambda_t(x) = \lambda^0(x) + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots.$$

Then the requirements are that $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$. In this form it is evident that $\lambda_t(0) = 1$ and $\lambda_t(-x) = 1/\lambda_t(x)$.

Let R be a pre-λ-ring: an element $x \in R$ is *of finite degree n* if $\lambda_t(x)$ is a polynomial of degree n; in other words, if $\lambda^m(x) = 0$ for all $m > n$, but $\lambda^n(x) \neq 0$.

DEFINITION 4.7. A λ-ring R is a pre-λ-ring in which

(1) $\lambda_t(1) = 1 + t$,

(2) For all x, y in R, $n \geqslant 0$, $\lambda^n(xy) = P_n(\lambda^1 x, \lambda^2 x, ..., \lambda^n x, \lambda^1 y, ..., \lambda^n y)$,

(3) For all x in R, and $n, m \geqslant 0$, $\lambda^m(\lambda^n(x)) = Q_{mn}(\lambda^1 x, ..., \lambda^{mn} x)$,

for some polynomials P_n and Q_{mn} (see [6]).

DEFINITION 4.8. A *splitting* λ-ring R is a λ-ring in which

(i) $\lambda_t(1) = 1 + t$.

(ii) Each element $r \in R$ is expressible in the form $r = \sum \pm a_i$ (a finite sum in which each summand is plus-or-minus an element a_i of degree 1).

(iii) The product of two elements of degree one is again of degree one.

The *splitting principle* [6] says that every λ-ring can be embedded in a splitting λ-ring.

DEFINITION 4.9. Let A be a splitting λ-ring. An A-species is an element of $A\langle \mathcal{M} \rangle$, i.e., a formal linear combination of the molecular species with coefficients in A.

Similar to (4.1), (4.2), (4.3), and (4.4), we can extend the operations $(+, \cdot, \times, \text{and } ')$ to the set

$$A\langle M \rangle = \left\{ \sum_{T \in M} a_T T \mid a_T \in A \right\}.$$

If we want to extend the operator $\circ$ to a splitting ring A, then the key point is to define e^{aX} for any $a \in A$ and the exponential species e^X (see [18] for more details).

For any $a \in A$, $a = \sum a_i$ for some finite N and all elements a_i of degree 1. Then

$$e^{aX} = \prod_{i=1}^{N} e^{a_i X} = \prod_{i=1}^{N} \sum_{k \geqslant 0} a_i^k \frac{X^k}{k!} = \sum_{n \geqslant 0} \sum_{\lambda} m_\lambda \frac{X^\lambda}{\lambda!},$$

where m_λ is the monomial symmetric polynomial in the variables $a_1, a_2, ..., a_N$ (see [6]).

As in the case of $\mathbb{K}$-species, propositions (4.1), (4.2), and (4.3) can then be used to define substitution of λ-species (see Definition 4.5). Note that all the identities involving $+$, $\cdot$, $\times$, $\circ$, $'$, 0, and 1, which are true for species remain true for λ-species. For example, the proof of Theorem IV still works.

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