Distribution of the Number of Consecutive Records

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ABSTRACT: We study the distribution of the number $\xi_{n,r}$ of times that r consecutive records occur in a sequence of n independent and identically distributed random variables from a common continuous distribution, or equivalently, in a random permutation of n elements. We show that the asymptotic distribution of $\xi_{n,r}$ exists and is Poisson for $r = 1, 2$ and non-Poisson for $r \geq 3$. Precise asymptotic results are derived for four probability distances of the associated approximations: Fortet-Mourier, total variation, Kolmogorov, and point metric. In particular, the distributions of $\xi_{n,r}$ have the specific property that the last three distances are asymptotically of the same behavior for $r \geq 2$. We also provide interesting combinatorial bijections for $\xi_{n,2}$ and compute explicitly the limiting law for $\xi_{n,3}$ in terms of Kummer's confluent hypergeometric functions. © 2000 John Wiley & Sons, Inc. Random Struct. Alg., 17, 169–196, 2000

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1. INTRODUCTION

Introduce the uniform probability distribution on the set $\mathscr{S}_n$ of permutations of n elements. Given a permutation π, we say that $\pi(j)$ is a *record* (or *left-to-right maximum, upper record, outstanding element*, etc.) if $\pi(i) < \pi(j)$ for all $1 \le i < j$. It is well known that the distribution of the number of records ξ_n in a random permutation involves the signless Stirling numbers of the first kind:

$$\mathrm{E}(u^{\xi_n}) = \frac{u(u+1)\cdots(u+n-1)}{n!} \qquad (n \ge 1).$$

This relation is equivalent to the fact that $\xi_n = \sum_{1\le j\le n} X_j$, where X_j denotes a Bernoulli random variable with mean $1/j$, a result due independently to Dwass [8] and to Rényi [20]. We note that since there is a well-known bijection between the number of cycles and the number of records in permutations (cf. Knuth [17]), the above decomposition is equivalent to the corresponding one for cycles first given in Feller [11]; see also Arratia et al. [1].

Let $Y_h(\lambda)$ be a Poisson random variable with mean $\lambda > 0$, shifted by h:

$$\mathrm{P}(Y_h(\lambda) = m) = e^{-\lambda}\frac{\lambda^{m-h}}{(m-h)!} \qquad (m = h, h+1, \ldots),$$

where h is a nonnegative integer. From the above representation of ξ_n, it can be shown (cf. Hwang [15]) that the total variation distance d_{TV} and the Fortet-Mourier (or Wasserstein) distance d_{FM} between the distribution of ξ_n and that of $Y_1(\lambda)$, where $\lambda = \log n + \gamma - 1$, satisfy

$$\begin{aligned}
d_{\mathrm{TV}}(\mathscr{L}(\xi_n), \mathscr{L}(Y_1(\lambda))) &:= \frac{1}{2}\sum_{k\ge 0} |\mathrm{P}(\xi_n = k) - \mathrm{P}(Y_1(\lambda) = k)| \\
&= \frac{\frac{\pi^2}{6} - 1}{\sqrt{2\pi e}\,\log n}\big(1 + O\big((\log n)^{-1/2}\big)\big), \\
d_{\mathrm{FM}}(\mathscr{L}(\xi_n), \mathscr{L}(Y_1(\lambda))) &:= \sum_{k\ge 0} |\mathrm{P}(\xi_n \le k) - \mathrm{P}(Y_1(\lambda) \le k)| \\
&= \frac{\frac{\pi^2}{6} - 1}{\sqrt{2\pi \log n}}\big(1 + O\big((\log n)^{-1/2}\big)\big),
\end{aligned}$$

as $n \to \infty$; see also Barbour et al. [5] and Borovkov and Pfeifer [6]. Note that if we use $Y_0(\lambda)$ instead of $Y_1(\lambda)$ then the constant $\frac{\pi^2}{6} - 1$ on the right-hand member is replaced by $\frac{\pi^2}{6}$.

A considerable number of extensions and generalizations of the above problem have been studied in the literature; see the monograph by Arnold et al. [2], Bai et al. [3, 4], Nevzorov [19], Goldie and Resnick [14], and the references therein for further information.

We consider in this paper the number of times $\xi_{n,r}$ that r consecutive records (possibly overlapping) occur in a sequence of n independent and identically distributed random variables from the same continuous distribution,[1] or equivalently, in a random permutation π of n elements; that is, the number of j's for which $\pi(j), \pi(j+1), \ldots, \pi(j+r-1)$ are all records, where $r \geq 1$ is a fixed integer. Another equivalent way of describing $\xi_{n,r}$ is in terms of the number of r consecutive ones in a sequence of n independent Bernoulli random variables $\{X_1, X_2, \ldots\}$, where $\mathrm{E}(u^{X_j}) = 1 + (u-1)/j$:

$$\xi_{n,r} \overset{d}{=} \sum_{1 \leq j \leq n-r+1} X_j X_{j+1} \cdots X_{j+r-1}. \tag{1}$$

Denote the right-hand side of (1) by $\tilde{\xi}_{n,r}$. As $n \to \infty$ with r fixed, $\tilde{\xi}_{n,r}$ increases to a limit random variable $\tilde{\xi}_r := \sum_{j \geq 1} X_j \cdots X_{j+r-1}$, and for $r \geq 2$, the fact that $E(\tilde{\xi}_r) < \infty$ (see Theorem 3 below) shows that $\tilde{\xi}_r < \infty$ with probability 1. Thus $\xi_{n,r}$ has a limit distribution as $n \to \infty$, which is the distribution of $\tilde{\xi}_r$.

For more quantitative results, we start from $\xi_{n,2}$ which has several specific properties not enjoyed by $\xi_{n,r}$ for $r \geq 3$. We first show that the limiting distribution of $\xi_{n,2}$ is still Poisson in the following sense.

Theorem 1. *For $n \geq 2$*

$$\begin{aligned} d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) &= \frac{e^{-1}}{n} + \Delta_{\mathrm{TV}}, \\ d_{\mathrm{FM}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) &= \frac{1}{n}, \end{aligned}$$

where

$$\Delta_{\mathrm{TV}} = \begin{cases} \dfrac{n+1}{n} \displaystyle\sum_{j \geq n+2} \frac{(-1)^{j+1}}{j!}, & \text{if } n \text{ is odd}; \\ \dfrac{1}{(n+1)!} - \left(2 + \dfrac{1}{n}\right) \displaystyle\sum_{j \geq n+2} \frac{(-1)^j}{j!}, & \text{if } n \text{ is even}. \end{cases} \tag{2}$$

In particular,

$$|\Delta_{\mathrm{TV}}| \leq \begin{cases} \dfrac{(n+1)(n+3)}{n(n+4)} \cdot \dfrac{1}{(n+2)!}, & \text{if } n \text{ is odd}; \\ \dfrac{n^2+n-1}{n(n+3)} \cdot \dfrac{1}{(n+1)!}, & \text{if } n \text{ is even}. \end{cases} \tag{3}$$

Surprisingly, the Fortet-Mourier (or Wasserstein) distance of the distributions of $\xi_{n,2}$ and $Y_0(1)$ is *identically* $1/n$ for $n \geq 2$! (It also holds for $n = 1$.) Also *the Kolmogorov distance and the total variation distance are identical*; see Corollary 4. A

[1] The number of times *exactly* r but not $r+1$ consecutive records occur, namely, for $n > r$

$$X_1 \cdots X_r(1 - X_{r+1}) + \sum_{1 \leq j \leq n-r-1} (1 - X_j) X_{j+1} \cdots X_{j+r}(1 - X_{j+r+1}) + (1 - X_{n-r}) X_{n-r+1} \cdots X_n$$

is equal to $X_{n,r} - 2X_{n,r+1} + X_{n,r+2}$.

similar result for the point metric is given in Corollary 5. Both corollaries are in Section 2.2.

Originally, we learned about the problem of $\xi_{n,2}$ from Yuan Shih Chow who attributed the problem to P. Diaconis, and we were also aware of a few proofs of the asymptotic Poisson distribution of $\xi_{n,2}$. Our precise (and exact) results for $\xi_{n,2}$ are new. Also to the best of our knowledge, no result on distribution of $\xi_{n,r}$ for $r \geq 3$ seems to be known.

The proof of Theorem 1 is based on the following explicit expression of the probability generating function of $\xi_{n,2}$.

Proposition 1. *The probability generating function of $\xi_{n,2}$ satisfies*

$$\mathrm{E}(u^{\xi_{n,2}}) = \sum_{0\leq j<n} \frac{(u-1)^j}{j!}\left(1-\frac{j}{n}\right), \tag{4}$$

for $n \geq 1$.

Corollary 1. *For $0 \leq k < n$ and $n \geq 2$*

$$\mathrm{P}(\xi_{n,2} = k) = \frac{1}{k!}\sum_{0\leq j<n-k} \frac{(-1)^j}{j!}\left(1-\frac{k+j}{n}\right). \tag{5}$$

The right-hand side of (4) is reminiscent of the number of derangements in random permutations (cf. p. 181, Knuth [17] or (10) below). Indeed, the distribution of $\xi_{n,2}$ is identical to that of η_n, the number of singletons (cycles of unit length) in a random permutation π, not counting $\pi(n) = n$ if it occurs (i.e., the number of j's for which $\pi(j) = j$ and $1 \leq j < n$). There is yet another equi-distributed random variable ζ_n, the number of successions in a random permutation (i.e., the number of j's for which $\pi(j+1) = \pi(j) + 1$).

Theorem 2.

$$\xi_{n,2} \stackrel{d}{=} \eta_n \stackrel{d}{=} \zeta_n.$$

Derangements and successions of permutations were first studied by Montmort [18] and Whitworth [22], respectively. For more information on η_n and ζ_n, see in particular Dwass [9], Comtet [7], Fu [13], and the references therein.

The preceding results can be summarized by saying that $\xi_{n,r}$ is asymptotically Poisson for $r = 1, 2$. We will show that the limiting distribution of $\xi_{n,r}$ is no longer Poisson for $r \geq 3$. Our first result in this direction is the following recurrence for the probability generating function of $\xi_{n,r}$.

Proposition 2. *Let $F_{n,r}(u) = n!\mathrm{E}(u^{\xi_{n,r}})$. Then $F_{n,r}$ satisfies for $r \geq 1$*

$$F_{n,r}(u) = \begin{cases} n!, & \text{if } 0 \leq n < r; \\ (n+u-1)F_{n-1,r}(u) + (1-u)\sum_{2\leq j\leq r}(n-j)F_{n-j,r}(u), & \text{if } n \geq r. \end{cases} \tag{6}$$

From this recurrence, we can derive expressions for the moments of $\xi_{n,r}$. Let $\mu_r = \frac{1}{(r-1)(r-1)!}$ and

$$\sigma_r^2 = \frac{1}{(r-1)(r-1)!} - \frac{1}{(r-1)^2(r-1)!^2} + \frac{1}{(r-1)^2(2r-2)!} + \sum_{r\le j\le 2r-2} \frac{2}{j\cdot j!}.$$

The first few values of σ_r^2 are given by

$$\{\sigma_r^2\}_{r\ge 2} = \left\{1, \frac{95}{288}, \frac{2503}{32400}, \frac{959767}{67737600}, \frac{25038079}{11430720000}, \dots\right\}.$$

Theorem 3. *Let $r \ge 2$. The mean and the variance of $\xi_{n,r}$ satisfy* $\mathrm{E}(\xi_{n,r}) = \mathrm{Var}(\xi_{n,r}) = 0$ *for $0 \le n < r$ and*

$$\mathrm{E}(\xi_{n,r}) = \mu_r - \frac{(n-r+1)!}{n!(r-1)} \qquad (n \ge r),$$

$$\mathrm{Var}(\xi_{n,r}) = \begin{cases} 2\sum_{r\le j\le n}\left(\frac{1}{j\cdot j!} - \frac{(n-j)!}{j\cdot n!}\right) - (\mathrm{E}(\xi_{n,r}))^2 + \mathrm{E}(\xi_{n,r}), & \textit{if } r \le n \le 2r; \\ \sigma_r^2 - \frac{(n-r+1)!}{n!(r-1)} + \frac{(n-2r+2)!}{(r-1)^2 n!} & \\ \quad -\left(\frac{(n-r+1)!}{(r-1)n!}\right)^2 - 2\sum_{r\le j\le 2r-2}\frac{(n-j)!}{j\cdot n!}, & \textit{if } n \ge 2(r-1). \end{cases}$$

Note that the two expressions for $\mathrm{Var}(\xi_{n,r})$ coincide for $n = 2r-2, 2r-1, 2r$.

Corollary 2. *For $r \ge 2$*

$$\mathrm{E}(\xi_{n,r}) = \mu_r - \frac{n^{1-r}}{r-1} + O(n^{-r}),$$

$$\mathrm{Var}(\xi_{n,r}) = \sigma_r^2 - \frac{n^{1-r}}{r-1} + O(n^{-r}),$$

as $n \to \infty$.

To proceed further, we observe that the recurrence (6) leads to the following ordinary differential equation (ODE)

$$\begin{aligned}(1-z)A^{(r)}(z) &= (r-(1-z)(1-u))A^{(r-1)}(z) \\ &\quad + (1-u)\sum_{1\le j\le r-2}(z+j)A^{(j)}(z),\end{aligned} \tag{7}$$

with initial conditions $A^{(j)}(0) = j!, 0 \le j \le r-1$, where

$$A(z) = A(z,u;r) = 1 + \sum_{n\ge 1}\mathrm{E}(u^{\xi_{n,r}})z^n.$$

A detailed study of the ODE (7) yields the existence of the limiting law of $\xi_{n,r}$ for each fixed r, as $n \to \infty$.

Proposition 3. *For $r \geq 3$*

$$\mathrm{E}(u^{\xi_{n,r}}) = \phi_r(u) + \frac{1-u}{r-1}\,\phi_r(u)n^{1-r} + O(n^{-r}),$$

uniformly for all finite $u \in \mathbb{C}$, where $\phi_r(u)$ is an entire probability generating function.

Since $\mu_r = \sigma_r^2$ only for $r = 2$, we deduce the following corollary.

Corollary 3. *The limiting distribution of $\xi_{n,r}$ exists and is not Poisson for $r \geq 3$.*

Let ξ_r be a random variable distributed according to the limiting law of $\xi_{n,r}$, namely, $\mathrm{E}(u^{\xi_r}) = \phi_r(u)$. Without explicit characterization of the limiting law, we can still derive precise asymptotics for the corresponding probability metrics.

Theorem 4. *For $r \geq 3$*

$$d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,r}), \mathscr{L}(\xi_r)) = \frac{\phi_r(0)}{r-1}\,n^{1-r} + O(n^{-r}),$$
$$d_{\mathrm{FM}}(\mathscr{L}(\xi_{n,r}), \mathscr{L}(\xi_r)) = \frac{1}{r-1}\,n^{1-r} + O(n^{-r}),$$

The Kolmogorov distance and the point metric have the same asymptotic behavior as d_{TV}; see Corollaries 6 and 7, at the end of Section 3. Note that $\phi_r(0) = \mathrm{P}(\xi_r = 0) = \lim_{n\to\infty} \mathrm{P}(\xi_{n,r} = 0)$; see Table 1 for numeric values. Also Theorem 1 can be stated (up to error terms) in the same forms (with $r = 2$).

But curiosities remain: what is then the limiting law of $\xi_{n,r}$? We have the following partial result for $\xi_{n,3}$, which essentially involves the resolution of a second-order ODE of confluent hypergeometric type. Let

$$\Phi(a, c; z) = \sum_{j\geq 0} \frac{(a)_j z^j}{(c)_j j!} \tag{8}$$

denote Kummer's confluent hypergeometric function (cf. [10, Chap. 6]), where $(x)_j = \prod_{0\leq i\leq j-1}(x+i)$ denotes the Pochhammer symbol (an empty product is to be interpreted as unity).

Theorem 5. *The probability generating function of the limiting law of $\xi_{n,3}$ satisfies*

$$\phi_3(u) = \frac{e^{h-\Delta}}{2}(2\Phi(a, 2; \Delta) - h\Phi(a, 3; \Delta)), \tag{9}$$

where $\Delta = \pm\sqrt{(u-1)(u+3)}$,

$$a = \frac{3}{2} + \frac{1}{\Delta}\left(1 - \sqrt{1 + \frac{\Delta^2}{4}}\right), \qquad h = \frac{\Delta}{2} - 1 + \sqrt{1 + \frac{\Delta^2}{4}}.$$

TABLE 1 Numeric Values of the Limiting Probabilities $\lim_{n\to\infty}\mathrm{P}(\xi_{n,r} = 0)$ for r from 3 to 14

3	4	5	6
0.80800 91253	0.95489 72634	0.99125 03641	0.99856 48174
7	8	9	10
0.99979 68632	0.99997 47555	0.99999 72059	0.99999 97213
11	12	13	14
0.99999 99747	0.99999 99979	0.99999 99998	0.99999 99999 88

(We can define either branch of the square root without any inconsistency.) Different expressions of $\phi_3(u)$ can be obtained by the integral representations and recurrence relations of $\Phi(a, c; z)$. We leave open the characterization of $\phi_r(u)$ for $r \geq 4$.

Amazingly, the complicated form (as a function of u) of the right-hand side of (9) gives rise to positive numbers for nonnegative u. For example, when $u = 0$, $\Delta = \pm\sqrt{3}i$, $a = 3/2 \mp \sqrt{3}i/6$, and $h = (-1 \pm \sqrt{3}i)/2$,

$$\mathrm{P}(\xi_{n,3} = 0) \to \phi_3(0) \approx 0.80800\,91253\,46368\,57744\,07540\,78478\ldots$$

(Choice of either "+" or "−" leads to the same numeric value, provided the choice is consistent.)

In view of Theorem 2, it is interesting to compare our results for $\xi_{n,r}$ with the results in Fu [13] for the number $\zeta_{n,r}$ of r successions in a random permutation π (i.e., number of occurrences of the pattern $\pi(j+i) = \pi(j) + i$ for $i = 1, 2, \ldots, r-1$). His result states that

$$\mathrm{P}(\zeta_{n,r} = 0) \to 1 \qquad (n \to \infty),$$

for $r \geq 3$. Thus *there are considerably fewer r successions than the number of r consecutive records for $r \geq 3$.*

On the other hand, the limit of $\mathrm{P}(\xi_{n,r} = 0)$ as $n \to \infty$, although different from 1 for each fixed r, tends very fast toward 1 as r increases; see Table 1. Thus the limiting laws are "almost degenerate" for moderate r. This partly explains why the three distances d_{TV}, d_{K}, and d_{L} have the same asymptotic behaviors.

2. TWO CONSECUTIVE RECORDS

2.1. Probability Generating Function

Although Proposition 1 can be obtained by solving the recurrence (6) for $r = 2$, we first give an independent proof for more methodological interest.

Let $\xi'_{n,2}$ be the number of occurrences of two consecutive records in a permutation chosen uniformly at random from the subset of $\mathscr{S}_n$ for which $\pi(n) = n$. Set $q_n(u) = \mathrm{E}(u^{\xi'_{n,2}})$ and $f_n(u) = \mathrm{E}(u^{\xi_{n,2}})$. Then, by conditioning on the location of n

in a random permutation of n elements, we have the recurrence

$$f_n(u) = \frac{1}{n} \sum_{1 \le j \le n} q_j(u) \qquad (n \ge 2),$$

and similarly,

$$q_n(u) = \frac{u}{n-1}\, q_{n-1}(u) + \frac{n-2}{n-1}\, f_{n-2}(u) \qquad (n \ge 2).$$

with the initial conditions $f_0(u) = q_0(u) = 0$ and $f_1(u) = q_1(u) = 1$.

Thus $(n-1)q_n(u) = uq_{n-1}(u) + \sum_{1 \le j \le n-2} q_j(u)$. Subtracting this equation with n replaced by $n-1$, we obtain

$$(n-1)(q_n(u) - q_{n-1}(u)) = (u-1)(q_{n-1}(u) - q_{n-2}(u)),$$

and consequently,

$$q_n(u) = \sum_{0 \le j < n} \frac{(u-1)^j}{j!} \qquad (n \ge 1). \tag{10}$$

Hence

$$f_n(u) = \frac{1}{n} \sum_{0 \le j < n} \frac{(u-1)^j}{j!}(n-j),$$

from which (4) follows. ∎

2.2. Poisson Approximation

We prove Theorem 1 in this subsection.

Total Variation Distance. Set

$$\delta_k = \mathrm{P}(\xi_{n,2} = k) - \mathrm{P}(Y_0(1) = k).$$

By definition,

$$d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) = \frac{1}{2} \sum_{k \ge 0} |\delta_k| = \frac{1}{2} \sum_{0 \le k \le n} |\delta_k| + \frac{1}{2} \sum_{k > n} \frac{e^{-1}}{k!}. \tag{11}$$

By (5), we have

$$\delta_k = -\frac{e^{-1}}{n}\left(\frac{1}{(k-1)!} - \frac{1}{k!}\right) + E_k \qquad (0 \le k \le n),$$

where $1/j! = 0$ for negative integer j and

$$\begin{aligned} E_k &= -\frac{1}{k!} \sum_{j > n-k} \frac{(-1)^j}{j!}\left(1 - \frac{j+k}{n}\right) \\ &= \frac{1}{nk!} \sum_{j \ge 1} \frac{(-1)^{n-k+j} j}{(n-k+j)!}. \end{aligned}$$

By rearranging the series of E_k, we can write it in an alternative form

$$E_k = \frac{n-k+1}{nk!} \sum_{j \geq n-k+2} \frac{(-1)^{j+1}}{j!}. \tag{12}$$

We now show that for $0 \leq k \leq n$ and $k \neq 1$

$$|E_k| \leq \frac{e^{-1}}{n} \left| \frac{1}{(k-1)!} - \frac{1}{k!} \right|. \tag{13}$$

For, by (12),

$$\begin{aligned}
|E_k| &= \frac{n-k+1}{nk!} \sum_{j \geq 0} \frac{(-1)^j}{(n-k+j+2)!} \\
&= \frac{n-k+1}{nk!} \sum_{j \geq 0} \frac{n-k+2j+2}{(n-k+2j+3)!} \\
&\leq \frac{1}{nk!} \sum_{j \geq 0} \frac{n-k+2j+1}{(n-k+2j+2)!} \\
&= \frac{1}{nk!} \sum_{j \geq 0} \frac{(-1)^j}{(n-k+j+1)!} \\
&\leq \frac{|k-1|}{nk!} e^{-1},
\end{aligned}$$

for $k \neq 1$ and $n \geq 2$. Thus

$$|\delta_k| = \begin{cases} \dfrac{e^{-1}}{n} + E_0, & \text{if } k = 0; \\ |E_1|, & \text{if } k = 1; \\ \dfrac{e^{-1}}{n} \left(\dfrac{1}{(k-1)!} - \dfrac{1}{k!} \right) - E_k, & \text{if } 2 \leq k \leq n. \end{cases}$$

Consequently,

$$\begin{aligned}
\sum_{0 \leq k \leq n} |\delta_k| &= \frac{2e^{-1}}{n} - \frac{e^{-1}}{n \cdot n!} + E_0 + |E_1| - \sum_{2 \leq k \leq n} E_k \\
&= \frac{2e^{-1}}{n} - \frac{e^{-1}}{n \cdot n!} + 2E_0 + |E_1| + E_1 - \sum_{0 \leq k \leq n} E_k.
\end{aligned}$$

Consider first the last sum, which by (13) can be simplified as follows.

$$\begin{aligned}
-\sum_{0 \leq k \leq n} E_k &= -\frac{1}{n} \sum_{j \geq 1} \frac{(-1)^{n+j} j}{(n+j)!} \sum_{0 \leq k \leq n} \binom{n+j}{k} (-1)^k \\
&= \frac{1}{n} \sum_{j \geq 1} \frac{(-1)^{n+j} j}{(n+j)!} \sum_{n+1 \leq k \leq n+j} \binom{n+j}{k} (-1)^k
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n} \sum_{k \ge 1} \frac{1}{(k+n)!} \sum_{j \ge 0} \frac{(-1)^j}{j!} (j+k) \\
&= \frac{e^{-1}}{n} \sum_{j \ge 1} \frac{j}{(n+j+1)!} \\
&= \frac{e^{-1}}{n} \sum_{j \ge n+2} \frac{j-1-n}{j!} \\
&= \frac{e^{-1}}{n(n+1)!} - \sum_{j \ge n+2} \frac{e^{-1}}{j!}.
\end{aligned}
$$

From this it follows that

$$
\sum_{k>n} \frac{e^{-1}}{k!} - \frac{e^{-1}}{n \cdot n!} - \sum_{0 \le k \le n} E_k = \frac{e^{-1}}{(n+1)!} - \frac{e^{-1}}{n \cdot n!} + \frac{e^{-1}}{n(n+1)!} = 0. \tag{14}
$$

Thus, by (11) and (14),

$$
d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) = \frac{e^{-1}}{n} + \Delta_{\mathrm{TV}},
$$

where

$$
\Delta_{\mathrm{TV}} = \begin{cases} E_0, & \text{if } n \text{ is odd;} \\ E_0 + E_1, & \text{if } n \text{ is even.} \end{cases} \tag{15}
$$

This together with (12) proves (2). For the proof of (3), we need the inequality

$$
\sum_{j \ge 0} \frac{(-1)^j}{(N+j)!} \le \frac{N+1}{N+2} \cdot \frac{1}{N!}, \tag{16}
$$

for nonnegative integer N. This can be seen as follows. First, the case $N = 0$ is trivial. Assume $N \ge 1$. We have

$$
\sum_{j \ge 0} \frac{(-1)^j}{(N+j)!} = \sum_{j \ge 0} \frac{N+2j}{(N+2j+1)!}.
$$

Then by induction

$$
\frac{N+2j}{(N+2j+1)!} \le \frac{N}{(N+1)!(N+1)^{2j}} \qquad (j \ge 0).
$$

Summing over all j yields (16).

The first inequality in (3) (odd n) follows from (2) and (16). When n is even, we use (3) and rearrange terms as follows

$$
\Delta_{\mathrm{TV}} = \frac{1}{(n+1)!} - \left(2 + \frac{1}{n}\right) \frac{1}{(n+2)!} + \left(2 + \frac{1}{n}\right) \sum_{j \ge 0} \frac{(-1)^j}{(n+3+j)!}.
$$

Applying (16) to the last sum yields

$$|\Delta_{\mathrm{TV}}| \leq \frac{1}{(n+1)!} \cdot \frac{n^4+8n^3+16n^2+n-11}{n(n+2)(n+3)(n+5)}.$$

Now

$$\frac{n^4+8n^3+16n^2+n-11}{n(n+2)(n+3)(n+5)} = \frac{n^2+n-1}{n(n+3)} - \frac{2n+1}{n(n+2)(n+3)(n+5)} \leq \frac{n^2+n-1}{n(n+3)}.$$

This completes the proof of Theorem 1. ∎

Remark. Although the proof of Theorem 1 can be simplified slightly by using the relation

$$d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) = \sum_{k\geq 0} \Big(\mathrm{P}(\xi_{n,2}=k) - \mathrm{P}(Y_0(1)=k)\Big)^+,$$

our proof is easily amended for other distances.

Fortet-Mourier Distance. Consider now the Fortet-Mourier distance. Arguing as above, we have for $0 \leq k \leq n$

$$\mathrm{P}(\xi_{n,2} \leq k) - \mathrm{P}(Y_0(1) \leq k) = \sum_{0\leq \ell\leq k} \delta_\ell = \frac{e^{-1}}{nk!} + \sum_{0\leq \ell \leq k} E_\ell. \tag{17}$$

By induction using (13), we have the inequality

$$\left| \sum_{0\leq \ell\leq k} E_\ell \right| \leq \frac{e^{-1}}{nk!},$$

for $0 \leq k \leq n$. Thus

$$|\mathrm{P}(\xi_{n,2} \leq k) - \mathrm{P}(Y_0(1) \leq k)| = \frac{e^{-1}}{nk!} + \sum_{0\leq \ell\leq k} E_\ell.$$

Summing over k gives, as the derivations in (14),

$$\begin{aligned}
\sum_{0\leq k\leq n} \sum_{0\leq \ell\leq k} E_\ell &= \sum_{0\leq k\leq n} E_k(n+1-k) \\
&= \frac{1}{n}\sum_{j\geq 1} \frac{(-1)^{n+j} j}{(n+j)!} \sum_{n+2\leq k\leq n+j} \binom{n+j}{k} (-1)^k (k-n-1) \\
&= \frac{1}{n} \sum_{k\geq n+2} \frac{k-n-1}{k!} \sum_{j\geq 0} \frac{(-1)^j}{j!}(j+k-n) \\
&= \frac{e^{-1}}{n} \sum_{j\geq n+2} \frac{(j-n-1)^2}{j!},
\end{aligned}$$

where we used the identity

$$\sum_{0\le k\le n}\binom{n+j}{k}(-1)^k(n+1-k)=\sum_{n+2\le k\le n+j}\binom{n+j}{k}(-1)^k(k-n-1).$$

Accordingly,

$$\sum_{0\le k\le n}|\mathrm{P}(\xi_{n,2}\le k)-\mathrm{P}(Y_0(1)\le k)|=\frac{1}{n}-\frac{e^{-1}}{n}\sum_{j\ge n+1}\frac{1}{j!}+\frac{e^{-1}}{n}\sum_{j\ge n+2}\frac{(j-n-1)^2}{j!}.$$

Thus

$$d_{\mathrm{FM}}(\mathscr{L}(\xi_{n,2}),\mathscr{L}(Y_0(1)))=\frac{1}{n}+\Delta_{\mathrm{FM}},$$

where

$$\Delta_{\mathrm{FM}}=-\frac{e^{-1}}{n}\sum_{j\ge n+1}\frac{1}{j!}+\frac{e^{-1}}{n}\sum_{j\ge n+2}\frac{(j-n-1)^2}{j!}+\sum_{k>n}\sum_{j>k}\frac{e^{-1}}{j!}.$$

The right-hand side is identically zero and this completes the proof of Theorem 1. ∎

Kolmogorov Distance.

Corollary 4. *The Kolmogorov distance*

$$d_{\mathrm{K}}(\mathscr{L}(\xi_{n,2}),\mathscr{L}(Y_0(1))):=\sup_{k\ge 0}|\mathrm{P}(\xi_{n,2}\le k)-\mathrm{P}(Y_0(1)\le k)|$$

and the total variation distance are identical for $n\ge 2$:

$$d_{\mathrm{K}}(\mathscr{L}(\xi_{n,2}),\mathscr{L}(Y_0(1)))=d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,2}),\mathscr{L}(Y_0(1))).$$

Proof. First we have the (general) inequality

$$d_{\mathrm{K}}(\mathscr{L}(\xi_{n,2}),\mathscr{L}(Y_0(1)))\le d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,2}),\mathscr{L}(Y_0(1))).$$

By (17),

$$|\mathrm{P}(\xi_{n,2}=0)-\mathrm{P}(Y_0(1)=0)|=\frac{e^{-1}}{n}+E_0,$$

$$|\mathrm{P}(\xi_{n,2}\le 1)-\mathrm{P}(Y_0(1)\le 1)|=\frac{e^{-1}}{n}+E_0+E_1.$$

The result then follows from (15). ∎

We actually proved more: namely, the supremum in the definition of d_{K} is attained at $k=0$ for n odd and at $k=1$ for n even.

Remark. From Hwang [15], we have

$$d_{\mathrm{K}}(\mathscr{L}(\xi_n),\mathscr{L}(Y_1(\lambda)))=\frac{\frac{\pi^2}{6}-1}{2\sqrt{2\pi e}\,\log n}\bigl(1+O\bigl((\log n)^{-1/2}\bigr)\bigr).$$

Thus $d_{\mathrm{K}}\sim d_{\mathrm{TV}}/2$.

Point Metric. Still another consequence of our proof of Theorem 1 is the following result for the point metric:

$$d_{\mathrm{L}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) := \sup_{k\geq 0} |\mathrm{P}(\xi_{n,2}=k) - \mathrm{P}(Y_0(1)=k)|.$$

Corollary 5. *The point metric satisfies, for $n \geq 2$,*

$$d_{\mathrm{L}}(\mathscr{L}(\xi_{n,2}), \mathscr{L}(Y_0(1))) = \frac{e^{-1}}{n} + E_0,$$

where $E_0 = (1+1/n)\sum_{j\geq n+2}(-1)^{j+1}/j!$. In particular,

$$|E_0| \leq \frac{(n+1)(n+3)}{n(n+4)} \cdot \frac{1}{(n+2)!}.$$

Thus $d_{\mathrm{L}} = d_{\mathrm{K}} = d_{\mathrm{TV}}$ for odd values of n and $d_{\mathrm{L}} \sim d_{\mathrm{K}} = d_{\mathrm{TV}}$ for n even.

Remark. Again from Hwang [15], we have

$$d_{\mathrm{L}}(\mathscr{L}(\xi_n), \mathscr{L}(Y_1(\lambda))) = \frac{\frac{\pi^2}{6}-1}{2\sqrt{\pi}\,(\log n)^{3/2}}\left(1+O\big((\log n)^{-1/2}\big)\right).$$

Thus d_{L} is much smaller than d_{TV} in this case.

2.3. Equi-Distribution

A bijection proof of $\xi_{n,2} =^{\mathrm{d}} \eta_n$ is easy, but a direct bijection for $\xi_{n,2} =^{\mathrm{d}} \zeta_n$ is much harder. We give instead an inductive bijection.

Bijection of $\xi_{n,2} =^{\mathrm{d}}= \eta_n$. Given a permutation $\pi \in \mathscr{S}_n$, we rewrite it in the following canonical form (cf. [17, pp. 176 et seq.]): write all singletons explicitly; within each cycle, put the largest number first; sort the cycles in increasing order of the first number in the cycle; remove the parentheses. Denote the resulting permutation by π', namely, the result of the procedure is to be considered as the list $(\pi'(1), \ldots, \pi'(n))$. Conversely, each permutation $\pi \in \mathscr{S}_n$ has an "inverse" σ such that $\sigma' = \pi$. In this way, it is easily seen that the number of singletons up to $n-1$ in π is exactly the number of occurrences of two consecutive records in π'.

Inductive Bijection. Note that

$$F_{n,2}(u) = n! \sum_{0\leq j<n} \frac{(u-1)^j}{j!}\left(1-\frac{j}{n}\right)$$

satisfies

$$F_{n,2}(u) = (n+u-1)F_{n-1,2}(u) - (u-1)F'_{n-1,2}(u). \tag{18}$$

We now show that the generating polynomials of η_n and ζ_n both satisfy (18).

Consider first η_n. Assume that $\sigma \in \mathscr{S}_{n-1}$ has k singletons (the singleton $\sigma(n-1) = n-1$ is not counted if any). Write σ in the form of product of cycles. Insert n in the cycles of σ (there being n different possibilities and each cycle having a number equal to its size of producing different cycles) and call the new permutation π. Now *exchange the positions of* n *and* $n-1$ in π. There are three cases:

(a) $\pi(n) = n$. The number of singletons is increased by 1.
(b) $\sigma(j) = j \neq n-1$ and $\pi(j) = n, \pi(n) = j$. The number of singletons is decreased by 1.
(c) The number of singletons remains the same for other cases.

Thus rewriting the equation from σ to π, we have

$$z^k \mapsto z^{k+1} + kz^{k-1} + (n-k-1)z^k$$

corresponding, respectively, to the above cases. This together with the initial conditions proves that the generating polynomial $n!\mathrm{E}(u^{\eta_n})$ satisfies (18).

The proof for ζ_n is much simpler. Indeed, if a random permutation in $\mathscr{S}_{n-1}$ has k successions, then (i) inserting n right after $n-1$ increases the number of successions by 1, (ii) inserting n in a succession reduces the number of successions by 1, and (iii) the number of successions is not changed for other cases.

3. r CONSECUTIVE RECORDS

3.1. Generating Function

We prove Proposition 2 by induction. Let $\pi \in \mathscr{S}_n$ be a generic permutation. We consider two cases. (i) If j, $1 \leq j \leq r$, is the least index for which

$$\pi(n) = n, \quad \pi(n-1) = n-1, \ldots, \pi(n-j+2) = n-j+2,$$
$$\pi(n-j+1) \neq n-j+1,$$

then the total number of r consecutive records of permutations $\in \mathscr{S}_n$ with this pattern is enumerated by $(n-j)F_{n-j,r}(u)$ since there are $n-j$ possible choices for $\pi(n-j+1)$. (ii) On the other hand, if $\pi(n-i) = n-i$ for $i = 0, 1, \ldots, r-1$, then the total number of r consecutive records of permutations $\in \mathscr{S}_n$ with this pattern is given by

$$u\left(F_{n-1,r}(u) - \sum_{2\leq j\leq r} (n-j)F_{n-j,r}(u)\right),$$

by an argument similar to case (i). ∎

3.2. Mean and Variance

We first state a simple lemma.

Lemma 1. *For any integers* $0 \le a \le b$, *the identity*

$$\sum_{b\le j\le n} \frac{(j-a)!}{j!} = \frac{1}{a-1}\left(\frac{(b-a)!}{(b-1)!} - \frac{(n-a+1)!}{n!}\right) \tag{19}$$

holds for $n \ge b$.

Proof (Sketch). For $a > 0$, we use the beta integral

$$\frac{(j-a)!}{j!} = \frac{1}{\Gamma(a)}\int_0^1 t^{a-1}(1-t)^{j-a}\mathrm{d}t.$$

The identity (19) is trivial for $a = 0$. ∎

The mean of $\xi_{n,r}$ is easily obtained from (1) via linearity of expectation and (19)

$$\mathrm{E}(\xi_{n,r}) = \sum_{1\le j\le n-r+1} \frac{1}{j(j+1)\cdots(j+r-1)} = \frac{1}{(r-1)(r-1)!} - \frac{(n-r+1)!}{(r-1)n!}.$$

For the variance, we first apply (6) to obtain a general expression for $\varpi_{n,k}$ defined by

$$f_{n,r}(u) := \mathrm{E}(u^{\xi_{n,r}}) = \sum_{0\le k\le n-r+1} \varpi_{n,k}(u-1)^k.$$

Rewrite (6) as

$$f_{n,r}(u) = f_{n-1,r}(u) + (u-1)\left(\frac{f_{n-1,r}(u)}{n} - \sum_{2\le j\le r}\frac{n-j}{n!}(n-j)!f_{n-j,r}(u)\right).$$

Iterating the recurrence yields

$$f_{n,r}(u) = 1 + \frac{u-1}{r!} + (u-1)\sum_{r<m\le n}\left(\frac{f_{m-1,r}(u)}{m} - \sum_{2\le j\le r}\frac{m-j}{m!}(m-j)!f_{m-j,r}(u)\right),$$

since $f_{r,r}(u) = 1 + (u-1)/r!$. This can be rewritten, using the identity $\sum_{a\le i\le b} i\cdot i! = (b+1)! - a!$ for $b \ge a$, in the more convenient form

$$f_{n,r}(u) = 1 + \frac{u-1}{r!} + (u-1)\sum_{r<m\le n}$$
$$\times\left(\frac{(m-r)!}{m!}f_{m-r,r}(u) + \frac{1}{m!}\sum_{m-r<j<m} j!\left(f_{j,r} - f_{j-1,r}\right)\right).$$

This implies that $\varpi_{n,0} = 1$,

$$\varpi_{n,1} = \frac{1}{(r-1)(r-1)!} - \frac{(n-r+1)!}{(r-1)n!} \qquad (n \ge r-1),$$

and for $k \geq 2$

$$\varpi_{n,k} = \sum_{r<m\leq n} \left(\frac{(m-r)!}{m!} \varpi_{m-r,k-1} + \frac{1}{m!} \sum_{m-r<j<m} j! \Big(\varpi_{j,k-1} - \varpi_{j-1,k-1} \Big) \right). \quad (20)$$

In particular for $k = 2$ and $n \geq 2(r-1)$, we have

$$\begin{aligned}\varpi_{n,2} &= \sum_{r<m\leq n} \left(\frac{(m-r)!}{m!} \left(\frac{1}{(r-1)(r-1)!} - \frac{(m-2r+1)!}{(r-1)(m-r)!} \right) + \frac{1}{m!} \sum_{m-r<j<m} (j-r)! \right) \\ &= \rho_r - \frac{(n-r+1)!}{(r-1)^2(r-1)!n!} + \frac{(n-2r+2)!}{2(r-1)^2 n!} - \sum_{r\leq j\leq 2r-2} \frac{(n-j)!}{j \cdot n!},\end{aligned}$$

where

$$\rho_r = \frac{1}{2(r-1)^2(2r-2)!} + \sum_{r\leq j\leq 2r-2} \frac{1}{j \cdot j!}.$$

The variance of $\xi_{n,r}$ for $n \geq 2(r-1)$ then follows from the right-hand side using the relation

$$\mathrm{Var}(\xi_{n,r}) = 2\varpi_{n,2} - \varpi_{n,1}^2 + \varpi_{n,1}.$$

The other case $n \leq 2r$ is similar. This completes the proof of Theorem 3. ∎

By extending the computations further, we have a general expression for $\varpi_{n,k}$ when $n \geq k(r-1)$.

Proposition 4. *For $r \geq 2$ and $n \geq k(r-1)$*

$$\varpi_{n,k} = \nu_k(r) + \sum_{r-1\leq m\leq k(r-1)} e_{k,m} \frac{(n-m)!}{n!},$$

where

$$e_{k,m} = \begin{cases} -\dfrac{\nu_{k-1}(r)}{r-1}, & \text{if } m = r-1; \\ \dfrac{1}{m} \displaystyle\sum_{r-1\leq j<m} j e_{k-1,j}, & \text{if } r \leq m < 2(r-1); \\ -\dfrac{e_{k-1,m-r+1}}{m} + \dfrac{1}{m} \displaystyle\sum_{j=m-r+1}^{\min\{m-1,(k-1)(r-1)\}} j e_{k-1,j}, & \text{if } 2(r-1) \leq m \\ & \quad \leq k(r-1), \end{cases}$$

and

$$\nu_k(r) = \varpi_{k(r-1),r-1} - \sum_{r<m\leq k(r-1)} e_{k,m} \frac{(k(r-1)-m)!}{(k(r-1))!}.$$

Proof. (Sketch). By induction and (19). ∎

In particular, for $r = 2$, one easily checks that $\nu_k(r) = 1/k!$, $e_{k,1} = -1/(k-1)!$, and $e_{k,m} = 0$ otherwise.

Thus the sequences $\nu_k(r)$ and $e_{k,m}$ can be successively computed by means of the values of $\varpi_{j(r-1),r-1}$. The "full history" recurrence (20) for $\varpi_{j(r-1),r-1}$ is, however, not easily solved. The delicate part is that for general k, $\varpi_{n,k}$ has $k+1$ different expressions depending on which interval (formed by multiples of $r-1$) n lies and each expression contributes a term in the leading coefficient $\nu_k(r)$ of $\varpi_{n,k}$.

In particular, a lengthy computation leads to

$$\varpi_{n,3} = \kappa_r - \frac{\rho_r}{r-1} n^{1-r} + O(n^{-r}),$$

where

$$\kappa_r = \frac{1}{6(r-1)^3(3r-3)!} + \frac{1}{r-1} \sum_{r \le j \le 2r-2} \frac{1}{j(r+j-1)!} + \sum_{r \le j \le 2r-2} \sum_{1 \le \ell \le r-1} \frac{1}{(\ell+j)(\ell+j)!}.$$

3.3. Differential Equation

In this subsection, we apply tools from the theory of ODE to study the probability generating function of the limiting law of $\xi_{n,r}$ for $r \ge 3$.

Before going further, we note that from (7) with $r = 2$, $A'(z)$ can be explicitly solved as

$$A'(z) = \frac{e^{z(u-1)}}{(1-z)^2}.$$

Thus the limiting Poisson distribution of $\xi_{n,2}$ also follows from this explicit solution. For $r = 3$, an explicit solution is also available; see the next section. In general, (7) is harder to solve explicitly for $r \ge 4$.

For notational convenience, we use the symbol Z to denote $1 - z$ throughout this section. It is simpler to consider the following equation

$$Z^{r-1}B^{(r-1)} + Z^{r-2}(r - Z(1-u))B^{(r-2)} + (1-u) \sum_{0 \le j \le r-3} (-1)^{r+j}(j+2-Z)Z^{r-2}B^{(j)} = 0, \tag{21}$$

where $B(Z) = A'(z)$. The indicial polynomial of (21) is (cf. [16])

$$I_0(\rho) = (\rho+2)\rho(\rho-1)\cdots(\rho-r+3) = 0.$$

Note that this algebraic equation is independent of u. Arrange the zeros of $I_0(\rho)$ as

$$\rho_0 = r-3 > \rho_1 = r-4 > \cdots > \rho_{r-4} = 1 > \rho_{r-3} = 0 > \rho_{r-2} = -2.$$

By standard theory in ODE (cf. [16, Chap. 16])

(P1) the point $z = 1$ is a regular singular point and a real singularity; and
(P2) the general solutions of (21) should contain the logarithmic function for $u \neq 1$.

From **(P1)** and the indicial polynomial $I_0(\rho)$, we predict the solution to be of the form

$$B(Z) = \frac{\phi_r(u)}{Z^2} + \mathscr{R}(z),$$

where $\mathscr{R}(z)$ is composed of terms with powers of $\log Z$ and regular functions.

The proof of this prediction is technically more involved because *all roots of the indicial polynomial are integers* (so are their differences) on the one hand, and only $\log Z$ appears in the solution *without higher powers of* $\log Z$ on the other hand. Our proof uses the Frobenius method with suitable modifications (cf. [16, Chap. 16]).

Proposition 5. *For any fixed complex u and $r \geq 3$, the solution to Eq.* (21) *with the initial conditions $B^{(j)}(1) = (-1)^j (j+1)!$, $j = 0, 1, \ldots, r-2$, satisfies*

$$B(Z) = \frac{\phi_r(u)}{Z^2} + \Xi(Z) Z^{r-3} \log Z + \Upsilon(Z), \tag{22}$$

where $\Xi(Z)$ and $\Upsilon(Z)$ are entire functions in Z. In particular, we have

$$\Xi(0) = (-1)^r \frac{(1-u)\phi_r(u)}{(r-1)(r-3)!}. \tag{23}$$

Proof. Let $\tilde{B}(Z, \rho) = \sum_{k \geq 0} c_k(\rho) Z^{k+\rho}$, where $c_0(\rho) \neq 0$. Define the operator

$$\begin{aligned}\mathscr{L} := {} & Z^{r-1} \frac{\mathrm{d}^{r-1}}{\mathrm{d}Z^{r-1}} + Z^{r-2}(r - Z(1-u)) \frac{\mathrm{d}^{r-2}}{\mathrm{d}Z^{r-2}} \\ & + (1-u) \sum_{0 \leq j \leq r-3} (-1)^{r+j} (j + 2 - Z) Z^{r-2} \frac{\mathrm{d}^j}{\mathrm{d}Z^j}.\end{aligned}$$

It is not difficult to show that a solution of $\mathscr{L}[B] = 0$ is of the form $\sum_{k \geq r-3} c_k(\rho) Z^k$ with $c_{r-3}(\rho) \neq 0$, which we denote by $B(Z, \rho_0)$. The precise form of this solution is, however, immaterial. We apply the modified Frobenius method described in [16] to find the other $r-2$ linearly independent solutions. Define $I(\rho) = I_0(\rho+1) \cdots I_0(\rho + r - 1)$. Let $B(Z, \rho) = I(\rho)\tilde{B}(Z, \rho)$ and $J_k(\rho) = I(\rho) c_k(\rho)$ for $k \geq 0$. Then

$$\mathscr{L}[B(Z, \rho)] = c_0(\rho) I_0(\rho) I(\rho) Z^\rho. \tag{24}$$

Define

$$I_j(\rho) = \frac{I_0(\rho)}{\prod_{0 \leq i \leq j-1} (\rho - \rho_i)} = \begin{cases} (\rho+2) \prod_{0 \leq i \leq r-3-j} (\rho - i), & \text{if } 1 \leq j \leq r-2; \\ 1, & \text{if } j = r-1. \end{cases}$$

By substituting the series representation of $B(Z, \rho)$ into (24) and equating the same powers of Z on both sides of (24), we obtain the following recurrence relations for $J_k(\rho)$. For any ρ satisfying $|\rho - \rho_j| < \delta$,

$$\begin{cases} I_0(\rho+1)J_1(\rho) - (1-u)I_1(\rho)J_0(\rho) & = 0, \\ I_0(\rho+2)J_2(\rho) - (1-u)I_1(\rho+1)J_1(\rho) + (1-u)I_2(\rho)J_0(\rho) & = 0, \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\ I_0(\rho+r-1)J_{r-1}(\rho) - (1-u)\Big(I_1(\rho+r-2)J_{r-2}(\rho) & \\ \quad + \cdots + (-1)^{r-2}(1-u)I_{r-2}(\rho+1)J_1(\rho) + (-1)^{r-1}I_{r-1}(\rho)J_0(\rho)\Big) & = 0, \end{cases} \tag{25}$$

and for $s \geq r-1$

$$I_0(\rho+s)J_s(\rho) + (1-u) \sum_{1\leq k\leq r-1} (-1)^k I_k(\rho+s-k)J_{s-k}(\rho) = 0. \tag{26}$$

An important property enjoyed by $I(\rho)$ is the decomposition

$$I(\rho) = K(\rho)(\rho+2)^{r-2} \prod_{1\leq j\leq r-3} (\rho-\rho_j)^j,$$

where $K(\rho)$ has no divisors of the form $\rho - \rho_j$, $j \geq 0$.

With these relations, we now show by induction that

(Q1) $(\rho+2)^{r-2}\prod_{0\leq j\leq r-3}(\rho-\rho_j)^j$ divides $J_k(\rho)$ for $0 \leq k \leq r-2$;
(Q2) $(\rho+2)^{r-3}\prod_{0\leq j\leq r-3}(\rho-\rho_j)^j$ divides $J_k(\rho)$ for $k \geq r-1$.

Assume for each $0 \leq j \leq r-3$, (Q1) and (Q2) hold for $k = r-1$. Then $(\rho - \rho_j)^j(\rho+2)^{r-3}$ divides $J_k(\rho)$ for $0 \leq k \leq r-1$. By (26), $(\rho-\rho_j)^j(\rho+2)^{r-3}$ divides $J_r(\rho)$ since $I_0(\rho+r)$ and $I_{r-1}(\rho)$ have no divisors of the form $\rho - \rho_j$; on the other hand, $(\rho-\rho_j)^j(\rho+2)^{r-3}$ divides $J_k(\rho)$ for $1 \leq k \leq r-1$. By induction on k, we arrive at the assertions (Q1) and (Q2) for $k \geq r$.

It remains to show that (Q1) and (Q2) hold for $k = r-1$. To this aim we first have to show that the same properties hold for J_k for $0 \leq k \leq r-1$ because by (25) $J_k(\rho)$ is recursively defined in nature. By the definition of $J_0(\rho) = c_0(\rho)I(\rho)$, $J_0(\rho)$ satisfies (Q1). For $J_1(\rho)$, by (25),

$$\begin{aligned} J_1(\rho) &= (1-u)\frac{c_0(\rho)I_1(\rho)I(\rho)}{I_0(\rho+1)} = c_0(\rho)(1-u)\frac{(\rho+2)(\rho)_{r-3}I(\rho)}{(\rho+3)(\rho+1)_{r-2}} \\ &= (1-u)K_1(\rho)(\rho+2)^{r-1} \prod_{1\leq j\leq r-3} (\rho-\rho_j)^j, \end{aligned} \tag{27}$$

where $K_1(\rho)$ has no divisors of the form $\rho - \rho_j$, $j \geq 0$. Thus $J_1(\rho)$ satisfies (Q1). Now

$$J_2(\rho) = (1-u)\frac{I_1(\rho+1)J_1(\rho)}{I_0(\rho+2)} - (1-u)\frac{c_0(\rho)I_2(\rho)I(\rho)}{I_0(\rho+2)}.$$

We divide the analysis into two cases: $r = 3$ and $r > 3$. For notational convenience, we use the generic symbol $K_{ij}(\rho)$ to denote polynomials of ρ free from the factors $\rho - \rho_j$. If $r = 3$, then $I_2(\rho) = 1$ and

$$J_2(\rho) = c_0(\rho)(1-u)^2(\rho+2)(\rho+3) - c_0(\rho)(1-u)(\rho+1)(\rho+3).$$

Thus (Q2) holds. If $r > 3$, then

$$\begin{aligned} J_2(\rho) &= (1-u)\frac{(\rho+3)(\rho+1)_{r-3}J_1(\rho)}{(\rho+4)(\rho+2)_{r-2}} - (1-u)c_0(\rho)\frac{(\rho+2)(\rho)_{r-4}I(\rho)}{(\rho+4)(\rho+2)_{r-2}} \\ &= (1-u)\frac{(\rho+3)J_1(\rho)}{(\rho+4)(\rho+2)} - (1-u)\frac{c_0(\rho)I(\rho)}{(\rho+4)(\rho+1)} \\ &= (\rho+2)^{r-2}((1-u)^2K_{21}(\rho) - (1-u)K_{22}(\rho)) \prod_{1\le i\le r-3}(\rho-\rho_i)^i, \end{aligned}$$

which implies (Q1).

We have proved that for $r > 3$ J_1 and J_2 both have the divisors $(\rho + 2)^{r-2}\prod_{1\le j\le r-3}(\rho-\rho_j)^j$. The other cases of (Q1) can be proved similarly.

Divisors differ slightly for $k = r - 1$: by (25), $I_{r-1}(\rho) = 1$ has no longer the factor $\rho + 2$. Therefore, $J_{r-1}(\rho)$ admits the decomposition

$$J_{r-1}(\rho) = (\rho+2)^{r-3}K_{r-1}(\rho) \prod_{1\le i\le r-3}(\rho-\rho_i)^i.$$

With these properties at hand, we can find the other $r - 2$ linear independent solutions of $\mathscr{L}[B] = 0$. In fact, to each $\rho = \rho_j$, $1 \le j \le r-2$, there corresponds a solution of the form

$$\begin{aligned} B(Z, \rho_j) &= \frac{\partial^j}{\partial\rho^j}\left(Z^\rho \sum_{k\ge 0} J_k(\rho)Z^k\right)\Bigg|_{\rho=\rho_j} \\ &= \sum_{k\ge 0} Z^k\left(\sum_{s=0}^{j}\binom{j}{s}J_k^{(s)}(\rho)\frac{\partial^{j-s}}{\partial\rho^{j-s}}Z^\rho\right)\Bigg|_{\rho=\rho_j} \\ &= \sum_{k\ge 0} J_k^{(j)}(\rho_j)Z^{k+\rho_j}, \end{aligned}$$

because $J_k^{(s)}(\rho_j) = 0$ for $0 \le s < j$ by (Q1) and (Q2).

Since ρ_j are nonnegative integers for $0 \le j \le r-3$, we have constructed $r - 2$ entire solutions of the DE $\mathscr{L}[B] = 0$. The last solution is $B(Z, \rho_{r-2}) = B(Z, -2)$. Again by (Q1) and (Q2) of J_k, we have

$$\begin{aligned} B(Z, -2) &= \sum_{k\ge 0} J_k^{(r-2)}(-2)Z^{k-2} + (r-2)\sum_{k\ge r-1} J_k^{(r-3)}(-2)Z^{k-2}\log Z \\ &= \frac{J_0^{(r-2)}(-2)}{Z^2} + R(Z) + (r-2)Z^{r-3}\sum_{k\ge 0} J_{k+r-1}^{(r-3)}(-2)Z^k \log Z, \end{aligned}$$

because by (27) $J_1^{(r-2)}(-2) = 0$. Here $R(Z)$ is entire.

By the uniqueness theorem of ODE, the DE $\mathscr{L}[B] = 0$ with the initial values $B^{(j)}(1) = (-1)^j (j+1)!$, $j = 0, 1, \ldots, r-2$, has a solution of the form

$$B(Z) = \sum_{0 \le i \le r-2} \alpha_i B(Z, \rho_i) = \frac{\phi_r(u)}{Z^2} + \Xi(Z) Z^{r-3} \log Z + \Upsilon(Z),$$

where Υ is entire and α_i are determined by the initial values. In particular, we have

$$\alpha_{r-2} = \frac{\phi_r(u)}{J_0^{(r-2)}(-2)},$$

and hence

$$\Xi(Z) = (r-2)\phi_r(u) \sum_{k \ge 0} \frac{J_{k+r-1}^{(r-3)}(-2)}{J_0^{(r-2)}(-2)} Z^k.$$

By this expression, we have

$$\Xi(0) = (r-2)\phi_r(u) \frac{J_{r-1}^{(r-3)}(-2)}{J_0^{(r-2)}(-2)}. \tag{28}$$

It remains to compute $J_{r-1}^{(r-3)}(-2)$. To this end, we first observe that $(\rho+2)$ divides $I_j(\rho + r - 1 - j)$, $0 \le j \le r-2$. By this and (26) with $s = r-1$,

$$J_{r-1}(\rho) = (\rho+2)^{r-2} K_{r-1,1}(\rho) + (-1)^r (1-u) \frac{J_0(\rho)}{I_0(\rho + r - 1)}.$$

Thus

$$J_{r-1}^{(r-3)}(-2) = (-1)^r (1-u) \frac{\partial^{r-3}}{\partial \rho^{r-3}} \left(\frac{J_0(\rho)}{I_0(\rho + r - 1)} \right) \Bigg|_{\rho=-2}. \tag{29}$$

Let $\widetilde{J}(\rho) = J_0(\rho)/I_0(\rho + r - 1)$. Then $(\rho+2)^{r-3}$ divides $\widetilde{J}(\rho)$ since $\rho + 2$ divides $I_0(\rho + r - 1)$. Thus

$$\begin{aligned}
J_0^{(r-2)}(-2) &= \frac{\partial^{r-2}}{\partial \rho^{r-2}} \left(\widetilde{J}(\rho) I_0(\rho + r - 1) \right) \Bigg|_{\rho=-2} \\
&= \sum_{0 \le s \le r-2} \binom{r-2}{s} \widetilde{J}^{(s)}(-2) \frac{\partial^{r-2-s}}{\partial \rho^{r-2-s}} I_0^{(\rho-2-s)}(\rho + r - 1) \Bigg|_{\rho=-2} \\
&= (r-2) \widetilde{J}^{(r-3)}(-2) \frac{\partial}{\partial \rho} I_0(\rho + r - 1) \Bigg|_{\rho=-2}.
\end{aligned}$$

By (28) and (29), we have

$$\begin{aligned}
\Xi(0) &= \frac{(-1)^r (r-2)(1-u)\phi_r(u)}{(r-2) \frac{\partial}{\partial \rho} I_0(\rho + r - 1)|_{\rho=-2}} \\
&= \frac{(-1)^r (1-u)\phi_r(u)}{(\rho + r + 1) \prod_{3 \le i \le r-1} (\rho + i)|_{\rho=-2}} \\
&= \frac{(-1)^r (1-u)\phi_r(u)}{(r-1)(r-3)!}.
\end{aligned}$$

Finally, when $u = 1$, the solution to (21) is $1/Z^2$ because $\phi_r(u)$ cannot be identically zero for otherwise this would contradict the initial conditions. Thus $\phi_r(1) = 1$.

Since all coefficients of the DE (21) are analytic with respect to u, $\phi_r(u)$ is entire in u. Its Taylor coefficients, being the limits of positive sequences, are *a fortiori* nonnegative. This together with $\phi_r(1) = 1$ proves that $\phi_r(u)$ is an entire probability generating function. This completes the proof of Proposition 5. ∎

3.4. Asymptotics of $\mathrm{E}(u^{\xi_{n,r}})$

We prove Proposition 3 in this subsection.

By Proposition 5, we have

$$A'(z) = \frac{\phi_r(u)}{(1-z)^2} + \Xi(1-z)(1-z)^{r-3}\log(1-z) + \Upsilon(1-z).$$

Since the series $\sum_n \mathrm{E}(u^{\xi_{n,r}})z^n$ converges uniformly for z in the unit circle and satisfies (7), we obtain, by unicity,

$$A(z) = \frac{\phi_r(u)}{1-z} + \Lambda(z)(1-z)^{r-2}\log(1-z) + \Omega(z),$$

where $\Lambda(z)$ and $\Omega(z)$ are entire functions and

$$\Lambda(1) = (-1)^r \frac{u-1}{(r-1)!}\,\phi_r(u).$$

From this and the relation

$$[z^n](1-z)^{k-1}\log(1-z) = \frac{(-1)^k(k-1)!(n-k)!}{n!} \qquad (n > k),$$

where $[z^n]f(z)$ denotes the coefficient of z^n in the Taylor expansion of $f(z)$, we deduce that (cf. [12])

$$\mathrm{E}(u^{\xi_{n,r}}) = \phi_r(u) + (1-u)\phi_r(u)\frac{(n-r+1)!}{(r-1)n!} + O(n^{-r}).$$

This completes the proof. ∎

Note that this expansion as well as more smaller order terms can also be "guessed" from Proposition 4 since $e_{k,r-1} = -\nu_{k-1}(r)/(r-1)$ and $\phi_r(u) = \sum_{k\ge 0} \varpi_{n,k}(u-1)^k$. However, this approach is only formal in nature since we have no *a priori* information on the growth order of $\varpi_{n,k}$ for large k.

3.5. Asymptotics of Probability Metrics

We first derive some properties of the probabilities $\mathrm{P}(\xi_{n,r} = k)$ and then prove Theorem 4 in this subsection.

Monotonicity of $P(\xi_{nr}=k)$. We first observe that the sequence

$$\{\mathrm{P}(\xi_{n,r}=k)\}_{k=0,1,\dots}$$

is nonincreasing for $r \geq 2$. (Switching the last record in the last block of r consecutive records with its preceding record gives rise to a permutation with the number of r consecutive records reduced by unity.) From this we deduce that the same property also holds for the sequence of the limiting probabilities

$$\{\mathrm{P}(\xi_r=k)\}_{k=0,1,\dots}. \tag{30}$$

On the other hand, the sequence

$$\{\mathrm{P}(\xi_{n,r}=k)\}_{r=2,3,\dots}$$

is nondecreasing in r for $k \geq 1$ and nonincreasing for $k=0$. (Switching the last record with its preceding record in each block of consecutive records that may contain up to $r+k$ records produces a permutation with the same number of $r-1$ consecutive records.) This implies that

$$\mathrm{P}(\xi_{n,r}=k) \leq \mathrm{P}(\xi_{n,2}=k) = O(1/k!),$$

uniformly for $1 \leq k \leq n$. Thus the probabilities $\mathrm{P}(\xi_{n,r}=k)$ decrease very fast.

Actually, the probabilities $\mathrm{P}(\xi_{n,r}=k)$ decrease *factorially* fast both in k and in r.

Proposition 6. *The inequality*

$$\mathrm{P}(\xi_{n,r}=k) \geq (k+r-1)\mathrm{P}(\xi_{n,r}=k+1)$$

holds for $k \geq 0$ and $r \geq 2$.

Proof. Assume that $\pi \in \mathscr{S}_n$ and $\xi_{n,r}(\pi)=k+1$. Call the block of consecutive records

$$\pi(i) < \pi(i+1) < \cdots < \pi(i+v)$$

maximal if $\pi(i)$ is a record but $\pi(i-1)$ and $\pi(i+v+1)$ are not. In each maximal block of r consecutive records of π that contains s records, $r \leq s \leq k+r$, we can move any of the first $s-1$ records to the end of the block without changing the relative orders of other elements, the resulting block reducing the number of r consecutive records by unity. The total number of such rearrangements for this block is $s+r-2$. If the sizes of the maximal blocks of r consecutive records of π are $s_1, s_2, \dots, s_m$ (from left to right) then the number of all such rearrangements (of reducing the number of r consecutive records by 1) is

$$\sum_{1 \leq j \leq m} (s_j+r-2) = k+1+m(r-2) \geq k+r-1.$$

This proves the required inequality. ∎

Total Variation Distance. By Proposition 3

$$\mathrm{E}(u^{\xi_{n,r}}) = \phi_r(u) + \frac{1-u}{r-1}\,\phi_r(u)n^{1-r} + T_{n,r}(u), \tag{31}$$

where $T_{n,r}(u) = O(n^{-r})$ uniformly for $|u| \le 2$. As in the proof of Theorem 1, let

$$\delta_k = \mathrm{P}(\xi_{n,r} = k) - \mathrm{P}(\xi_r = k).$$

Then

$$d_{\mathrm{TV}}(\mathscr{L}(\xi_{n,r}), \mathscr{L}(\xi_r)) = \frac{1}{2}\sum_{0\le k\le n-r+1} |\delta_k| + \frac{1}{2}\mathrm{P}(\xi_r > n-r+1).$$

The last term is exponentially small

$$\begin{aligned}\frac{1}{2}\mathrm{P}(\xi_r > n-r+1) &= \frac{1}{4\pi i}\oint_{|u|=2} u^{-n+r-2}(u-1)^{-1}\phi_r(u)\,\mathrm{d}u\\ &= O(2^{-n+r}) \;=\; O(n^{-r}),\end{aligned}$$

since $\phi_r(u)$ is entire.

By taking the coefficient of u^k on both sides of (31), we obtain

$$\delta_k = \frac{n^{1-r}}{r-1}\,(\mathrm{P}(\xi_r = k) - \mathrm{P}(\xi_r = k-1)) + \varepsilon_k, \tag{32}$$

where

$$\varepsilon_k = \frac{1}{2\pi i}\oint_{|u|=2} u^{-k-1} T_{n,r}(u)\,\mathrm{d}u = O(2^{-k}n^{-r}),$$

uniformly for $0 \le k \le n-r+1$. Using the inequality

$$|x| - |y| \le |x+y| \le |x| + |y| \qquad (x, y \in \mathbb{R}),$$

we have

$$\begin{aligned}\frac{1}{2}\sum_{0\le k\le n-r+1}|\delta_k| &= \frac{n^{1-r}}{2(r-1)}\sum_{0\le k\le n-r+1}\bigl|\mathrm{P}(\xi_r = k) - \mathrm{P}(\xi_r = k-1)\bigr|\\ &\quad + O\left(\sum_{0\le k\le n-r+1} 2^{-k}n^{-r}\right)\\ &= \frac{n^{1-r}}{2(r-1)}\left(\mathrm{P}(\xi_r = 0) + \sum_{1\le k\le n-r+1}(\mathrm{P}(\xi_r = k-1) - \mathrm{P}(\xi_r = k))\right)\\ &\quad + O\left(n^{-r}\right)\\ &= \frac{n^{1-r}}{r-1}\,\mathrm{P}(\xi_r = 0) + O\left(n^{-r}\right),\end{aligned}$$

where we used the monotonicity of the sequence (30). ∎

Fortet-Mourier Distance. The proof for the Fortet-Mourier distance is similar, starting from the estimates

$$\begin{aligned}
\mathrm{P}(\xi_{n,r} \le k) - \mathrm{P}(\xi_r \le k) &= \frac{1}{2\pi i}\oint_{|u|=v<1} u^{-k-1}\frac{\mathrm{E}(u^{\xi_{n,r}}) - \phi_r(u)}{1-u}\mathrm{d}u \\
&= \frac{1}{2\pi i}\oint_{|u|=v<1} u^{-k-1}\left(\frac{n^{1-r}}{r-1}\,\phi_r(u) + \frac{T_{n,r}(u)}{1-u}\right)\mathrm{d}u \\
&= \frac{n^{1-r}}{r-1}\mathrm{P}(\xi_r = k) + O(2^{-k}n^{-r}), \qquad (33)
\end{aligned}$$

uniformly for $0 \le k \le n-r+1$, where we used the property that $T_{n,r}(1) = 0$. The remaining analysis is similar and is omitted here.

Kolmogorov Distance. From (33), we deduce the following corollary.

Corollary 6. *For $r \ge 3$,*

$$d_{\mathrm{K}}(\mathscr{L}(\xi_{n,r}), \mathscr{L}(\xi_r)) = \frac{n^{1-r}}{r-1}\mathrm{P}(\xi_r = 0) + O(n^{-r}).$$

Point Metric. Since $\mathrm{P}(\xi_r = 0) > 2/3$, we obtain from (32) the following approximation.

Corollary 7. *For $r \ge 3$,*

$$d_{\mathrm{L}}(\mathscr{L}(\xi_{n,r}), \mathscr{L}(\xi_r)) = \frac{n^{1-r}}{r-1}\mathrm{P}(\xi_r = 0) + O(n^{-r}).$$

4. THREE CONSECUTIVE RECORDS

We solve the DE (7) for $r = 3$ in this section. With slight abuse of notation, we write

$$B(z) = A'(z).$$

Then we are solving the ODE

$$(z-1)B''(z) + (3 + (z-1)U)B'(z) + U(1+z)B(z) = 0,$$

with $B(0) = 1$ and $B'(0) = 2$, where $U = 1-u$.

Define the many valued function $\Psi(a, c; z)$ for c a positive integer by (cf. [10, p. 261])

$$\begin{aligned}
\Psi(a, c; z) &= \frac{(c-2)!}{\Gamma(a)}\sum_{0\le j\le c-2}\frac{(a-c+1)_j z^{j-c+1}}{(2-c)_j j!} \\
&\quad + \frac{(-1)^c}{(c-1)!\Gamma(a-c+1)}\Phi(a, c; z)\log z \\
&\quad + \frac{(-1)^c}{(c-1)!\Gamma(a-c+1)}\sum_{j\ge0}\frac{(a)_j z^j}{(c)_j j!}\left(\psi(a+j) - \psi(1+j) - \psi(c+j)\right),
\end{aligned}$$

where the principal branch is given by $-\pi < \arg z \leq \pi$. Here ψ denotes the logarithmic derivative of the gamma function.

Proposition 7. *With the notations of Theorem* 5, *B satisfies*

$$B(z) = \frac{e^{hz-\Delta}}{2}\,\Gamma(a)\Delta^2\Big(\alpha(u)\Phi(a,3;\Delta(1-z)) + \beta(u)\Psi(a,3;\Delta(1-z))\Big), \quad (34)$$

where

$$\alpha(u) = -(2-a)\Psi(a,2;\Delta) + h\Psi(a,3;\Delta),$$
$$\beta(u) = 2\Phi(a,2;\Delta) - h\Phi(a,3;\Delta).$$

From the local expansion [by (34) and (8) with $c = 3$]:

$$\Psi(a,3;z) = \frac{z^{-2}}{\Gamma(a)} - \frac{a-2}{\Gamma(a)}z^{-1} - \frac{\log z}{2\Gamma(a-2)} + O(|z||\log z|),$$

as $z \to 0$, $z \in \mathbb{C} \setminus \{z : -\infty < z \leq 0\}$, it follows, by singularity analysis (cf. [12]), that

$$[z^n]B(z) = \frac{e^{h-\Delta}}{2}\,\beta(u)\left(n + 1 - h - (a-2)\Delta + \frac{(a-1)(a-2)\Delta^2}{2n} + O(n^{-2})\right).$$

Note that $h = (2-a)\Delta$. This together with the relation

$$\mathrm{E}(u^{\xi_{n,r}}) = [z^n]A(z) = \frac{1}{n}[z^{n-1}]B(z)$$

yields

$$\mathrm{E}(u^{\xi_{n,r}}) = \phi_3(u)\left(1 + \frac{(a-1)(a-2)\Delta^2}{2n^2} + O\left(n^{-3}\right)\right)$$

uniformly for finite u. Note that $(a-1)(a-2)\Delta^2 = 1 - u$. This is in accordance with Proposition 3.

Proof of Proposition 7. We follow the standard procedure for solving a second-order ODE with linear coefficients and sketch only the derivations; see, for example, [10, Section 6.2] or [21, pp. 90 et seq.] for details.

Let $B(z) = e^{h\lambda w}C(w)$, where $w = (z-1)/\lambda$. Here the two constants h and λ are chosen to satisfy the equations

$$\begin{cases} h^2 + Uh + U = 0, \\ (2h + U)\lambda = -1, \end{cases}$$

so that C satisfies the DE of confluent hypergeometric type:

$$wC''(w) + (3-w)C'(w) - aC(w) = 0.$$

Thus

$$h = \frac{-U \pm \sqrt{4U - U^2}}{2} = \frac{\Delta}{2} - 1 + \sqrt{1 + \frac{\Delta^2}{4}},$$

and $\lambda = -1/\Delta$. If $\Delta \neq 0$, then the solution of C is of the form

$$C(w) = K_1\Phi(a, 3; w) + K_2\Psi(a, 3; w),$$

or, equivalently,

$$B(z) = e^{h(z-1)}\Big(K_1\Phi(a, 3; \Delta(1-z)) + K_2\Psi(a, 3; \Delta(1-z))\Big).$$

The two constants K_1, K_2 are determined by the initial conditions $B(0) = 1$ and $B'(0) = 2$, giving

$$\begin{cases} K_1 = -\dfrac{e^h((2-h)\Psi(a,3;\Delta) + \Delta\Psi'(a,3;\Delta))}{\Delta(\Psi(a,3;\Delta)\Phi'(a,3;\Delta) - \Phi(a,3;\Delta)\Psi'(a,3;\Delta))}, \\ K_2 = \dfrac{e^h((2-h)\Phi(a,3;\Delta) + \Delta\Phi'(a,3;\Delta))}{\Delta(\Psi(a,3;\Delta)\Phi'(a,3;\Delta) - \Phi(a,3;\Delta)\Psi'(a,3;\Delta))}. \end{cases}$$

Applying the formula for the Wronskian of Φ and Ψ (cf. [10, p. 259]):

$$\Delta(\Psi(a,3;\Delta)\Phi'(a,3;\Delta) - \Phi(a,3;\Delta)\Psi'(a,3;\Delta)) = \frac{2e^\Delta}{\Gamma(a)\Delta^2},$$

we obtain (34). The solution is easily seen to hold also for $\Delta = 0$ by continuity. ▮

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