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On the sum of all distances in composite graphs

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Abstract

Let $G_1 \times G_2$, $G_1 + G_2$, $G_1[G_2]$, $G_1 \circ G_2$ and $G_1\{G_2\}$ be the product, join, composition, corona and cluster, respectively, of the graphs G_1 and G_2. We compute the sum of distances between all pairs of vertices in these composite graphs.

1. Introduction

In this paper we are concerned with finite undirected graphs without loops or multiple edges. If G is such a graph, let its vertex- and edge-sets be $V(G)$ and $E(G)$, respectively. The numbers of vertices and edges of G will be denoted by $|G|$ and $|E(G)|$, respectively.

Let $d(u,v) = d(u,v \mid G)$ be the distance between the vertices u and v of a connected graph G. Then we define

$$d(v \mid G) = \sum_{u \in V(G)} d(u,v),$$

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} d(v \mid G) = \frac{1}{2} \sum_{u,\, v \in V(G)} d(u,v).$$

If G is disconnected then by setting $d(v \mid G) = \infty$ and $W(G) = \infty$ the above definitions remain applicable. Then also all the results stated below remain true. In what follows we shall, nevertheless, restrict our considerations to connected graphs.

Hence, $W(G)$ is just the sum of distances between all pairs of vertrices of G. This quantity has been extensively studied in graph theory [1] and was named 'gross status' [5], 'total status' (see [1] p. 42), 'graph distance' [3] and 'transmission' [10, 13].

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Instead of $W(G)$, some authors studied the closely related 'mean distance' $\overline{W(G)} = \binom{|G|}{2}^{-1} W(G)$ [2, 7].

Doyle and Graver [2] mention an application of $\overline{W(G)}$ in architecture (as a tool for the evaluation of floor plans). Another use of $W(G)$, namely in chemistry, seems to have eluded the attention of mathematicians.

As a matter of fact, as early as in 1947 in American chemist Harold Wiener proposed the modeling of certain physico-chemical properties of hydrocarbon molecules by means of $W(G)$ [14]. Wiener himself published several other papers on this matter, showing that $W(G)$ is a quite successful tool for designing quantitative structure-property relations in organic chemistry. In the chemical literature $W(G)$ is nowadays known exclusively under the name 'Wiener number' or 'Wiener index'; the publications dealing with it are legion (for review and further references see [4, 9, 11]). For a mathematical work mentioning the Wiener index see [8].

In this paper we examine the quantity W in the case of graphs that are obtained by means of certain binary operations on pairs of graphs. We shall study the following types of composite graphs [6].

(a) The (Cartesian) *product* $G_1 \times G_2$:

$$V(G_1 \times G_2) = V(G_1) \times V(G_2);$$

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 \times G_2$ are adjacent iff $[u_1 = v_1$ and $(u_2, v_2) \in E(G_2)]$ or $[u_2 = v_2$ and $(u_1, v_1) \in E(G_1)]$.

(b) The *join* $G_1 + G_2$:

$$V(G_1 + G_2) = V(G_1) \cup V(G_2);$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u_1, u_2) | u_1 \in V(G_1), u_2 \in V(G_2)\}.$$

(c) The *composition* $G_1[G_2]$:

$$V(G_1[G_2]) = V(G_1) \times V(G_2);$$

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1[G_2]$ are adjacent iff $[u_1 = v_1$ and $(u_2, v_2) \in E(G_2)]$ or $[(u_1, v_1) \in E(G_1)]$.

(d) The *corona* $G_1 \circ G_2$ is obtained by taking one copy of G_1 and $|G_1|$ copies of G_2, and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1, $i = 1, 2, \ldots, |G_1|$.

(e) The *cluster* $G_1\{G_2\}$ is obtained by taking one copy of G_1 and $|G_1|$ copies of a rooted graph G_2, and by identifying the root of the ith copy of G_2 with the ith vertex of G_1, $i = 1, 2, \ldots, |G_1|$.
The composite graph $G_1\{G_2\}$ was studied by Schwenk [12]; the name 'cluster' is proposed here for the first time.

2. Statement of the results

Theorem 1. *Let G_1 and G_2 be connected graphs. Then*

$$W(G_1 \times G_2) = |G_2|^2 W(G_1) + |G_1|^2 W(G_2).$$

Theorem 2. *For any two graphs* G_1 *and* G_2,

$$W(G_1 + G_2) = |G_1|^2 - |G_1| + |G_2|^2 - |G_2| + |G_1||G_2| - |E(G_1)| - |E(G_2)|.$$

Theorem 3. *Let* G_1 *be a connected graph. Then,*

$$W(G_1[G_2]) = |G_2|^2 [W(G_1) + |G_1|] - |G_1| [|E(G_2)| + |G_2|].$$

Theorem 4. *Let* G_1 *be a connected graph. Then,*

$$W(G_1 \circ G_2) = (|G_2| + 1)^2 W(G_1) + |G_1|[|G_2|^2 - |E(G_2)|] + (|G_1|^2 - |G_1|)|G_2||G_2| + 1).$$

Theorem 5. *Let* G_1 *and* G_2 *be connected graphs. Let r be the root-vertex of* G_2. *Then,*

$$W(G_1\{G_2\}) = |G_2|^2 W(G_1) + |G_1| W(G_2) + (|G_1|^2 - |G_1|)|G_2| d(r|G_2).$$

3. Proofs

The demonstration of the validity of Theorems 2 and 3 is elementary. First, note that any two vertices of $G_1 + G_2$ are either adjacent or at distance two. The distance-two pairs are those corresponding to nonadjacent vertices in either G_1 or G_2. Hence,

$$W(G_1 + G_2) = [|E(G_1)| + |E(G_2)| + |G_1||G_2|] + 2[\tbinom{|G_1|}{2} - |E(G_1) + \tbinom{|G_2|}{2} - |E(G_2)|]$$

and we arrive at Theorem 2.

Second, note that $G_1[G_2]$ can be understood as obtained by taking $|G_1|$ copies of G_2 and by joining all the vertices of the ith and the jth copy of G_2 iff the ith and the jth vertices of G_1 are adjacent. Hence, the vertices in each copy of G_2 of $G_1[G_2]$ are either adjacent (if they were adjacent in G_2) or are at distance two (if they were non-adjacent in G_2). A pair of vertices of $G_1[G_2]$ belonging to different copies of G_2 has the same distance as the corresponding two vertices of G_1. A total of $|G_2|^2$ such vertex pairs of $G_1[G_2]$ correspond to one vertex-pair of G_1. Consequently,

$$W(G_1[G_2]) = |G_1| [|E(G_2)| + 2 [\tbinom{|G_2|}{2} - |E(G_2)|]] + |G_2|^2 W(G_1)$$

resulting in Theorem 3.

Proof of Theorem 1. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$, $u_1, v_1 \in V(G_1)$, $i = 1,2$, be two vertices of $G_1 \times G_2$. We first demonstrate that

$$d(u,v|G_1 \times G_2) = d(u_1, v_1|G_1) + d(u_2, v_2|G_2). \tag{1}$$

Define two projection operators π_1 and π_2, such that for any $(x,y) \in V(G_1 \times G_2)$, $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

Consider now a path P in $G_1 \times G_2$, connecting the vertices u and v. Let the vertices of P be $p_0 = u, p_1, \ldots, p_{k-1}, p_k = v$. The sequence of vertices $\pi_i(p_0), \pi_i(p_1), \ldots, \pi_i(p_k)$ of the graph G_1 represents a walk between u_i and v_i, $i = 1, 2$. From the definition of the product $G_1 \times G_2$ it is clear that $\pi_1(p_{j-1}) = \pi_1(p_j)$ if and only if

$$\pi_2(p_{j-1}) \neq \pi_2(p_j), \quad j = 1, 2, \ldots, k.$$

Denote by $S_i(P)$ number of vertex-pairs $\pi_i(p_{j-1}), \pi_i(p_j)$ for which $\pi_i(p_{j-1}) \neq \pi_i(p_j)$, $j = 1, 2, \ldots, k$, $i = 1, 2$. Then

$$S_1(P) + S_2(P) = k.$$

Since, on the other hand, $S_i(P) \geqslant d(u_i, v_i \,|\, G_i)$, $i = 1, 2$, we conclude that

$$d(u_1, v_1 \,|\, G_1) + d(u_2, v_2 \,|\, G_2) \leqslant k.$$

Choosing P to be the minimum-length path between u and v, we have $k = d(u,v)$ i.e.,

$$d(u_1, v_1 \,|\, G_1) + d(u_2, v_2 \,|\, G_2) \leqslant d(u,v). \tag{2}$$

On the other hand, the sequence of vertices $(\pi_1(p_0), \pi_2(p_0))$, $(\pi_1(p_1), \pi_2(p_0)), \ldots, (\pi_1(p_k), \pi_2(p_0)), (\pi_1(p_k), (\pi_2(p_1)), \ldots, \pi_1(p_k), \pi_2(p_k))$ induces a path in $G_1 \times G_2$ between $u = (\pi_1(p_0), \pi_2(p_0))$ and $v = (\pi_1(p_k), \pi_2(p_k))$ whose length is at least $d(u_1, v_1 \,|\, G_1) + d(u_2, v_2 \,|\, G_2)$. Hence,

$$d(u_1, v_1 \,|\, G_1) + d(u_2, v_2 \,|\, G_2) \geqslant d(u,v). \tag{3}$$

The identity (1) follows now from (2) and (3).

In order to complete the proof of Theorem 1 note that

$$W(G_1 \times G_2) = \frac{1}{2} \sum_{u_1, v_1 \in V(G_1)} \sum_{u_2, v_2 \in V(G_2)} d(u,v) \tag{4}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Theorem 1 is easily deduced when (1) is substituted back into (4). □

Proof of Theorem 5. If two vertices u and v belong to the same copy of G_2, then

$$d(u, v \,|\, G_1\{G_2\}) = d(u, v \,|\, G_2).$$

The respective contribution to $W(G_1\{G_2\})$ is clearly,

$$W_{\text{same}} = |G_1| \, W(G_2).$$

If, however, the vertices u and v of $G_1\{G_2\}$ belong to different copies of G_2, then

$$d(u, v \,|\, G_1\{G_2\}) = d(u, r \,|\, G_2) + d(i, j \,|\, G_1) + d(v, r \,|\, G_2)$$

where i and j denote the vertices of G_1 to which the copies of G_2 are attached. For each fixed pair i, j there are $|G_2|^2$ such pairs u, v and their contribution to $W(G_1\{G_2\})$ amounts $2|G_2|\,d(r|G_2) + |G_2|^2\, d(i,j|G_1)$. Summing these contributions over all the $\binom{|G_1|}{2}$ distinct pairs i, j we arrive at

$$W_{\text{diff}} = 2|G_2|\binom{|G_1|}{2} d(r|G_2) + |G_2|^2\, W(G_1).$$

Adding W_{same} and W_{diff} we obtain $W(G_1\{G_2\})$. □

Proof of Theorem 4. Theorem 4 is a special case of Theorem 5. We have $G_1 \circ G_2 \equiv G_1\{G_2 + K_1\}$, where K_1 is the one-vertex graph and where the root of $G_2 + K_1$ is chosen to be the vertex belonging to $V(K_1)$. □

4. Examples

In this section we report the W-values for several distinguished types of composite graphs, that can be obtained from simple graphs by applying the binary operations considered. Let, as usual, S_n, P_n, C_n and K_n denote the star, path, circuit and complete graph, respectively, on n vertices. Let further $\bar{G}$ be the complement of G. It is well known [1, 3] that

$$W(S_n) = (n-1)^2,$$

$$W(P_n) = \tfrac{1}{6}\, n(n^2 - 1),$$

$$W(C_n) = \tfrac{1}{8}\, n^3 \quad \text{if } n \text{ is even,}$$

$$= \tfrac{1}{8}\,(n^3 - n) \quad \text{if } n \text{ is odd.}$$

We call the graphs $\mathrm{Grd}_{m,n} = P_m \times P_n$, $\mathrm{Cyl}_{m,n} = P_m \times C_n$, $\mathrm{Tor}_{m,n} = C_m \times C_n$, $\mathrm{Sun}_{m,n} = C_m\{P_{n+1}\}$ and $\mathrm{Whl}_{m,n} = \bar{K}_m + C_n$ the grid, cylinder, torus, sun and generalized wheel, respectively. In the case of the sun it is assumed that P_{n+1} is rooted at vertex of degree one. The sums of all distances in these graphs are given by

$$W(\mathrm{Grd}_{m,n}) = \tfrac{1}{6}\, mn(m+n)(mn-1),$$

$$W(\mathrm{Grd}_{n,n}) = \tfrac{1}{3}\, n^3(n^2 - 1) \quad \text{for the square grid } (m = n),$$

$$W(\mathrm{Cyl}_{m,n}) = \frac{1}{8} m^2 n^3 + n^2 \binom{m+1}{3} \quad \text{if } n \text{ is even,}$$

$$= \frac{1}{8} m^2 (n^3 - n) + n^2 \binom{m+1}{3} \quad \text{if } n \text{ is odd,}$$

$$W(\mathrm{Tor}_{m,n}) = \tfrac{1}{8} m^2 n^2 (m+n) \quad \text{if } m \text{ and } n \text{ are even,}$$
$$= \tfrac{1}{8}[m^2 n^2 (m+n) - m^2 n] \quad \text{if } m \text{ is even, } n \text{ is odd,}$$
$$= \tfrac{1}{8}[m^2 n^2 (m+n) - n^2 m] \quad \text{if } m \text{ is odd, } n \text{ is even,}$$
$$W(\mathrm{Whl}_{m,n}) = m^2 + n^2 + mn - m - 2n,$$
$$W(\mathrm{Whl}_{1,n}) = n(n-1) \quad \text{for the simple wheel } (m=1),$$
$$W(\mathrm{Sun}_{m,n}) = \frac{1}{24} m(n+1)[3m(n+1)(m+4n) + 16n^2 + 20n] \quad \text{if } n \text{ is even,}$$
$$= \frac{1}{24} m(n+1)[3m(n+1)(m+4n) + 16n^2 + 17n - 3] \quad \text{if } n \text{ is odd.}$$

The complete multipartite graph $K_{a_1,a_2,\ldots,a_p}$, $p>1$, on $a_1+a_2+\cdots+a_p=n$ vertices is presented as $\overline{K_{a_1}} + \overline{K_{a_2}} + \cdots + \overline{K_{a_p}}$. This yields

$$W(K_{a_1,a_2,\ldots,a_p}) = n^2 - n - \sum_{1 \leqslant i < j \leqslant p} a_1 a_j.$$

For the complete bipartite graph ($p=2$) the above result reduces to the previously known formula (cf. [1, p. 203]).

$$W(K_{a,b}) = a^2 - a + b^2 - b + ab = n^2 - n - ab \quad \text{where } n = a+b.$$

If $a_1 = a_2 = \cdots = a_p = a$, then

$$W(K_{a,a,\ldots,a}) = \tfrac{1}{2}(n^2 - 2n + na) \quad \text{where } n = ap.$$

The t-fold bristled graph G, denoted by $\mathrm{Brs}_t(G)$, is defined as $G\{S_{t+1}\}$ where the root of S_{t+1} is on its vertex of degree t. Then

$$W(\mathrm{Brs}_t(G)) = (t+1)^2 W(G) + t(t+1)|G|^2 - t|G|.$$

For the simple bristled graph ($t=1$),

$$W(\mathrm{Brs}_1(G)) = 4W(G) + 2|G|^2 - |G|.$$

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