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Measures of Distinctness for Random Partitions and Compositions of an Integer

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1. INTRODUCTION

Partitions and compositions of integers are, besides their intrinsic interests, usually used as theoretical models for evolutionary processes in different contexts: statistical mechanics, theory of quantum strings, population biology, nonparametric statistics, etc.; cf. [1, 4, 8, 10, 12, 30, 49, 54]. Also parameters in partitions often have natural interpretations in terms of characters in symmetric groups; cf. [15, 47]. Thus properties (statistical, algebraic, analytic, ...) of these objects received constant attention in the literature.

In many situations, the notion of "degree of distinctness" naturally arises. The classical birthday paradox states that one needs on the average > 23 people to discover two that have the same birthday with probability $> 1/2$, assuming all birth dates to be equally likely; cf. [16]. The coupon collector problem is similar: what is the expected number of coupons one needs to gather before a full collection, under suitable probability assumptions on the issuing of the coupons? In applications in which only the first product (element, particle, ...) is "expensive" and the "cost" of the remaining reproductions is negligible, the study of the measures of distinctness becomes meaningful and important. The number of distinct outcomes in a sequence of multinomial trials (the classical occupancy problem) has wide applications; see, for example, Knuth [34], Johnson and Kotz [28], Kolchin et al. [35], Arató and Benczúr [5], and Vitter and Chen [50]. The number of distinct sites visited by a random walk plays an important role

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in a number of applications in physics and chemistry; see Larralde and Weiss [36] and the references therein. Finally, since distinct irreducible factors over finite fields are important in most algorithms for factorizing polynomials, it is also of interest to investigate the measures of distinctness (in order to determine the complexity of the algorithms); cf. [34, 46, 18].

Such a metric notion is useful and widely used in different fields. In number theory, the number of prime factors (with or without multiplicity) has long been used as a measure of *compositeness* of an integer; cf. [25, 40]. In algorithmic theory, the introduction of the measures of *presortedness*, like inversions, runs, total variations, etc., for sorting problems is well justified by its practical applications; cf. [37]. In information theory, the Shannon entropy is the standard measure of *information* of a source code. The study of measures of *association* for data is an important issue in many quantitative problems arising in diverse disciplines such as political science, psychology, and sociology, and in rank statistics, Kendall's τ and Spearman's ρ are commonly used measures of *association* (*or disarray*) of data; cf. [11, 22, 29]. In probability theory, the Kolmogorov distance and the total variation distance between two distributions are frequently used measures of *closeness*.

This paper is concerned with problems of the following type: *Given a random* (*under a suitable probability model*) *partition or composition*, *study quantitatively the measures of the degree of distinctness of its parts*.

Erdös and Lehner [14] were the first, from a probabilistic point of view, to study, in their classical and concise paper, the number and the sum of distinct parts in partitions (into positive integers). More precisely, if

$$\Pi\colon r_1 \cdot 1 + r_2 \cdot 2 + \cdots + r_n \cdot n = n; \qquad r_j \ge 0,\ j = 1, 2, \dots, n,$$

denotes a partition of n, they considered the two quantities

$$\sum_{1 \le j \le n} 1_{\{r_j \ge 1\}} \quad \text{and} \quad \sum_{1 \le j \le n} j \cdot 1_{\{r_j \ge 1\}}, \tag{1}$$

where $1_A = 1$ if property A holds and $1_A = 0$ otherwise. These two typical quantities measure to some extent the distinctness of the parts in Π. Their results state that a randomly given partition of n has about $\sqrt{6n}/\pi$ distinct parts whose sum is asymptotic to $6n/\pi^2$, where a uniform probability measure on the set of partitions of n is assigned.[1]

Only quite recently did these measures receive further attention. Wilf [55] studied the number of distinct components in general (decomposable) combinatorial structures. In particular, he rederived the result of Erdős

[1] The constant $6/\pi^2$ is also the density of square-free positive integers and the probability that two integers should be prime to each other; cf. [25, Chap. XVIII].

and Lehner for the mean number of distinct summands in a random partition. Central limit theorems for the number of distinct summands in partitions were derived by Goh and Schmutz [21]; see also Schmutz [45]. Local limit theorems were studied by Hwang [26].

The corresponding problems for compositions are, unlike most other ones, more complicated and first treated by Knopfmacher and Mays [31, 32]. They derived some combinatorial properties of the number and sum of distinct parts using an elementary approach. Another paper by Richmond and Knopfmacher [44] is also interesting since the results there further reveal the intricacy of the composition structure when studied from a "distinct" viewpoint. See also Warlimont [51] for a multiplicative counterpart.

A closely related stochastic model to integer composition is the one studied by Chen [9], who considered the number of distinct values assumed by a (finite) sequence of discrete random variables with total sum n. Although limit theorems for the number of distinct parts in general compositions can be formulated into this model with suitably chosen distribution, his general limit theorems are not useful for our purposes.

The purpose of this paper is twofold. First, the measures of distinctness for partitions and compositions are studied in a more general framework: We develop general methods for deriving generating functions for measures in partitions and different compositions—ordinary, cyclic, and branching—with parts belonging to any specific subset Λ of positive integers. Each of these types of compositions has its own operational characteristic and specific analytic properties. Next, we use probabilistic and analytic methods to investigate in detail the asymptotic behavior of a general weighted sum, including the number and sum of distinct parts in a random partition and composition as special cases.

Our results reveal, in particular, a general phenomenon of "logarithmic transfer" from the asymptotic behavior of the given counting sequence to that of the mean number of distinct summands. Also of special interest is the periodic oscillation in the asymptotic expansion of the number of distinct summands in compositions (but not in partitions), a result demanding further structural interpretations.

Some extensions will be briefly discussed in Section 5.

Notation. Throughout this paper, we denote by $\Lambda = \{\lambda_j\}_{j \geq 1}$ an infinite sequence of positive integers such that $1 \leq \lambda_1 < \lambda_2 < \cdots$. The generating function of Λ is denoted by $\Lambda(z) = \sum_{j \geq 1} z^{\lambda_j}$. The symbol $[z^n]f(z)$ represents the coefficient of z^n in the Taylor expansion of $f(z)$. The Vinogradov symbol $\ll$ is used as a synonym of Landau's $O(\cdot)$ symbol. All unspecified limits (including O, o, $\sim$, $\approx$, and $\ll$) are taken to be $n \to \infty$. The symbol $w(x)$ denotes a certain *real-valued* weight function. We use

Λ-partition of Λ-composition to mean partition or composition of an integer into parts λ_j. The symbol ε always denotes arbitrarily small but fixed quantity whose value vary from one occurrence to another.

2. MEASURES OF DISTINCTNESS

Let

$$\Pi = \Pi(n) \colon r_1 \cdot \lambda_1 + r_2 \cdot \lambda_2 + \cdots = n; \qquad r_j \geq 0,\ j = 1, 2, \ldots, \tag{2}$$

denote a Λ-partition or Λ-composition of n. Consider the following ways to measure the degree of distinctness of Π generalizing (1):

$$X_n(w) := \sum_{j \geq 1} w(\lambda_j) \cdot 1_{\{r_j \geq 1\}} \qquad (w \in \mathbf{R}). \tag{3}$$

The general measure $X_n(w)$ is a random variable when $\Pi(n)$ is chosen uniformly at random from the set of Λ-partitions (or Λ-compositions) of n. Thus X_n is the weighted sum of dependent Bernoulli random variables. The following probabilistic approach is especially useful for studying mean values of measures of the form (3).

By the linearity of expectation, we have

$$\begin{aligned} \mathrm{E}(X_n) &= \sum_{j \geq 1} w(\lambda_j) \mathrm{P}(r_j \geq 1) \\ &= \sum_{j \geq 1} w(\lambda_j) \mathrm{P}(\lambda_j \text{ appears in } \Pi). \end{aligned}$$

Let $p_\Lambda(n)$ and $c_\Lambda(n)$ denote, respectively, the number of Λ-partitions and Λ-compositions of n:

$$p_\Lambda(n) = [z^n] \prod_{j \geq 1} (1 - z^{\lambda_j})^{-1}, \qquad c_\Lambda(n) = [z^n] \frac{1}{1 - \Lambda(z)}.$$

Then

$$\mathrm{E}(X_n) = \sum_{j \geq 1} w(\lambda_j) \frac{p_\Lambda(n - \lambda_j)}{p_\Lambda(n)},$$

and

$$\mathrm{E}(X_n) = \sum_{j \geq 1} w(\lambda_j) \left(1 - \frac{[z^n](1 - \Lambda(z) + z^{\lambda_j})^{-1}}{c_\Lambda(n)} \right)$$

respectively. These expressions are useful for further asymptotics on $\mathrm{E}(X_n)$.

In what follows, when $w(x) = x^{\alpha}$, *we write* $\mathrm{E}(X_n^{(\alpha)})$.

Such an approach is, however, rather limited. Thus we shall derive the corresponding multivariate generating functions.

Besides X_n, one may also consider, for example, the following measures:

1. the number of odd (or even) parts or, in general, the number of parts that are $h \bmod k$, $k \geq 2$;

2. the number of distinct pairs of consecutive summands or the number of distinct "patterns" (with or without overlaps);

3. the total variation of parts, defined as the sum of the difference of all pairs of summands;

4. the order statistics of the parts (this being similar to order statistics of random variables);

5. the greatest common divisor or the least common multiple of the parts; and

6. other statistics on the parts like the number of inversions, runs, peaks, left-to-right maxima, etc.

Some of these quantities are difficult to work with. For other statistics, see Diaconis et al. [12].

3. COMPOSITIONS

In this section, we first derive generating functions for compositions with a given number of distinct parts; then we consider in detail the asymptotics of the mean measure $\mathrm{E}(X_n^{(\alpha)})$ when $\Lambda = \mathbf{Z}^+$ for all possible values of α. The case of general Λ is considered in Section 3.3. Finally, we conclude with cyclic and branching compositions.

3.1. *Generating Functions*

Let $u_0 = 1$ and

$$C(z; u_0, u_1, u_2, \ldots) = 1 + \sum_{n \geq 1} z^n \sum_{\substack{r_1\lambda_1 + r_2\lambda_2 + \cdots = n \\ r_l \geq 0}} \prod_{j \geq 1} u_{r_j}^{w(\lambda_j)},$$

where the inner summation runs over all composition of n into parts λ_j. If w assumes real values, we need to keep the u_js away from the origin to avoid possible ambiguity. (For computational purposes, one may take $u_j = e^{i\theta_j}$.)

THEOREM 1. *The generating function C satisfies*

$$C(z;u_0,u_1,u_2,\dots) = \int_0^\infty e^{-t}\prod_{j\geq 1}\left(1+\sum_{l\geq 1}\frac{u_l^{w(\lambda_j)}t^l z^{l\lambda_j}}{l!}\right)dt. \tag{4}$$

Proof. The presence of an infinite product in an integral indicates the basic principle: first consider the unordered counterpart and then incorporate the enumerating factor into the generating function.

To each partition of n of the type

$$\begin{aligned}\Pi = \Pi(n)\colon & s_1\cdot\lambda_{i_1} + s_2\cdot\lambda_{i_2} + \cdots + s_m\lambda_{i_m} = n;\\ & s_j\geq 1,\ j=1,2,\dots,m;\ \lambda_{i_1}<\lambda_{i_2}<\cdots<\lambda_{i_m},\end{aligned} \tag{5}$$

there correspond

$$\frac{(s_1+\cdots s_m)!}{s_1!s_2!\cdots s_m!}$$

compositions. Now we have

$$\prod_{j\geq 1}\left(1+\sum_{l\geq 1}\frac{u_l^{w(\lambda_j)}t^l z^{l\lambda_j}}{l!}\right) = 1+\sum_{n\geq 1} z^n \sum t^{s_1+\cdots s_m}\prod_{1\leq j\leq m}\frac{u_{s_j}^{w(\lambda_{i_j})}}{s_j!},$$

where the inner sum on the right-hand side is extended over all partitions of n of the form (5). Multiplying both sides by e^{-t} and integrating from 0 to ∞ gives the factor $(s_1+\cdots+s_m)!$. This completes the proof. ∎

Let $e_h(z) = \sum_{0\leq j\leq h} z^j/j!$. From (4), we obtain the following special cases. By taking $w(x)=1$ and $u_l = u$ for $l\geq h$, $h\geq 1$,

$$\int_0^\infty e^{-t}\prod_{j\geq 1}\Big(e_{h-1}(tz^{\lambda_j}) + u\big(e^{tz^{\lambda_j}} - e_{h-1}(tz^{\lambda_j})\big)\Big)\,dt,$$

where u "marks" the number of parts occurring $\geq h$ times, and, by taking $w(x)=x$ and $u_l=u$ for $l\geq h$,

$$\int_0^\infty e^{-t}\prod_{j\geq 1}\Big(e_{h-1}(tz^{\lambda_j}) + u^{\lambda_j}\big(e^{tz^{\lambda_j}} - e_{h-1}(tz^{\lambda_j})\big)\Big)\,dt,$$

where u "marks" the sum of those parts with frequencies $\geq h$.

Thus the expected number of parts with frequencies $\geq h$ in a random composition of n is given by

$$\frac{1}{c_\Lambda(n)}[z^n]\sum_{j\geq 1}\left(\frac{1}{1-\Lambda(z)} - \sum_{0\leq l\leq h-1}\frac{z^{l\lambda_j}}{\left(1-\Lambda(z)+z^{\lambda_j}\right)^{l+1}}\right)$$

$$= \frac{1}{c_\Lambda(n)}[z^n]\sum_{j\geq 1}\frac{z^{\lambda_j h}}{(1-\Lambda(z))\left(1-\Lambda(z)+z^{\lambda_j}\right)^h}.$$

3.2. *Asymptotics of the Mean Measure*

Let us first consider the case $\Lambda = \mathbf{Z}^+$ in some details for two reasons. First, this is a special case for which we can derive rather precise expressions for $\mathrm{E}(X_n^{(\alpha)})$. The problem becomes more involved for general Λ and requires different analysis. In particular, there is a logarithmic transfer between the asymptotic behavior of $\sum_{1\leq j\leq n} j^\alpha$ to that of $\mathrm{E}(X_n^{(\alpha)})$. Next, it is surprising that there appears a certain fluctuating phenomenon in the resulting formula. This seems rather unexpected and a possibly structural interpretation requires further investigation.

THEOREM 2. *As $n \to \infty$, the expectation of $X_n^{(\alpha)}$ satisfies*

(i) *if $\alpha < -1$,*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \zeta(-\alpha) + O\left((\log n)^{\alpha+1}\right); \tag{6}$$

(ii) *if $\alpha = -1$,*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \log\log n - \log\log 2 + O\left((\log n)^{-1}\right); \tag{7}$$

(iii) *if $\alpha > -1$,*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \left(\frac{\log n}{\log 2}\right)^{\alpha+1}\left(\frac{1}{\alpha+1} + \sum_{1\leq l\leq[\alpha]+2}\frac{\alpha(\alpha-1)\cdots(\alpha-l+1)}{(\log n)^l}\right.$$
$$\left.\times\left(e_l - \varpi_l(\log_2 n)\right)\right)$$
$$+ \zeta(-\alpha) + O\left((\log n)^{-1}\right), \tag{8}$$

where $e_l = [s^l]2^{-s}\Gamma(1-s)$, and

$$\varpi_l(u) = \sum_{j\in\mathbf{Z}\setminus\{0\}} e_{l,j}\Gamma(-\chi_j)e^{2j\pi i u} \qquad (\chi_j = 2j\pi i/\log 2),$$

with $e_{l,j} = [s^l]2^{-s}\Gamma(-s-\chi_j)/\Gamma(-\chi_j)$.

The error term in (8) can be replaced by $O(n^{-1}(\log n)^{\alpha+1})$ when α is a nonnegative integer. Note also that

$$\frac{1}{\alpha+1}\left(\frac{\log n}{\log 2}\right)^{\alpha+1} \to \log\log n - \log\log 2,$$

as $\alpha \to -1$.

Observe that the first term on the right-hand side of (8) may be roughly derived as

$$\mathrm{E}\big(X_n^{(\alpha)}\big) \approx 1^\alpha + \cdots + \big(\mathrm{E}(X_n)\big)^\alpha \sim \frac{1}{\alpha+1}\big(\mathrm{E}(X_n)\big)^{\alpha+1}.$$

This heuristic does not, however, apply for partitions; see Theorem 7.

COROLLARY 1. *The number and the sum of distinct parts in a random composition of n satisfy*

$$\mathrm{E}\big(X_n^{(0)}\big) = \log_2 n - \frac{3}{2} + \frac{\gamma}{\log 2} - p_1(\log_2 n) + O\big(n^{-1}\log n\big),$$

$$\mathrm{E}\big(X_n^{(1)}\big) = \frac{(\log n)^2}{2(\log 2)^2} + p_2(\log_2 n)\log n + p_3(\log_2 n) + O\big(n^{-1}(\log n)^2\big),$$

respectively, where γ denotes Euler's constant and the p_js are periodic functions of period 1 *whose Fourier series can be written as*

$$p_1(u) = \frac{\varpi_0(u)}{\log 2}, \qquad p_2(u) = \frac{\gamma - \log 2}{(\log 2)^2} - \frac{\varpi_0(u)}{(\log 2)^2},$$

$$p_3(u) = \frac{6\gamma^2 - 12\gamma\log 2 + 5(\log 2)^2 + \pi^2}{12(\log 2)^2} + \frac{1}{\log 2}\sum_{k\in\mathbf{Z}\setminus\{0\}}\left(1 + \frac{\psi(\chi_k)}{\log 2}\right)\gamma(\chi_k)e^{2k\pi i u},$$

ψ being the logarithmic derivative of the Gamma function.

Note that the expected value of the number of parts, counted with multiplicity, in a random composition of n is equal to $(n+1)/2$. Therefore, the preceding results imply roughly that *almost all compositions of n have many small parts with large multiplicities*.

A closely related quantity is the largest summand in a random composition of n whose mean value satisfies

$$2^{1-n}[z^n]\sum_{k\geq 1}\left(\frac{1-z}{1-2z} - \frac{1-z}{1-2z+z^k}\right)$$

and has asymptotically the same behavior as $\mathrm{E}(X_n^{(0)})$; see [24, 39].

Proof of Theorem 2. By definition,

$$\mathrm{E}\big(X_n^{(\alpha)}\big) = 2^{1-n}[z^n]F(z),$$

where

$$F(z) = \sum_{k\geq 1} k^{\alpha}\left(\frac{1-z}{1-2z} - \frac{1-z}{1-2z+z^k-z^{k+1}}\right). \tag{9}$$

We note that generating functions of the form (9) (with $\alpha = 0$) were encountered in different contexts; see Knuth [33] and Gourdon and Prodinger [23]. Our approach, which is an extension of theirs, yields more precise results.

Set $f_k(z) = 1 - 2z + z^k - z^{k+1}$. Then for $n \geq 1$,

$$\mathrm{E}\big(X_n^{(\alpha)}\big) = \sum_{1\leq k\leq n} k^{\alpha}\mu_{n,k},$$

where

$$\mu_{n,k} = 2^{1-n}[z^n]\left(\frac{1-z}{1-2z} - \frac{1-z}{f_k(z)}\right) = 1 - 2^{1-n}[z^n]\frac{1-z}{f_k(z)}.$$

By Rouché's theorem, each $f_k(z)$ has a unique zero in the unit circle (by considering $|1-2e^{i\theta}| = 5 - 4\cos\theta$ and $|e^{ik\theta} - e^{i(k+1)\theta}| = 2 - 2\cos\theta$). Moreover, by positivity of coefficients, this root, denoted by $\frac{1}{2}(1+\varepsilon_k)$, is real positive. By definition the ε_ks satisfy

$$2^{-k-1}(1+\varepsilon_k)^k = \frac{\varepsilon_k}{1-\varepsilon_k}.$$

Thus, setting $y = 2^{-k-1}$, we obtain by the Lagrange inversion formula

$$\varepsilon_k = \sum_{j\geq 1} c_j(k)y^j = y + (k-1)y^2 + \frac{3k^2-7k+2}{2}y^3 + \cdots, \tag{10}$$

where

$$c_j(k) = \frac{(-1)^{j-1}}{j}\sum_{0\leq i<j}\binom{j}{i+1}(-1)^i\binom{kj}{i} \qquad (j\geq 1).$$

Now by the residue theorem,

$$\mu_{n,k} = 1 + 2\frac{1-\varepsilon_k}{f_k'\big(\frac{1}{2}(1+\varepsilon_k)\big)(1+\varepsilon_k)^{n+1}} + R_k,$$

where

$$|R_k| = 2^{1-n}\left|\frac{1}{2i\pi}\oint_{|z|=1} z^{-n-1}\,\frac{1-z}{f_k(z)}\,dz\right| \leq 2^{2-n},$$

since

$$\left|f_k(e^{i\theta})\right| \geq |1-2e^{i\theta}| - \left|e^{ik\theta} - e^{i(k+1)\theta}\right| \geq 1.$$

Thus we obtain

$$\begin{aligned}\mu_{n,k} &= 1 - \frac{(1-\varepsilon_k)^2}{\left(1-(k-1)\varepsilon_k + k\varepsilon_k^2\right)(1+\varepsilon_k)^n} + R_k\\ &= 1 - (1+\varepsilon_k)^{-n} + r_{n,k} + R_k,\end{aligned}$$

where $r_{n,k} \ll k\varepsilon_k(1+\varepsilon_k)^{-n} \ll k2^{-k}e^{-n/2^{k+3}}$. Consequently,

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \sum_{1\leq k\leq n} k^{\alpha}\left(1-(1+\varepsilon_k)^{-n}\right) + \Sigma_1 + \Sigma_2,$$

where

$$\begin{aligned}\Sigma_1 &:= \sum_{1\leq k\leq n} k^{\alpha} r_{n,k} \ll \sum_{k\geq 1} k^{\alpha+1}2^{-k}e^{-n/2^{k+3}}\\ &\ll \int_1^{\infty} x^{\alpha+1}2^{-x}e^{-n/2^{x+1}}\,dx \ll n^{-1}(\log n)^{\alpha+1}\end{aligned}$$

and

$$\Sigma_2 := \sum_{1\leq k\leq n} k^{\alpha}R_k \ll 2^{-n}\sum_{1\leq k\leq n} k^{\alpha} \ll n^{-1}(\log n)^{\alpha+1}.$$

In a similar way, we have

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \sum_{k\geq 1} k^{\alpha}\left(1-e^{-n/2^{k+1}}\right) + O\left(n^{-1}(\log n)^{\alpha+1}\right).$$

It remains to evaluate the harmonic sum (cf. [17])

$$S := \sum_{k\geq 1} k^{\alpha}\left(1-e^{-n/2^{k+1}}\right).$$

Consider first the case $\alpha < -1$. Taking[2]

$$K = \left[\log_2 \frac{n}{-2(\alpha+1)} - 2\log_2 \log\log \frac{n}{-2(\alpha+1)}\right],$$

we obtain easily

$$\begin{aligned} S - \zeta(-\alpha) &= -\left(\sum_{1\le k\le K} + \sum_{k>K}\right) k^{\alpha} e^{-n/2^{k+1}} \\ &\ll e^{-n/2^{K+1}} + K^{\alpha+1} \\ &\ll \exp\left(-\left(\log\log \frac{n}{-2(\alpha+1)}\right)^2\right) + (\log n)^{\alpha+1} \\ &\ll (\log n)^{\alpha+1}, \end{aligned}$$

thus proving (6).

We are left with the case $\alpha \ge -1$ for which we apply the Mellin inversion formula (cf. [17])

$$1 - e^{-w} = \frac{-1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(s) w^{-s}\, ds \qquad (\Re w > 0),$$

giving

$$S = \frac{-1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(-s) n^s U_\alpha(s)\, ds, \tag{11}$$

where

$$U_\alpha(s) = \sum_{k\ge 1} k^{\alpha} 2^{-(k+1)s} \qquad (\Re s > 0).$$

The function U_α is essentially a special case of the Lerch zeta function (cf. [13]). From known properties of the Lerch zeta function, it follows that, for a fixed $\alpha \in \mathbf{R}$, the function $\sum_{k\ge 1} k^{\alpha} z^k$ is analytic in the unit circle and analytically continuable to the whole z-plane with the exception of a branch cut from 1 to ∞. Thus our $U_\alpha(s)$ admits analytic continuation to the whole s-plane with branch cuts from χ_j to $-\infty + \chi_j$ for each $j \in \mathbf{Z}$. (For $\alpha \in \mathrm{N}$, the branch points χ_j reduce to poles of order $\alpha + 1$.) Thus we deform the integration path in (11) into the contour $\mathcal{C}$ shown in Fig. 1. In symbols, $\mathcal{C} = \bigcup_{j\in\mathbf{Z}} (\mathcal{C}_j \cup \mathcal{V}_j)$. Here $\mathcal{C}_j$ consists of two straight lines—one

[2] The Mellin transform approach applies equally well to this case, but with a slightly worse error term.

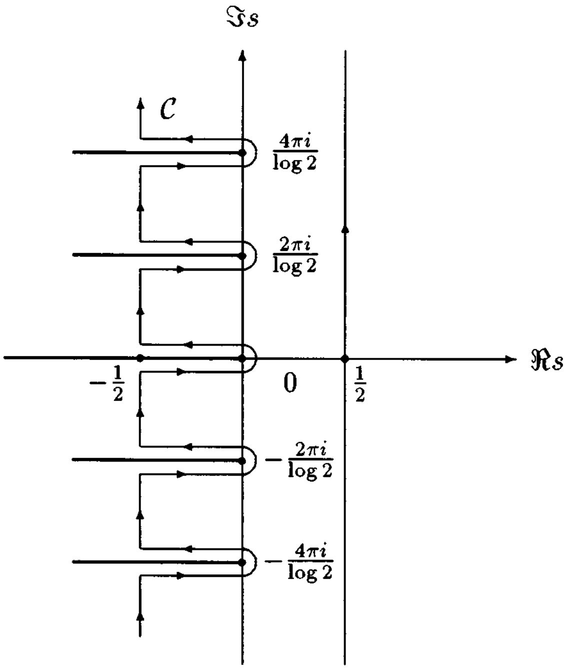

FIG. 1. The contour $\mathcal{C}$.

from $-1/2 + \chi_j - i/n$ to $\chi_j - i/n$ and the other from $\chi_j + i/n$ to $-1/2 + \chi_j + i/n$—and a semicircle $\{s: |s - \chi_j| = n^{-1}e^{i\theta}: -\pi/2 \le \theta \le \pi/2\}$ joins these two segments; $\mathcal{V}_j$ is the vertical segment joining $\mathcal{C}_{j-1}$ and $\mathcal{C}_j$.

From expansions for Lerch's zeta function, we easily deduce that (cf. [13])

$$U_\alpha(s) = \begin{cases} 2^{-s}\Gamma(\alpha+1)(s\log 2 - 2j\pi i)^{-\alpha-1} + 2^{-s}\zeta(-\alpha) + O\big(|s-\chi_j|\big), & \\ & \text{if } \alpha > -1, \\ -\log(s-\chi_j) - \log\log 2 + O\big(|s-\chi_j|\big), & \text{if } \alpha = -1, \end{cases} \tag{12}$$

for $s \sim 0$ in the cut plane and for $j \in \mathbf{Z}$.

We observe first that

$$\sum_{j\in\mathbf{Z}} \frac{1}{2\pi i}\int_{\mathcal{V}_j} \Gamma(-s) n^s U_\alpha(s)\, ds \ll n^{-(1/2)+\varepsilon},$$

since Γ is exponentially small at $\sigma \pm i\infty$;

$$\Gamma(\sigma + it) \ll |t|^{\sigma-1} e^{-\pi|t|/2} \qquad (|t| \to \infty), \tag{13}$$

for each fixed σ. Only the integrals over the curvilinear parts $\mathcal{C}_j$ are asymptotically significant.

Consider first the case $\alpha = -1$. For $j = 0$, by (12) and the expansion

$$\Gamma(-s) = -\frac{1}{s} + \gamma + O(|s|) \qquad (s \sim 0),$$

we have

$$\begin{aligned} S_0 &:= \frac{-1}{2\pi i}\int_{\mathcal{C}_0} \Gamma(-s) n^s U_\alpha(s)\, ds \\ &= \frac{-1}{2\pi i}\int_{\mathcal{C}_0} s^{-1} n^s (\log s + \log\log 2)\, ds + O\big((\log n)^{-1}\big) \\ &= \frac{-1}{2\pi i}\int_{-\infty}^{(0+)} s^{-1} n^s (\log s + \log\log 2)\, ds + O\big(n^{-(1/2)+\varepsilon} + (\log n)^{-1}\big), \end{aligned}$$

where the integration path $\int_{-\infty}^{(0+)}$ starts from $-\infty - i/n$, encircles the origin once counterclockwise, and then goes to $-\infty + i/n$. By the known integral representation (cf. [52, Sect. 12.22])

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^s s^{-z}\, ds = \frac{1}{\Gamma(z)} \qquad (z \in \mathbf{C}), \tag{14}$$

we easily obtain (7) [differentiating both sides of the representation with respect to z gives a logarithmic factor $\log(1/s)$].

For $j \in \mathbf{Z}\backslash\{0\}$, since $\Gamma(-s)$ is regular at $s = \chi_j$, we obtain

$$\begin{aligned} S_j &:= \frac{-1}{2\pi i}\int_{\mathcal{C}_0} \Gamma(-s-\chi_j) n^{s+\chi_j}\left(\log\left(\frac{1}{s}\right) + O(1)\right) ds \\ &= \frac{-n^{-\chi_j}}{2\pi i}\int_{\mathcal{C}_0'} \Gamma\left(-\frac{w}{\log n} - \chi_j\right) e^w (\log n - \log w + O(1))\, dw \\ &=: J_1 + J_2 + J_3, \end{aligned}$$

say, where $\mathcal{C}_0'$ is the transformed contour of $\mathcal{C}_0$ under the change of variables $w = s\log n$. For J_1, the integrand being analytic in the contour of integration, we have

$$J_1 \ll n^{-1/2}\big|\Gamma(-\chi_j)\big|;$$

the other two integrals are bounded above by

$$J_2 + J_3 \ll \frac{1}{\log n} \left| \Gamma(-\chi_j) \right|.$$

Therefore, we have, by (13),

$$\begin{aligned}
\mathrm{E}\left(X_n^{(\alpha)} \right) &= \sum_{j \in \mathbf{Z}} S_j + O(n^{-(1/2)+\varepsilon}) \\
&= \log \log n - \log \log 2 \\
&\quad + O\left(\frac{1}{\log n} \left(1 + \sum_{j \in \mathbf{Z} \setminus \{0\}} \left| \Gamma(-\chi_j) \psi(-\chi_j) \right| \right) \right),
\end{aligned}$$

which is (7).

The case $\alpha > -1$ is similar. We have for $j = 0$,

$$\begin{aligned}
S_0 = {} & \frac{\Gamma(\alpha+1)}{2\pi i} (\log 2)^{-\alpha-1} \int_{\mathcal{C}_0} \frac{\Gamma(1-s)}{s} 2^{-s} n^s s^{-\alpha-1}\, ds \\
& - \frac{\zeta(-\alpha)}{2\pi i} \int_{\mathcal{C}_0} \Gamma(-s) 2^{-s} n^s \, ds + O\left((\log n)^{-1} \right).
\end{aligned}$$

By the expansion $\Gamma(1-s)2^{-s} = \sum_{l \geq 0} e_l s^l$ and the integral representation (14),

$$\begin{aligned}
S_0 = {} & \left(\frac{\log n}{\log 2} \right)^{\alpha+1} \sum_{0 \leq l \leq [\alpha]+2} \frac{e_l}{(\log n)^l} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2-l)} \\
& + \zeta(-\alpha) + O\left((\log n)^{\{\alpha\}-2} \right).
\end{aligned}$$

For $j \in \mathbf{Z} \setminus \{0\}$,

$$\begin{aligned}
S_j = {} & -\frac{\Gamma(\alpha+1)}{2\pi i} (\log 2)^{-\alpha-1} \int_{\mathcal{C}_0} \Gamma(-s-\chi_j) s^{-\alpha-1} 2^{-s} n^{s+\chi_j} \, ds \\
& + O\left(\frac{\left| \Gamma(-\chi_j) \right|}{\log n} \right) \\
= {} & -\frac{(\log n)^{\alpha}}{(\log 2)^{\alpha+1}} \Gamma(-\chi_j) n^{\chi_j} \\
& \times \left(\sum_{0 \leq l \leq [\alpha]+1} \frac{e_l'}{(\log n)^l} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-l)} + O\left((\log n)^{-[\alpha]-2} \right) \right) \\
& + O\left(\frac{\left| \Gamma(-\chi_j) \right|}{\log n} \left(\log |\chi_j| \right)^{[\alpha]+2} \right),
\end{aligned}$$

from which (8) follows. This completes the proof of Theorem 2. ∎

Direct expansion of F gives for each $n \geq 1$,

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \sum_{1 \leq k \leq n} \sum_{0 \leq j \leq n-k} \sum_{d|k} (-1)^{j+d-1} \left(\frac{k}{d}\right)^{\alpha} \times \binom{d+n-k-j}{d} \binom{d+1}{j} 2^{-k-j-1}, \tag{15}$$

which is useful only for small values of n.

3.3. *A General Scheme*

We now consider general Λ, excluding the cases of finite Λ, since otherwise the problem is easy. We need a definition of Bateman and Erdős.

DEFINITION (Property P_k). A sequence $A = \{a_n\}_{n \geq 1}$ of positive integers is said to have *property* P_k if A has more than k elements and if we remove an arbitrary subset of k elements from A, the remaining elements have greatest common divisor unity.

Recall that

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \sum_{\lambda_j \leq n} \lambda_j^{\alpha} \left(1 - \frac{[z^n]\left(1 - \Lambda(z) + z^{\lambda_j}\right)^{-1}}{[z^n]\left(1 - \Lambda(z)\right)^{-1}}\right).$$

where $\alpha \in \mathbf{R}$. Instead of studying the asymptotics of this special generating function, we consider, motivated by branching and cyclic compositions (see Section 3.5), the more general form

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \sum_{\lambda_j \leq n} \lambda_j^{\alpha} \left(1 - \frac{[z^n] g\left(\Lambda(z) - z^{\lambda_j}\right)}{[z^n] g\left(\Lambda(z)\right)}\right),$$

where

$$g(z) = \left(\frac{1}{1-z}\right)^{\varpi} \left(\log \frac{1}{1-z}\right)^{\gamma},$$

with $\varpi\gamma \geq 0$ and $\varpi + \gamma > 0$.

We show that there is a general phenomenon of ‘‘logarithmic transfer’’ from the asymptotic behavior of $\sum_{\lambda_j \leq x} \lambda_j^{\alpha}$ to that of $\mathrm{E}(X_n^{(\alpha)})$.

THEOREM 3. *If* Λ *has property* P_1, *then*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) \sim \sum_{\lambda_j \leq L(n)} \lambda_j^{\alpha}, \tag{16}$$

where

$$L(n) = \frac{1}{\log(1/\rho)} \log \frac{n}{\rho\Lambda'(\rho)},$$

with $0 < \rho < 1$ *satisfying* $\Lambda(\rho) = 1$.

COROLLARY 2. *If* $\Lambda = \{1^h, 2^h, 3^h, \dots\}$, *where* $h \geq 2$ *is an integer, then*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) \sim \begin{cases} \zeta(-h\alpha), & \text{if } \alpha < -1/h, \\ \dfrac{1}{h} \log\log n, & \text{if } \alpha = -1/h, \\ \dfrac{1}{h\alpha + 1}\left(\dfrac{\log n}{\log(1/\rho)}\right)^{\alpha + 1/h}, & \text{if } \alpha > -1/h, \end{cases}$$

where $0 < \rho < 1$ *satisfies* $\sum_{j \geq 1} \rho^{j^h} = 1$.

COROLLARY 3. *Assume that* Λ *has the property* P_1 *and*

$$\sum_{\lambda_j \leq x} 1 \sim Kx^{\varpi}(\log x)^{\gamma} \qquad (x \to \infty).$$

Then

$$\mathrm{E}\left(X_n^{(0)}\right) \sim K\left(\frac{\log n}{\log(1/\rho)}\right)^{\varpi} (\log\log n)^{\gamma}.$$

The logarithmic transfer in the last formula is transparent.

In general, it is difficult to obtain more precise expansions than (16).

We first develop some lemmas and then prove the theorem.

3.3.1. *Lemmas*

Recall that $\rho \in (0,1)$ satisfy $\Lambda(\rho) = 1$. Let $\rho_j \in (0,1)$ satisfy $1 = \Lambda(\rho_j) - \rho_j^{\lambda_j}$. Then ρ_j tends to ρ exponentially fast.

LEMMA 1. *Let* $y = \rho^{\lambda_j - 1}/\Lambda'(\rho)$ *and write* $\rho_j = \rho(1 + \varepsilon_j)$. *Then*

$$\varepsilon_j = \sum_{h \geq 1} c_h(j) y^h,$$

where

$$c_h(j) = \frac{1}{h}[t^{h-1}]\left(\frac{(1+t)^{\lambda_j}}{1 + E(t)}\right)^h$$

$$\text{with} \quad E(t) = \frac{\Lambda(\rho(1+t)) - \Lambda(\rho)}{\rho\Lambda'(\rho)t} - 1.$$

Proof. The proof is a direct application of the Lagrange inversion formula. ∎

In particular, we have

$$\rho_j = \rho + \frac{\rho^{\lambda_j}}{\Lambda'(\rho)} + \left(\lambda_j - \frac{\rho\Lambda''(\rho)}{2\Lambda'(\rho)}\right)\frac{\rho^{2\lambda_j - 1}}{\Lambda'(\rho)^2} + \cdots . \tag{17}$$

Consider now the equation

$$\Lambda(z) = 1.$$

By our greatest common denominator (gcd) assumption, there is a $\delta > 0$ such that the preceding equation has only one solution ρ for z in $|z| \le \rho + \delta$. Similarly, by property P_1 there exists a $\delta_j > 0$ such that the equation $1 = \Lambda(z) - z^{\lambda_j}$ is uniquely solvable in $|z| \le \rho_j + \delta_j$ for each fixed j, say, $j \le j_0$, j_0 being sufficiently large but fixed. For $j \ge j_0$, by property P_1 and Lemma 1, we can find a sufficiently small but fixed $\delta' > 0$ such that each of the equations $1 = \Lambda(z) - z^{\lambda_j}$ has a unique zero in $|z| \le \rho + \delta'$. Since the proofs of these assertions follow standard lines and are not interesting, they are not given here. The uniformity will be needed subsequently.

LEMMA 2. *For $\rho^{\lambda_j} n \gg 1$, we have the uniform estimate*

$$\frac{[z^n]g(\Lambda(z) - z^{\lambda_j})}{[z^n]g(\Lambda(z))} \ll \exp\left(-\frac{\rho^{\lambda_j} n}{\rho\Lambda'(\rho)}\right). \tag{18}$$

Proof. For convenience, let $\lambda_0 = 0$ and $\rho_0 = \rho$. By the local expansion of $(1 - \Lambda(z) + z^{\lambda_j})^{-1}$ around the dominant singularity $z = \rho_j$,

$$\frac{1}{1 - \Lambda(z) + z^{\lambda_j}} = \frac{1}{\rho_j \Lambda'(\rho_j) - \lambda_j \rho_j^{\lambda_j}} \frac{1}{1 - z/\rho_j}\left(1 + O\left(\left|1 - \frac{z}{\rho_j}\right|\right)\right),$$

for $z \sim \rho_j$, we obtain for $j \ge 0$,

$$g(\Lambda(z) - z^{\lambda_j}) = \left(\frac{1}{\rho_j \Lambda'(\rho_j) - \lambda_j \rho_j^{\lambda_j}}\right)^{\varpi}\left(\frac{1}{1 - z/\rho_j}\right)^{\varpi}\left(\log\frac{1}{1 - z/\rho_j}\right)^{\gamma}$$
$$\times\left(1 + O\left(\frac{1}{|\log(1 - z/\rho_j)|}\right)\right).$$

By the foregoing remarks, we can apply the singularity analysis of Flajolet and Odlyzko [19],

$$[z^n]g(\Lambda(z) - z^{\lambda_j})$$
$$= \begin{cases} \rho_j^{-n}\left(\dfrac{1}{\rho_j \Lambda'(\rho_j) - \lambda_j \rho_j^{\lambda_j}}\right)^{\varpi} \dfrac{n^{\varpi - 1}}{\Gamma(\varpi)} (\log n)^{\gamma}\Big(1 + O\big((\log n)^{-1}\big)\Big), & \\ & \text{if } \varpi > 0, \\ \rho_j^{-n} \dfrac{\gamma}{n} (\log n)^{\gamma - 1}\Big(1 + O\big((\log n)^{-1}\big)\Big), & \text{if } \varpi = 0, \end{cases}$$

the O-term being uniform for $j \geq 1$. Consequently,

$$\frac{[z^n]g(\Lambda(z) - z^{\lambda_j})}{[z^n]g(\Lambda(z))} \ll \left(\frac{\rho_j}{\rho}\right)^{-n},$$

in both cases. Thus (18) follows from (17). ∎

The following inequality is useful in majorizing the sums for large λ_j.

LEMMA 3. *If $f(z) = \sum_{n \geq 0} a_n z^n$ is a formal power series with $a_n \geq 0$, then for each positive integer l,*

$$[z^n] \sum_{1 \leq h \leq 2l} (-1)^{h-1} z^{h\lambda_j} \frac{f^{(h)}(\Lambda(z))}{h!}$$
$$\leq [z^n]\big(f(\Lambda(z)) - f(\Lambda(z) - z^{\lambda_j})\big)$$
$$\leq [z^n] \sum_{1 \leq h \leq 2l-1} (-1)^{h-1} z^{h\lambda_j} \frac{f^{(h)}(\Lambda(z))}{h!}. \tag{19}$$

Proof. Since $\lambda_1 \geq 1$, we have

$$[z^n]\big(f(\Lambda(z)) - f(\Lambda(z) - z^{\lambda_j})\big)$$
$$= \sum_{1 \leq m \leq n} a_m [z^n]\Big(\Lambda^m(z) - \big(\Lambda(z) - z^{\lambda_j}\big)^m\Big).$$

The required inequality follows from a simple sieve argument. ∎

LEMMA 4. *For $\rho^{\lambda_j} n \ll 1$, we have*

$$1 - \frac{[z^n]g(\Lambda(z) - z^{\lambda_j})}{[z^n]g(\Lambda(z))} \ll \rho^{\lambda_j} n, \tag{20}$$

uniformly in j.

Proof. Applying the second inequality in (19) with $l = 1$, we obtain

$$[z^n]\big(g(\Lambda(z)) - g(\Lambda(z) - z^{\lambda_j})\big) \le [z^n] z^{\lambda_j} g'(\Lambda(z)),$$

where

$$g'(z) = \left(\frac{1}{1-z}\right)^{\varpi+1}\left(\log\frac{1}{1-z}\right)^{\gamma-1}\left(\varpi \log\frac{1}{1-z} + \gamma\right).$$

The estimate (20) follows then from applying the singularity analysis. ∎

Proof of Theorem 3. *For convenience, set* $\eta = 1/(\rho\Lambda'(\rho))$. *We divide the estimation of* $\mathrm{E}(X_n^{(\alpha)})$ *into three parts*:

$$\mathrm{E}\big(X_n^{(\alpha)}\big) = \xi_n^{(1)} + \xi_n^{(2)} + \xi_n^{(3)},$$

where [*recall that* $L(n) = \log_{1/\rho}(\eta n)$]

$$\xi_n^{(1)} := \sum_{\lambda_j \le L(n)} \lambda_j^{\alpha},$$

and by (18) *and* (20),

$$\xi_n^{(2)} := -\sum_{\lambda_j \le L(n)} \lambda_j^{\alpha} \frac{[z^n]g\big(\Lambda(z) - z^{\lambda_j}\big)}{[z^n]g(\Lambda(z))} \ll \sum_{\lambda_j \le L(n)} \lambda_j^{\alpha} e^{-\eta n \rho^{\lambda_j}},$$

$$\xi_n^{(3)} := \sum_{L(n) < \lambda_j \le n} \lambda_j^{\alpha}\left(1 - \frac{[z^n]g\big(\Lambda(z) - z^{\lambda_j}\big)}{[z^n]g(\Lambda(z))}\right) \ll \sum_{L(n) < \lambda_j \le n} \lambda_j^{\alpha} n \rho^{\lambda_j}.$$

Take $L_1(n) = L(n) - \log_{1/\rho}\omega_n$, *where* ω_n *is any sequence tending to infinity arbitrarily slowly*. *We have*

$$\begin{aligned}\xi_n^{(2)} &\ll e^{-\omega_n} \sum_{\lambda_j \le L_1(n)} \lambda_j^{\alpha} + \sum_{L_1(n) < \lambda_j \le L(n)} \lambda_j^{\alpha}\\ &= o\big(\xi_n^{(1)}\big).\end{aligned}$$

For $\xi_n^{(3)}$, *take* $L_2(n) = L(n) + \log_{1/\rho}\omega_n$. *Then*

$$\xi_n^{(3)} \ll \left(\sum_{L(n) < \lambda_j \le L_2(n)} + \sum_{L_2(n) < \lambda_j \le n}\right) \lambda_j^{\alpha} n \rho^{\lambda_j}.$$

The first sum on the right-hand side is $o(\xi_n^{(1)})$. *For the second sum, let* $A(x) = \sum_{\lambda_j \le x} 1$. *We have*

$$\sum_{L_2(n) < \lambda_j \le n} \lambda_j^{\alpha} n \rho^{\lambda_j} \ll \frac{1}{\omega_n} A\big(L_2(n)\big) L_2(n)^{\alpha} = o\big(\xi_n^{(1)}\big)$$

since

$$\xi_n^{(1)} \asymp 1 + L(n)^{\alpha} A(L(n)).$$

This completes the proof of Theorem 3. ∎

3.4. *Compositions with Small Number of Distinct Summands*

The foregoing approach may be extended to calculate the higher moments of, say, the number of distinct parts in a random Λ-composition of n. The calculation, however, becomes very complicated even for the simplest case $\Lambda = \mathbf{Z}^+$. Instead of pursuing this line further, we consider in this section the distribution of compositions with small number of distinct summands. The general intuition behind this problem is that only small parts give the principal contribution.

More precisely, let $\Lambda = \{\lambda_j\}_{j \geq 1}$ be a sequence of positive integers and let X_n denote the number of distinct summands in a random composition of n into parts λ_j, each being assigned the same probability. Then

$$\begin{aligned} \mathrm{P}(X_n = m) &= \frac{1}{[z^n](1 - \Lambda(z))^{-1}} [z^n u^m] \int_0^{\infty} e^{-t} \prod_{j \geq 1} \left(1 + u(e^{tz^{\lambda_j}} - 1)\right) dt \\ &= \frac{1}{[z^n](1 - \Lambda(z))^{-1}} \sum \frac{(s_1 + s_2 \cdots + s_m)!}{s_1! s_2! \cdots s_m!}, \end{aligned}$$

where the sum is extended over all partitions of the type (5). We shall show that if $\lambda_m \leq \mu_0 \log n$, for some $\mu_0 > 0$, then

$$\mathrm{P}(X_n = m) \sim \frac{1}{[z^n](1 - \Lambda(z))^{-1}} [z^n] \frac{1}{1 - z^{\lambda_1} - \cdots - z^{\lambda_m}},$$

namely, only the contribution of the first m numbers in Λ is asymptotically significant. Obviously, for $m = 1$, we have

$$\mathrm{P}(X_n = 1) = \frac{1}{[z^n](1 - \Lambda(z))^{-1}} [z^n] \sum_{j \geq 1} \frac{z^{\lambda_j}}{1 - z^{\lambda_j}}.$$

A precise statement of the result follows.

THEOREM 4. *We have*

$$\mathrm{P}(X_n = m) \sim \frac{1}{[z^n](1 - \Lambda(z))^{-1}} [z^n] \frac{1}{1 - z^{\lambda_1} - \cdots - z^{\lambda_m}}$$

uniformly for $m \geq 2$ *and*

$$\lambda_{m+1} \leq \frac{1-\varepsilon}{\log(1/\rho)} \log \frac{n}{\log n}, \tag{21}$$

with $0 < \rho < 1$ *satisfying* $\Lambda(\rho) = 1$.

Let ϱ_m denote the solution to the equation

$$1 - z^{\lambda_1} - \cdots - z^{\lambda_m}$$

in the unit interval.

COROLLARY 4. *If the greatest common divisor of* $\{\lambda_1, \ldots, \lambda_k\}$ *is* 1 *for* $k = k_0 \geq 2$, *then*

$$\mathrm{P}(X_n = m) \sim \frac{\rho^{-n-1}\varrho_m^{-n}}{\Lambda'(\rho)\left(\lambda_1 \varrho_m^{\lambda_1} + \cdots + \lambda_m \varrho_m^{\lambda_m}\right)},$$

uniformly for $m \geq k_0$ *in the range* (21).

Proof of Theorem 4. *By definition*

$$\begin{aligned}
&[u^m]\int_0^\infty e^{-t} \prod_{j\geq 1} \left(1 + u(e^{tz^{\lambda_j}} - 1)\right) dt \\
&\quad= \sum_{1\leq j_1<j_2<\cdots<j_m} \int_0^\infty e^{-t}(e^{tz^{\lambda_{j1}}} - 1)\cdots(e^{tz^{\lambda_{jm}}} - 1)\, dt \\
&\quad= \sum_{1\leq j_1<j_2<\cdots<j_m} \left(\frac{1}{1 - z^{\lambda_{j_1}} - \cdots - z^{\lambda_{j_m}}} \right. \\
&\qquad\qquad - \sum_{1\leq i\leq m} \frac{1}{1 - z^{\lambda_{j_1}} - \cdots - z^{\lambda_{j_m}} + z^{\lambda_{j_i}}} \\
&\qquad\qquad \left. + \sum_{\substack{1\leq i,h\leq m\\ i\neq h}} \frac{1}{1 - z^{\lambda_{j_1}} - \cdots - z^{\lambda_{j_m}} + z^{\lambda_{j_i}} + z^{\lambda_{j_h}}} - + \cdots \right).
\end{aligned}$$

Since the nearest pole(s) *of these fractions to the origin is on the circle* $|z| = \varrho_m$, *we deduce that for* $m \geq 2$,

$$\begin{aligned}
&[z^n u^m]\int_0^\infty e^{-t} \prod_{j\geq 1} \left(1 + u(e^{tz^{\lambda_j}} - 1)\right) dt \\
&\quad= [z^n]\frac{1}{1 - z^{\lambda_1} - \cdots - z^{\lambda_m}} + O(n^m \zeta_m^{-n}),
\end{aligned}$$

where $\zeta_m \in (0,1)$ *solves the equation*

$$1 - z^{\lambda_1} - \cdots - z^{\lambda_{m-1}} - z^{\lambda_{m+1}} = 0.$$

Obviously,

$$[z^n] \frac{1}{1 - z^{\lambda_1} - \cdots - z^{\lambda_m}} \ll \varrho_m^{-n}.$$

On the other hand, it is easily seen that

$$\frac{\zeta_m}{\varrho_m} \ll \exp\left(\frac{\varrho_m^{\lambda_{m+1}}}{\sum_{1 \le j < m} \lambda_j \varrho_m^{\lambda_j}} \right).$$

Consequently,

$$\left(\frac{\zeta_m}{\varrho_m} \right)^{-n} \le \exp\left(- \frac{n \rho^{\lambda_{m+1}}}{\rho \Lambda'(\rho)} \right).$$

It follows that for m satisfying (21),

$$n^m \zeta_m^{-n} = o(\varrho_m^{-n}).$$

This completes the proof. ▮

COROLLARY 5. *Let* $\Lambda = \mathbf{Z}^+$. *Then for*

$$2 \le m \le \frac{1 - \varepsilon}{\log 2} \log \frac{n}{\log n}$$

we have

$$\mathrm{P}(X_n = m) \sim 2^{-n-1} \frac{1 - \varepsilon_m}{1 - m\varepsilon_m} (1 + \varepsilon_m)^{-n},$$

where

$$\varepsilon_m = \sum_{j \ge 1} \frac{1}{j} \binom{(m+1)j}{j-1} 2^{-(m+1)j}.$$

Obviously, for $m = 1$ we have

$$\mathrm{P}(X_n = 1) = d(n) 2^{1-n} \qquad (n \ge 2),$$

where $d(n)$ denotes the number of divisors of n.

3.5. *Other Compositions*

In this section, we consider cyclic and branching compositions.

3.5.1. *Cyclic Compositions*

A composition of n into parts λ_j is an ordered sequence $(\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_m})$ such that $\sum_{1 \le k \le m} \lambda_{j_k} = n$. Two such compositions are equivalent if one can be obtained from the other by circular rotations of its elements. In symbols, $\pi = \pi'$ if

$$\pi = \left(\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_m}\right) \quad \text{and} \quad \pi' = \left(\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_m}\right),$$

and there exists a k, $1 \le k \le m$, such that

$$\lambda_{j_1} = \lambda_{i_k}, \lambda_{j_2} = \lambda_{i_{k+1}}, \dots, \lambda_{j_{m-k+1}} = \lambda_{i_m}, \lambda_{j_{m-k+2}} = \lambda_{i_1}, \dots, \lambda_{m_m} = \lambda_{i_{k-1}}.$$

Let $\mu(n)$ be the Möbius function, $u_0 = 1$, and

$$S(z; u_0, u_1, u_2, \dots) = \sum_{n \ge 1} z^n \sum_{\substack{r_1\lambda_1 + r_2\lambda_2 + \cdots = n \\ r_l \ge 0}} \prod_{j \ge 1} u_{r_j}^{w(\lambda_j)},$$

where the inner summation runs over all *cyclic compositions* of n into parts λ_j. Obviously,

$$S(z) := S(z; 1, 1, \dots) = \sum_{j \ge 1} \frac{\phi(j)}{j} \log \frac{1}{1 - \Lambda(z^j)},$$

where $\phi(k)$ is Euler's totient function.

THEOREM 5. *The generating function S satisfies*

$$S(z; u_0, u_1, u_2, \dots) = \sum_{m, k \ge 1} \mu(k) \int_0^1 \frac{1}{y} \int_0^\infty e^{-t} \times \left(\prod_{j \ge 1} \left(1 + \sum_{l \ge 1} \frac{u_l^{w(\lambda_j)} y^{kl} t^l z^{mkl\lambda_j}}{l!} \right) - 1 \right) dt\, dy.$$

Proof (Sketch). We use the approach of Flajolet and Soria [20] together with the proof method of Theorem 1. ∎

In particular, the number of distinct summands (marked by u) in cyclic compositions is given by

$$\sum_{m,k\geq 1}\mu(k)\int_0^1\frac{1}{y}\int_0^{\infty}e^{-t}\left(\prod_{j\geq 1}\left(1+u\left(e^{y^k t z^{mk\lambda_j}}-1\right)\right)-1\right)dt\,dy.$$

Let $s(n)$ denote the number of cyclic compositions of n;

$$s(n)=[z^n]\sum_{j\geq 1}\frac{\phi(j)}{j}\log\frac{1}{1-\Lambda(z^j)},$$

and let X_n denote the number of distinct summands in a random cyclic composition of n, where each cyclic composition of n is equally likely. We have, by the foregoing generating function or by the argument of Section 2,

$$\sum_{n\geq 1}s(n)\mathrm{E}\left(X_n^{(\alpha)}\right)z^n$$
$$=\sum_{k\geq 1}\frac{\phi(k)}{k}\sum_{j\geq 1}\left(\log\frac{1}{1-\Lambda(z^k)}-\log\frac{1}{1-\Lambda(z^k)+z^{k\lambda_j}}\right),$$

from which asymptotic estimates can be derived as in Section 3.3.

3.5.2. *Branching Compositions*

Arranging the parts of a partition of n in the leaves of, say, a binary tree, we obtain a *binary branching composition* of n (cf. [27, Chap. 8]). For example, when $\Lambda=\mathbf{Z}^+$ and $n=3$, we have the five different compositions shown in Fig. 2.

Let $u_0=1$ and

$$B(z;u_0,u_1,u_2,\ldots)=\sum_{n\geq 1}z^n\sum_{\substack{r_1\lambda_1+r_2\lambda_2+\cdots=n\\ r_l\geq 0}}\prod_{j\geq 1}u_{r_j}^{w(\lambda_j)},$$

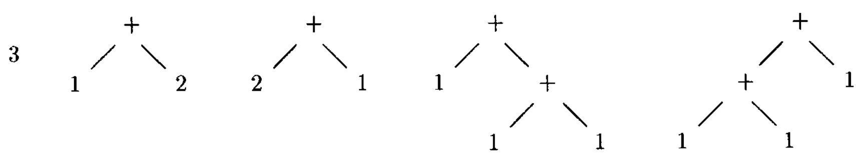

FIG. 2. Five branching compositions of 3 ($\Lambda=\mathbf{Z}^+$).

where the inner summation runs over all *binary branching compositions* of n into parts λ_j. Clearly,

$$B(z) := B(z;1,1,\dots) = \frac{1-\sqrt{1-4\Lambda(z)}}{2}.$$

THEOREM 6. *The generating function B satisfies*

$$B(z;u_0,u_1,u_2,\dots) = \frac{1}{2\pi}\int_0^1 y^{-3/2}\sqrt{1-y}\int_0^\infty e^{-t} \times\left(\prod_{j\geq 1}\left(1+\sum_{l\geq 1}\frac{u_l^{w(\lambda_j)}4^l y^l t^l z^{l\lambda_j}}{l!}\right)-1\right)dt\,dy.$$

Proof. The five compositions in Fig. 1 can be represented in an alternative way as

$$3 \quad (12) \quad (21) \quad (1(11)) \quad ((11)1),$$

by writing each

$$\begin{array}{ccc} & + & \\ / & & \backslash \\ A & & B \end{array}$$

as $(A \quad B)$. Now if π is a composition of n with m parts $(\lambda_{j_1},\dots,\lambda_{j_m})$, there are C_{m-1} different ways of bracketing the parts into branching compositions, where C_n is the Catalan numbers. The required result follows from the integral representation

$$C_n = [z^n]\frac{1-\sqrt{1-4z}}{2z} = \frac{4^{n+1}}{2\pi}\int_0^1 y^{n-1/2}\sqrt{1-y}\,dy \qquad (n\geq 0)$$

(by Cauchy's integral formula with suitable deformation of the integration contour). ∎

Thus we have for the number of distinct summands X_n,

$$\mathrm{E}(X_n) = \sum_{\lambda_j\leq n}\left(1-\frac{[z^n]\left(1-\sqrt{1-4\left(\Lambda(z)-z^{\lambda_j}\right)}\right)/2}{[z^n]\left(1-\sqrt{1-4\Lambda(z)}\right)/2}\right).$$

Asymptotic results can be obtained as in Section 3.3.

Compositions based on other types of trees and labelings as in [27, Chap. 8] can be similarly studied.

4. PARTITIONS

As for compositions, we first derive precise asymptotic results for our general measure of distinctness in the special case $\lambda = \mathbf{Z}^+$ and then derive weaker results in the general case.

4.1. *Asymptotics of the Mean Measure*

Instead of the logarithmic transfer for compositions, we have the following "polynomial transfer" for partitions. Furthermore, the fluctuating behavior disappears.

THEOREM 7. *We have*

(i) *for* $\alpha < -1$,

$$E\left(X_n^{(\alpha)}\right) = \zeta(-\alpha) + \begin{cases} O\left(n^{\max\{\alpha+1,-1\}/2}\right), & \text{if } \alpha \neq -2, \\ O\left(n^{-1/2}\log n\right), & \text{if } \alpha = -2; \end{cases} \tag{22}$$

(ii) *for* $\alpha = -1$,

$$\mathrm{E}\left(X_n^{(\alpha)}\right) = \frac{1}{2}\log n + \frac{1}{2}\log\frac{6}{\pi^2} + O\left(n^{-1/2}\right); \tag{23}$$

(iii) *for* $\alpha > -1$,

$$\begin{aligned} \mathrm{E}\left(X_n^{(\alpha)}\right) &= \Gamma(\alpha+1)\kappa^{-(\alpha+1)/2}\left(n - \frac{1}{24}\right)^{(\alpha+1)/2} \\ &\quad \times \left(1 + \sum_{1 \le h \le [\alpha]+2} \frac{b_h}{\left(\kappa(n-1/24)\right)^{h/2}}\right) \\ &\quad + \zeta(-\alpha) + O\left(n^{-1/2}\right), \end{aligned} \tag{24}$$

where $\kappa = \pi^2/6$ *and*

$$b_h = \sum_{0 \le j \le h} \frac{(-1)^j}{2^{h+3j} j!} \prod_{1 \le l \le j} \left(4\left(\alpha - \frac{1}{2}\right)^2 - (2l-1)^2\right).$$

One may further expand the factor $n - 1/24$ so that the expansion is in descending powers of $n^{1/2}$. Expressions for the coefficients become, however, more involved.

COROLLARY 6. *The expected number and sum of distinct parts in a random partition of n satisfy*

$$\mathrm{E}\big(X_n^{(0)}\big) = \frac{\sqrt{6n}}{\pi} + \frac{3}{\pi^2} + O(n^{-1/2}),$$

$$\mathrm{E}\big(X_n^{(1)}\big) = \frac{6n}{\pi^2} + \frac{3\sqrt{6n}}{\pi^3} - \frac{1}{4\pi^2} + \frac{9}{\pi^4} + O(n^{-1/2}).$$

An exact asymptotic formula for $\sum_{0\le j\le n} p(n-j)$ of Hardy–Ramanujan–Rademacher type can be obtained by applying a result of Almkvist [2].

Proof of Theorem 7 (Sketch). Let $p(n)$ denote the number of partitions of n. Then

$$F(z) := 1 + \sum_{n\ge 1} p(n) z^n = \prod_{j\ge 1} \frac{1}{1-z^j},$$

for $|z| < 1$. Set $E_\alpha(z) = \sum_{j\ge 1} j^\alpha z^j$. Then

$$\mathrm{E}\big(X_n^{(\alpha)}\big) = \frac{1}{p(n)} \sum_{1\le j\le n} j^\alpha p(n-j) = \frac{1}{p(n)}[z^n]E_\alpha(z)F(z).$$

Write $z = e^{-\tau}$ with $\tau = r + iy$, where $r > 0$ and $-\pi \le y \le \pi$. By Mellin transform and the functional equation of Riemann's zeta function (cf. [3, 17]),

$$F(e^{-\tau}) = \frac{\tau^{1/2}}{\sqrt{2\pi}} \exp\left(\frac{\pi^2}{6\tau} - \frac{\tau}{24}\right)\big(1 + O(e^{-4\pi^2/\tau})\big), \tag{25}$$

$$E_\alpha(e^{-\tau}) = \begin{cases} \Gamma(\alpha+1)\tau^{-\alpha-1} + \zeta(-\alpha) + O(|\tau|), & \text{if } \alpha > -1, \\ -\log \tau + O(|\tau|), & \text{if } \alpha = -1, \\ \zeta(-\alpha) + (|\tau|^{\alpha+1}), & \text{if } -2 < \alpha < -1, \\ \zeta(-\alpha) + (|\tau| \log |\tau|), & \text{if } \alpha = -2, \\ \zeta(-\alpha) + (|\tau|), & \text{if } \alpha < -2, \end{cases}$$

as $\tau \to 0$ and $|y| \le r$. On the other hand, we have, for $r \le |y| \le \pi$ and $0 < r < 1$,

$$\begin{aligned} \frac{|F(e^{-r-iy})|}{F(e^{-r})} &\le \exp\left(-\frac{1}{e^r - 1}\left(1 - \frac{e^r - 1}{|e^{r+iy} - 1|}\right)\right) \\ &\le \exp\left(-\frac{1}{er}\left(1 - \frac{1}{\sqrt{1 + 4e/\big(\pi^2(e-1)^2\big)}}\right)\right) \\ &< e^{-1/(20r)}. \end{aligned}$$

Take $r = \pi/\sqrt{6(n-1/24)}$ [approximate saddle point of $e^{nr}F(e^{-r})$]. From Cauchy's integral formula using the preceding estimate, it follows that

$$[z^n]E_\alpha(z)F(z) = \frac{1}{2\pi}\int_{-r}^{r} e^{nr+iny}E_\alpha(e^{-r-iy})F(e^{-r-iy})\,dy$$
$$+ O\left(e^{nr+\pi^2/6r-3\sqrt{n}/10\pi}\right).$$

Substituting (25) into the integral and suitably deforming the integration path (cf. [3, 7]), we can express the integral on the right-hand side in terms of the modified Bessel functions $I_\nu(z)$. The results of the theorem follow from the asymptotic expansion of $I_\nu(z)$ (cf. [52, p. 373]),

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 + \sum_{j\geq 1}\frac{(-1)^j}{j!8^j z^j}\prod_{1\leq l\leq j}\left(4\nu^2-(2l-1)^2\right)\right),$$

as $|z| \to \infty$ in the sector $|\arg z| < \pi/2$. ∎

4.2. *A General Scheme*

As in Section 3.3 for compositions we consider the asymptotics of $\mathrm{E}(X_n^{(\alpha)})$ for general Λ-partitions. Instead of property P_1, we need property P_0 in this section. Our investigation rely closely on a scheme of Richmond [41].

Let $\Lambda = \{\lambda_j\}_{j\geq 1}$ be an increasing sequence of positive integers satisfying

1. $\limsup_{n\to\infty} \log\log\lambda_n/(\log n) < \infty$;
2. $\sum_{\lambda_j\leq 2x} 1 \ll \sum_{\lambda_j \leq x} 1$; and
3. the sequence Λ has property P_0.

THEOREM 8. *Under the preceding assumptions, we have*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) \sim \sum_{j\geq 1}\lambda_j^\alpha e^{-\nu\lambda_j}, \tag{26}$$

where $\nu > 0$ *satisfies*

$$\sum_{j\geq 1}\frac{\lambda_j}{e^{\nu\lambda_j}-1} = n.$$

Proof (Sketch). Richmond showed that (cf. [41, 43])

$$p_{\Lambda}(n) = \frac{e^{n\nu + \Psi(\nu)}}{\sqrt{2\pi \Psi''(\nu)}}(1 + O(\nu)),$$

where

$$\Psi(t) = \sum_{j \geq 1} \log \frac{1}{1 - e^{-t\lambda_j}}.$$

Namely, the saddle-point approximation applies to the integral

$$p_{\Lambda}(n) = \frac{1}{2\pi i}\int_{r-\pi i}^{r+\pi i} e^{n\tau} F_{\Lambda}(e^{-\tau})\, d\tau,$$

where

$$F_{\Lambda}(z) = \prod_{j \geq 1} \frac{1}{1 - z^{\lambda_j}}.$$

Let $E_{\alpha}(z) = \sum_{j \geq 1} \lambda_j^{\alpha} z^{\lambda_j}$. The crucial observation leading to (26) is that $E_{\alpha}(e^{-\tau})$ is only a polynomial growth rate as $\tau \to 0$ and thus the saddle point ν will (asymptotically) not be changed by its presence. Following the proof of Richmond, we deduce that

$$\mathrm{E}\big(X_n^{(\alpha)}\big) = \frac{1}{p_{\Lambda}(n)}[z^n]E_{\alpha}(z)F_{\Lambda}(z) \sim E_{\alpha}(e^{-\nu}),$$

as required. ∎

COROLLARY 7. *Let* $\Lambda = \{1, 2^h, 3^h, \ldots\}$, *where* $h \geq 2$. *Then*

$$\mathrm{E}\big(X_n^{(\alpha)}\big) \sim \begin{cases} \zeta(-h\alpha), & \text{if } \alpha < -1/h, \\ \dfrac{h}{h+1} \log n, & \text{if } \alpha = -1/h, \\ \dfrac{\Gamma(\alpha + 1/h)}{h}\left(\dfrac{n}{\Gamma(1 + 1/h)\zeta(1 + 1/h)}\right)^{(\alpha h + 1)/(h+1)}, & \\ & \text{if } \alpha > -1/h. \end{cases}$$

Proof of Theorem 8. By Mellin transform,

$$\sum_{j \geq 1} j^{h\alpha} e^{-j^h \nu} = \frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} \nu^{-s}\Gamma(s)\zeta(h(s - \alpha))\, ds,$$

where $c = \max\{\alpha + 2 + 1/h, 1\}$, we easily deduce that

$$\sum_{j\geq 1} j^{h\alpha} e^{-j^h \nu} \sim \begin{cases} \zeta(-h\alpha), & \text{if } \alpha < -1/h, \\ \log\left(\dfrac{1}{\nu}\right), & \text{if } \alpha = -1/h, \\ \dfrac{\Gamma(\alpha + 1/h)}{h} \nu^{-\alpha - 1/h}, & \text{if } \alpha > -1/h, \end{cases}$$

as $\nu \to 0$. The saddle point ν satisfies, again by Mellin transform,

$$\nu \sim \left(\frac{\zeta(1 + 1/h)\Gamma(1 + 1/h)}{n} \right)^{h/(h+1)}.$$

from which the result follows. ∎

In a similar manner, we derive the following result; cf. [42]. Let

$$P(s) = \sum_{p \text{ prime}} p^{-s} \qquad (\Re s > 1).$$

COROLLARY 8. *If Λ is the set of prime numbers, then*

$$\mathrm{E}\left(X_n^{(\alpha)}\right) \sim \begin{cases} P(\alpha), & \text{if } \alpha < -1, \\ \log\log n, & \text{if } \alpha = -1, \\ \dfrac{2\Gamma(\alpha + 1)}{\pi^{\alpha+1}} (3n)^{(\alpha+1)/2} (\log n)^{(\alpha-1)/2}, & \text{if } \alpha > -1. \end{cases}$$

4.3. *Partitions with Small Number of Distinct Summands*

Central and local limit theorems for the number of distinct parts in a random Λ-partition have been studied in Hwang [26] under a scheme essentially due to Meinardus [38]; see also [21, 45]. We consider in this section the distribution of integer partitions with small number of distinct parts. Our arguments rely on some new combinatorial inequalities which are interesting per se.

Define two quantities by the generating functions

$$\sum_{n,m} q_\Lambda(n,m) u^m z^n = \prod_{j\geq 1} (1 + uz^{\lambda_j}),$$

$$\sum_{n,m} \pi_\Lambda(n,m) u^m z^n = \prod_{j\geq 1} \left(1 + \frac{uz^{\lambda_j}}{1 - z^{\lambda_j}}\right).$$

Let $\Lambda(z) = \sum_{j\geq 1} z^{\lambda_j}$ and

$$L_\mu(z) = \sum_{j\geq 1} \frac{z^{\mu\lambda_j}}{(1 - z^{\lambda_j})^\mu} \qquad (\mu \geq 1).$$

LEMMA 5. *For all positive integers n and m, we have*

$$[z^n]\frac{\Lambda^m(z)}{m!} - [z^n]\frac{\Lambda^{m-2}(z)\Lambda(z^2)}{2(m-2)!} \leq q_\Lambda(n,m) \leq [z^n]\frac{\Lambda^m(z)}{m!}, \quad (27)$$

$$[z^n]\frac{L_1^m(z)}{m!} - [z^n]\frac{L_1^{m-2}(z)L_2(z)}{2(m-2)!} \leq \pi_\Lambda(n,m) \leq [z^n]\frac{L_1^m(z)}{m!}. \quad (28)$$

Proof. Consider first (27). Observe that $[z^n]\Lambda^m(z)$ denotes the number of ordered partitions (compositions) of n with exactly m parts. Multiplying both sides of (27) by $m!$, we have

$$[z^n]\Lambda^m(z) - \binom{m}{2}[z^n]\Lambda^{m-2}(z)\Lambda(z^2) \leq m!q_\Lambda(n,m) \leq [z^n]\Lambda^m(z).$$

The middle term $m!q_\Lambda(n,m)$ denotes the number of compositions of n into m distinct parts. The second inequality is obvious since equality holds only if each part in a composition of n into m parts is distinct. The first inequality also follows from the fact that the coefficient

$$[z^n]\Lambda^{m-2}(z)\Lambda(z^2)$$

represents a quantity that is greater than or equal to the number of compositions of n with m parts in which at least two are identical.

The proof of the inequalities (28) is analogous and is omitted here. ∎

In the special case when $\Lambda = \mathbf{Z}^+$, the inequalities (27) imply that

$$[z^n]\frac{\Lambda^m(z)}{m!} - [z^n]\frac{\Lambda^{m-1}(z)}{2(m-2)!} \leq q_\Lambda(n,m) \leq [z^n]\frac{\Lambda^m(z)}{m!},$$

from which Stirling's formula yields $[\Lambda(z) = z/(1-z)]$

$$q_\Lambda(n,m) \sim \frac{n^{m-1}}{m!(m-1)!}, \quad (29)$$

uniformly for $1 \leq m = o(n^{1/3})$, a result of Erdős and Lehner [14]. Among the existing different proofs of the foregoing formula (cf. [14, 6, 48]), our derivation seems the simplest yet the most general.

To further demonstrate the usefulness of the inequality (27), let us consider the case when $\Lambda = \{1^h, 2^h, 3^h, \dots\}$, where $h \geq 2$. Although it is, in general, difficult to derive asymptotic estimates for $[z^n]\Lambda^m(z)$ for $m = O(1)$ (the Waring problem), we show that it is, however, possible to give useful estimates when $m \to \infty$ and $m = o(n^{1/(3h)})$ by applying the saddle-point method. From various estimates on the function $\Lambda(e^{-r-iy})$, as $r \to 0$, which is used in Waring's problem (cf. [7, Chap. IV]), we have, uniformly as $r \to 0$ and $r \le |y| \le \pi$,

$$\left|\Lambda(e^{-r-iy})\right| \le \vartheta\Lambda(e^{-r}), \tag{30}$$

for some absolute $\vartheta \in (0,1)$ depending only on h. In addition, by Mellin transform,

$$\Lambda(e^{-r-iy}) = \Gamma(1/h)(r+iy)^{-1/h} - \frac{1}{2} + O(|r|),$$

as $r \to 0$ an $|y| \le r$. These two estimates assure that we can apply the saddle-point method to the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{r+iy}\Lambda^m(e^{r+iy})\,dy,$$

giving

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{r+iy}\Lambda^m(e^{r+iy})\,dy \sim \frac{\Gamma(1/h)^m}{\Gamma(m/h)} n^{(m/h)-1},$$

uniformly for $m \to \infty$ and $m = o(n^{1/(2h)})$. It follows, by inequality (27), that the estimate

$$q_\Lambda(n,m) \sim \frac{\Gamma(1/h)^m}{\Gamma(m/h)m!} n^{(m/h)-1},$$

holds uniformly for $m \to \infty$ and $m = o(n^{1/(3h)})$.

In view of the complexity of the solution to Waring's problem (cf. [7, Chap. IV]), it is also difficult to expect general asymptotic results for $\pi_\Lambda(n,m)$ for small m as in the case of compositions. Consider, for example, the case $\Lambda = \mathbf{Z}^+$. We have $\pi_\Lambda(n,1) = d(n)$ and

$$\pi_\Lambda(n,2) = \frac{1}{2}\sum_{1\le j<n} d(j)d(n-j) - \frac{1}{2}\sum_{d|n}(d-1) \qquad (n \ge 2).$$

Asymptotic behavior of the first sum already presents technical difficulty.

The case when $m \to \infty$ can be treated as for q_Λ by the saddle-point method, but in a rather limited range. Set

$$W(z) = \sum_{j\ge 1} d(j)z^j = \sum_{j\ge 1}\frac{z^j}{1-z^j} \qquad (|z|<1),$$

and write $\tau = r + iy$, where $r > 0$ and $-\pi \le y \le \pi$. By Mellin transform, one easily derives that

$$W(e^{-\tau}) = \frac{\gamma - \log \tau}{\tau} + \frac{1}{4} + O(|r|),$$

as $r \to 0$ and $|y| \le r$. To derive an inequality of the type (30), we apply the Euler–Maclaurin summation formula to the function $f(x) = 1/(e^{x\tau} - 1) - 1/(x\tau)$ (cf. [53]),

$$W(e^{-\tau}) = \frac{\gamma - \log \tau}{\tau} + \frac{1}{4} + R,$$

where for $-\pi \le y \le \pi$,

$$\begin{aligned} |R| &\le \frac{1}{2}\int_0^\infty \left| \frac{\tau e^{x\tau}}{(e^{x\tau} - 1)^2} - \frac{1}{x^2\tau} \right| dx \\ &\ll \int_0^{1/r} |\tau|\, dx + \int_{1/r}^\infty \frac{|\tau| e^{rx}}{(e^{rx} - 1)^2}\, dx + \int_{1/r}^\infty \frac{1}{x^2|\tau|}\, dx \\ &\ll \frac{|\tau|}{r}. \end{aligned}$$

Thus, uniformly for $r > 0$ and $-\pi \le y \le \pi$,

$$W(e^{-\tau}) = \frac{\gamma - \log \tau}{\tau} + O\left(\frac{|\tau|}{r}\right),$$

from which we deduce the existence of an absolute constant $\vartheta \in (0,1)$ such that

$$\left| W(e^{-r-iy}) \right| \le \vartheta W(e^{-r}) \qquad (r \to 0,\ r \le |y| \le \pi).$$

By the saddle-point method using the preceding estimates and (28), we obtain

$$\pi_\Lambda(n, m) \sim \frac{n^{m-1}(\log n)^m}{m!(m-1)!}, \tag{31}$$

uniformly as $m \to \infty$ and $m = o((\log n)/(\log\log n))$. The case for slightly larger m can also be dealt with by the same method, but the results will be less implicit in nature. The appearance of the logarithmic factor in result (31) reflects the effect of repetitions of summands, in contrast to (29).

In general, the lower inequality in (28) is rather poor since the quantity soon takes on negative values as m increases.

5. EXTENSIONS

Since our treatment in this paper is completely general, many problems naturally arise. Let us briefly discuss some of them.

Our methods are useful for asymptotics of the number of parts with frequencies $\geq h$, $h \geq 1$; and similar results as those in Section 3.3 hold.

The most interesting open problem is the limit distribution of the number of distinct summands, X_n, in a random Λ-composition of n under suitable conditions on Λ. Intuitively, since X_n is a sum of dependent Bernoulli random variables,

$$X_n = X_n(\Pi) = \sum_{\lambda_j \leq n} \mathrm{P}\big(\lambda_j \text{ is a summand of } \Pi\big),$$

and no summand can give a preponderant contribution to the counting function, one expects that the limit distribution would be Gaussian for a large class of Λ-compositions, in accordance with the classical law of errors. This is so for partitions under an analytic scheme of Meinardus as already established in [26].

The fluctuating phenomena encountered in Theorems 2 demand further combinatorial interpretations.

Another problem of more (analytic) number-theoretic interest is an effective version of our Theorem 3. More generally, derive more precise asymptotics for general Λ-compositions. Besides the leading term, fluctuating functions as in Theorem 2 would be present.

Asymptotics of the convolutions of $d(n)$ seems another difficult problem. Besides its number-theoretic interest, its application to $\pi_\Lambda(n, m)$ [see (28)] gives a more convincing motivation.

As we have seen, combinatorial inequalities such as (19), (27), and (28), when combined with analytic methods, are very useful in giving asymptotic estimates. Such an approach, of both combinatorial and analytic interests, should be further developed.

Finally, our treatment can be extended to arithmetic problems. Consider, for example, the number of distinct factors in ordered factorizations (into, say, positive integers ≥ 2). We have the Dirichlet generating function

$$D(s, u) = \int_0^\infty e^{-t} \prod_{n \geq 2} \big(1 + u(e^{t/n^s} - 1)\big)\, dt,$$

where u marks the number of distinct factors. In particular,

$$D(s, 1) = \sum_{n \geq 1} \kappa_n n^{-s} = \frac{1}{2 - \zeta(s)}$$

and

$$\frac{\partial}{\partial u} D(s,u)\bigg|_{u=1} = \sum_{n\geq 1} K_n n^{-s} = \sum_{n\geq 2} \left(\frac{1}{2-\zeta(s)} - \frac{1}{2-\zeta(s)+n^{-s}} \right),$$

where K_n is the sum of the number of distinct factors of all factorizations of n. Using the zero-free region for Riemann's zeta function, we can study the asymptotics of the quantity

$$\mu_n := \frac{\sum_{1\leq j\leq n} K_j}{\sum_{1\leq j\leq n} \kappa_j},$$

which is the expected number of distinct factors in a random factorization ϕ, where each factorization ϕ of $1, 2, \ldots, n$ is equally likely. Asymptotically, we have $\mu_n \sim R n^{1/\rho}$, where $\rho > 1$ solves $\zeta(s) = 2$ and

$$R = -\Gamma\left(-\frac{1}{\rho}\right)\left(\frac{-1}{\rho\zeta'(\rho)}\right)^{1/\rho} = 1.8740221836\ldots.$$

Thus the transfer is no longer logarithmic, which means that the small factors play a role less significant than in the case of compositions. For related materials and lemmas, see [27, Chap. 11].

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