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Minimal graphs of a torus, a projective plane and spheres and some properties of minimal graphs of homotopy classes

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Abstract

Contractible transformations of graphs consist of contractible gluing and deleting of vertices and edges of graphs. They partition all graphs into the family of homotopy classes. Contractible transformations do not change the Euler characteristic and the homology groups of graphs. In this paper we describe the minimal representatives of some homotopy classes and find the formula for computing the Euler characteristic of partite and some other graphs.

We also describe the minimal graphs of a projective plane, a torus and a sphere.

1. Introduction

Contractible transformations of graphs are defined and based on contractible transformations of molecular spaces. In the simplest case a molecular space is a family of unit cubes in Euclidean space E^n. Similar spaces are used in digital topology for the study of the topological properties of image arrays. We discovered a surprising fact. Suppose that S_1 and S_2 are topologically equivalent surfaces in E^n. Divide E^n into a set of unit cubes and call the molecular spaces M_1 and M_2 representing the families of unit cubes intersecting S_1 and S_2, respectively. When a division was small enough, it was revealed that M_1 and M_2 could be transformed from one to the other using four kinds of transformations. This allows us to assume that the molecular space contains

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the topological and, perhaps, the geometrical characteristics of a continuous surface. Otherwise, the molecular space would be a discrete counterpart of continuous space. Any molecular space can be described by its intersection graph. In our previous paper [3] we defined the contractible transformations of graphs and proved that these transformations did not change the Euler characteristic of a graph. Since any point of S belongs to at least one cube of the molecular space M, the intersection region of S with the unit cube is a subset of S. The collection of all these subsets forms a map of S.

Barnette [1] found a set of seven complexes which can be used to generate a projective plane polyhedral map using face and vertex splitting. These complexes are minimal maps of a projective plane. Taylor [4] introduced three hypotheses which a crystalline minimal surface S must satisfy. He bounded the number of plane segments in this surface based on its Euler characteristic, and energy function and gave several examples of such surfaces.

In this paper we will give the definition of the minimal representative of a homotopy class of graphs. We will find the minimal graph of the sphere, the torus and the projective plane. We will prove that partite graphs are minimal, and give the simple formula of the Euler characteristic for such graphs.

Since we use only induced subgraphs, we shall use the word subgraph for an induced subgraph. The s-partite graph is denoted as $K(n_1, n_2, \ldots, n_s)$. Therefore, the designation $K(n)$ refers to the graph on n vertices with no edges, but not the complete graph, which is denoted as $C(n)$. For any terminology used but not defined here, see [2].

2. Contractible transformations of graphs

In [3] we introduced contractible transformations of graphs, based on four operations. Let G be a graph with a set of vertices, $V=(v_1, v_2, \ldots, v_n)$ and a set of edges E. Let G_1, $G_1 \subseteq G$, be a subgraph of the graph G. The subgraph $O(G_1)$, containing all neighbors of all vertices of G_1 except vertices of G_1, is called the rim of G_1. The subgraph $U(G_1)$, containing the vertices of G_1, as well as $O(G_1)$, is called the ball of the graph G_1. Clearly, $U(G_1)-G_1=O(G_1)$.

If G_1 is a vertex v, then $O(v)$ and $U(v)$ are called the rim and the ball of the vertex v, respectively. The subgraph $O(v_1 v_2 \cdots v_k)=O(v_1)\cap O(v_2)\cap\cdots\cap O(v_k)$ is called the joint rim of the vertices $v_1, v_2, \ldots, v_k$.

Definition 2.1. The following transformations are said to be contractible:

(1) Deleting of a vertex v. A vertex v of a graph G can be deleted if the rim $O(v)$ is contractible, $O(v)\in T$.

(2) Gluing of a vertex v. If a subgraph G_1 of the graph G is contractible, $G_1 \in T$, then the vertex v can be glued to the graph G in such a manner that $O(v)=G_1$.

(3) Deleting of an edge $(v_1 v_2)$. The $(v_1 v_2)$ of a graph G can be deleted if the joint rim $O(v_1 v_2)$ is contractible, $O(v_1 v_2)\in T$.

(4) Gluing of an edge $(v_1 v_2)$. Let two vertices v_1 and v_2 of a graph G be nonadjacent. The edge $(v_1 v_2)$ can be glued to the graph G if the joint rim $O(v_1 v_2)$ is contractible.

Definition 2.2. The family T of graphs $G_1, G_2, \ldots, G_n, \ldots$, $T=(G_1, G_2, \ldots, G_n, \ldots)$, is called contractible if

(1) the trivial graph $K(1)$ belongs to T,

(2) any graph of T can be obtained from the trivial graph by contractible transformations.

All graphs of the family T are called contractible and the family T of contractible graphs is determined by inductive application of operations (1)–(4).

The vertex v and the edge $(v_1 v_2)$ of a graph G are called contractible if $O(v) \in T$ and $O(v_i v_2) \in T$. Obviously, the contractible transformations divide the set of all graphs into homotopy classes [3]. Two graphs G_1 and G_2 are called homology-equivalent if G_1 can be mapped to G_2 by contractible transformations (1)–(4).

Let G be a finite graph. Denote n_p as the number of its complete subgraphs $C(p)$. The vector

$$f(G)=(n_1, n_2, \ldots, n_n)$$

is the f-vector of the graph G. The Euler characteristic of a graph G, $F(G)$, was defined in [3] as

$$F(G)=\sum_{k=1}^{s} (-1)^{k+1} n_k.$$

3. Minimal graphs

Definition 3.1. Let G be a graph with a set of vertices $V=(v_1, v_2, \ldots, v_n)$ and a set of edges E. The graph G is called minimal if

(1) The rim $O(v)$ of any vertex v of G is not contractible, $O(v) \notin T$.

(2) For any edge $(v_i v_k) \in E$, the joint rim $O(v_i v_k)$ is not contractible, $O(v_i v_k) \notin T$.

(3) If we glue any contractible edge e to G, the obtained graph, $G+e$, will have, as before, all vertices noncontractible.

Criteria (1) and (2) mean that G does not have contractible vertices or edges which can be deleted. Criterion (3) means that the vertices retains contractibility in spite of the fact that we glue the contractible edge to G.

Theorem 3.2. *Let $K(n_1, n_2, \ldots, n_s)$ be the s-partite graph. The Euler characteristic of the graph is*

$$F(K(n_1, n_2, \ldots, n_s))=1-\prod_{i=1}^{s} (1-n_i). \tag{1}$$

Proof. In [3] it was shown that

$$F(K(n_1,n_2,\dots,n_s))=n_1-(n_1-1)F(K(n_2,\dots,n_s)). \tag{2}$$

Suppose that for $s\leqslant p$ formula (1) is correct, i.e.

$$F(K(n_1,n_2,\dots,n_p))=1-\prod_{i=1}^{p}(1-n_i).$$

Then

$$\begin{aligned}F(K(n_1,n_2,\dots,n_{p+1}))&=n_{p+1}-(n_{p+1}-1)F(K(n_1,n_2,\dots,n_p))\\&=n_{p+1}-(n_{p+1}-1)\left(1-\prod_{i=1}^{p}(1-n_i)\right)\\&=1-\prod_{i=1}^{p+1}(1-n_i).\qquad\square\end{aligned} \tag{3}$$

Corollary 3.3. *If $n_i\geqslant 2$, $i=1,2,\dots,s$, then $K(n_1,n_2,\dots,n_s)$ is noncontractible.*

Proof. It was shown in [3] that $F(G)=1$ for any contractible graph G. It follows from (1) that

$$F(K(n_1,n_2,\dots,n_s))\neq 1$$

if $n_i\neq 1$, $i=1,2,\dots,s$. Therefore, $K(n_1,n_2,\dots,n_s)$ is noncontractible. $\square$

Theorem 3.4. *Any s-partite graph $K(n_1,n_2,\dots,n_s)$, where $n_i\geqslant 2$, $i=1,2,\dots,s$, is minimal.*

Proof. We have to prove that criteria (1)–(3) of the definition can be realized on any s-partite graph. For $K(n_1)$, $K(n_1,n_2)$, and $K(n_1,n_2,n_3)$ the theorem is verified directly. Suppose that all $K(n_1,n_2,\dots,n_s)$, $s\leqslant p$, are minimal. Let $s=p+1$ and $V=V_1\cup V_2\cup\dots\cup V_p\cup V_{p+1}$, $|V_1|=n_1$, $|V_2|=n_2,\dots,|V_p|=n_p$, $|V_{p+1}|=n_{p+1}$, be the set of vertices of the graph. If $v_i\in V_i$ then $O(v_i)=K(n_1,\dots,\check{n}_i,\dots,n_{p+1})$. Through inductive reasoning, we may assume that $O(v_i)$ is noncontractible. In exactly the same way, we prove that for the edge (v_iv_k), $v_i\in V_i$, $v_k\in V_k$, the joint rim $O(v_iv_k)$ is equal to $K(n_1,\dots,\check{n}_i,\dots,\check{n}_k,\dots,n_{p+1})$, $O(v_iv_k)=K(n_1,\dots,\check{n}_i,\dots,\check{n}_k,\dots,n_{p+1})$, and $O(v_iv_k)$ is noncontractible.

Criterion (3) in this case is satisfied automatically because for any two vertices v_i and v_k, $O(v_iv_k)$ is noncontractible, and we cannot glue any edges. Here, as usual, the check indicates that $\check{v}_k$ is understood to be deleted. $\square$

Let $G(p,g)$ and $H(p',g')$ be two graphs on p and p' vertices, and g and g' edges, respectively. We denote $G<H$ if

(1) $p<p'$, or

(2) $p=p'$, $g<g'$.

Theorem 3.5. *Let G be a graph. If $F(H) \neq F(G)$ for all $H < G$, then G is minimal.*

Proof. (i) If $G(p, g)$ contains a contractible vertex v, then v can be deleted and $G - v = H < G$. and $F(H) = F(G)$. This is a contradiction to the given fact $F(H) \neq F(G)$. Hence, G does not have contractible vertices.

(ii) Suppose that the edge $(v_1 v_2)$ of G is contractible. As before, this is proved similarly.

(iii) Suppose that the joint rim $O(v_1 v_2)$ of two nonadjacent vertices v_1 and v_2 is contractible. Glue the edge $e = (v_1 v_2)$ to G. Suppose that the vertex v_i becomes contractible, we can then delete this vertex from $G + e$ and obtain the graph $H = G + e - v_i$. Since $H < G$, then $F(H) = F(G)$, and we obtain the same contradiction. This completes the proof. □

This theorem can help us find minimal graphs. We start from the graph with $p = 1$ and enumerate the set of graphs by increasing p and g, respectively, and computing their Euler characteristics. Any graph in this enumeration, which satisfies Theorem 3.5, is a minimal one.

4. Some minimal connected graphs

(1) Here we enumerate all connected minimal graphs with the number of vertices $n \leqslant 6$ (Fig. 1). Their Euler characteristics are

$$F(K(1)) = 1, \qquad F(K(2,2)) = 0, \qquad F(K(2,3)) = -1,$$

$$F(K(2,2,2)) = 2, \qquad F(K(2,4)) = F(K(3,3)) - e = -2,$$

$$F(K(3,3)) = -3.$$

Here is a good example of how two different minimal graphs can be in the same homotopic class. Let $K(3,3) - e$ be the graph obtained from $K(3,3)$ by deleting of one edge e (Fig. 1). It is easy to see that $K(3,3) - e$ and $K(4,2)$ are both minimal and homotopy-equivalent to each other. Obviously, they are not isomorphic.

(2) In the paper [3], n-partite graphs $K(2, 2, \ldots, 2)$ were considered as the discrete counterparts of continuous $(n-1)$-dimensional spheres S^{n-1}. With respect to Theorem 3.4 all of them are minimal, and

$$F(K(2, 2, \ldots, 2)) = \begin{cases} 0, & n = 2p, \\ 2, & n = 2p + 1. \end{cases}$$

(3) The graph G from [3], which is the discrete model of the torus (Fig. 2), is not minimal. The following is easily verified: we can delete all right-oblique (or

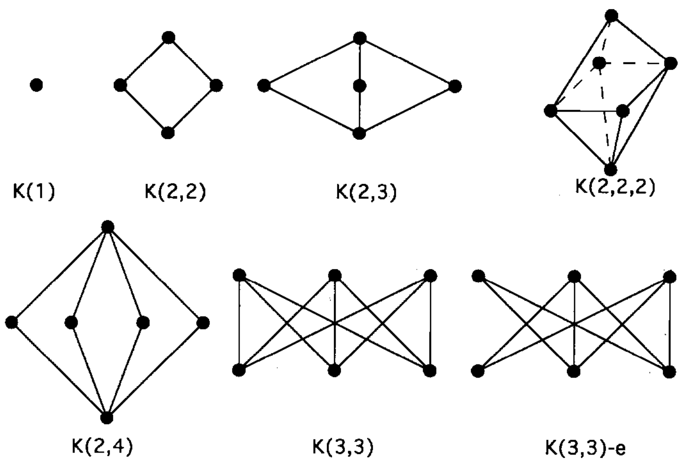

Fig. 1.

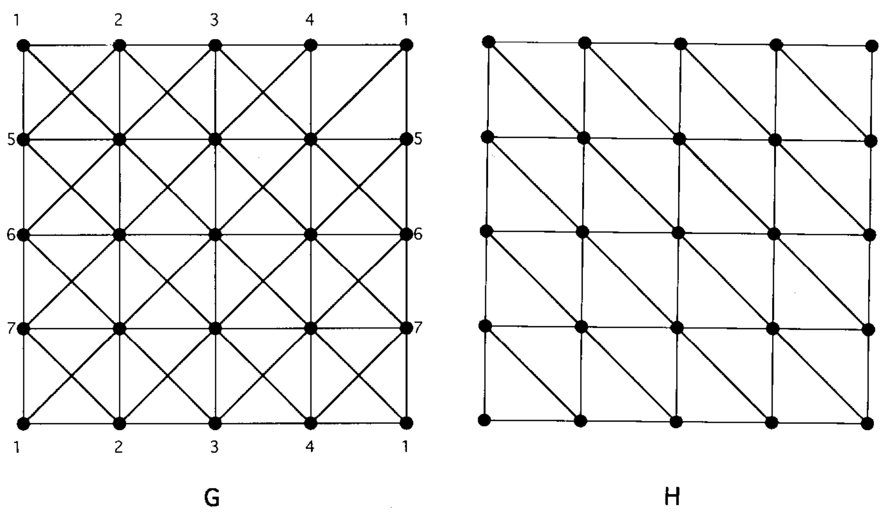

Fig. 2.

left-oblique) edges and obtain the graph H, which is minimal, and

$$F(G)=F(M)=0.$$

(4) The graph G, which is the digital counterpart of the projective plane [3], can be reduced to the graph H (Fig. 3), which is minimal, and

$$F(G)=F(H)=1.$$

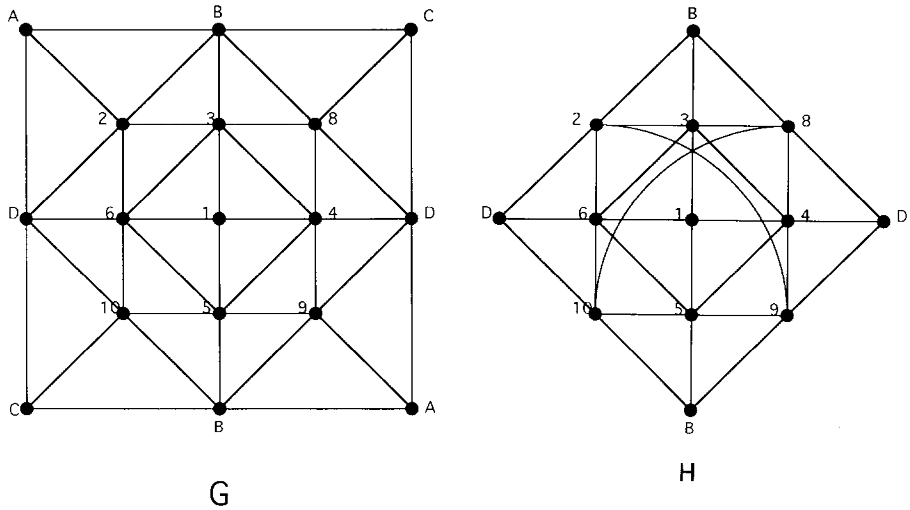

Fig. 3.

5. Problems

It is clear that in the homotopy class of contractible graphs the only one-point graph $K(1)$ is the minimal one. On the other hand, there exist two minimal graphs $K(4,2)$ and $K(3,3)-e$ in homotopy class of $K(4,2)$. The problem is how many minimal graphs exist in the given homotopy class. In addition, we can give one more definition of the minimal graph, which works as well as the first one.

Definition 5.1 (*$*$-definition*). Let G be a graph with a set of vertices $V=(v_1, v_2, \ldots, v_n)$ and a set of edges E. The graph G is called $*$-minimal if

(1) the rim $O(v)$ of any vertex v of G is not contractible, $O(v) \in T$;

(2) for any edge $(v_i v_k) \in E$, the joint rim $O(v_i v_k)$ is not contractible, $O(v_i v_k) \in T$;

(3) if we glue in turn any number of contractible edges $e_1, e_2, \ldots, e_p$ to G, the obtained graph $G+e_1+e_2+\cdots+e_p$ will have, as before, all vertices noncontractible.

It is easy to show that the icosahedron can be mapped to the octahedron by contractible transformations. Obviously, the icosahedron is minimal with respect to the first definition of the minimality, but non $*$-minimal with respect to the second definition.

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