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Graph homotopy and Graham homotopy

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Abstract

Simple-homotopy for cell complexes is a special type of topological homotopy constructed by elementary collapses and elementary expansions. In this paper, we introduce graph homotopy for graphs and Graham homotopy for hypergraphs and study the relation between the two homotopies and the simple-homotopy for cell complexes. The graph homotopy is useful to describe topological properties of discretized geometric figures, while the Graham homotopy is essential to characterize acyclic hypergraphs and acyclic relational database schemes. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Contractible transformations on graphs were introduced in [4,5] to study molecular spaces. The simplest model of a so-called molecular space is a family of unit cubes in a Euclidean space. These models are useful in digital topology for image processing and computer graphics. Therefore, the study of combinatorial, topological, and geometric properties of molecular spaces should be useful to understand molecular objects. It has been illustrated that, for a continuous bounded closed surface, the induced intersection graph from the molecular space usually preserves essential topological properties of the original surface. For example, given two topologically equivalent bounded closed surfaces Σ_1 and Σ_2 of $\mathbf{R}^3$, if $\mathbf{R}^3$ is divided into a set of congruent cubes by coordinate hyperplanes, we have one family, L_1, of cubes intersecting Σ_1 and another family, L_2, of cubes intersecting Σ_2. If the division is refined enough, the intersection graphs $G(L_1)$ and $G(L_2)$ will be graph homotopy equivalent, i.e., $G(L_1)$ can be transformed into $G(L_2)$ by a sequence of contractible graph transformations (see below). This means that some topological properties of the continuous surfaces have been encoded

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into the molecular spaces. Thus it should be interesting and important to find out the topological properties that are preserved by the induced molecular spaces. In a series of papers, Ivashchenko and Yeh studied some of these preserved properties such as the Euler characteristic and homology groups, see [4–8,16]. They showed that graph transformations do not change the Euler characteristic and the homology group of graphs for some special cases. This leads us to ask whether the graph transformations are actually topological homotopy equivalence. The first half of this paper is to show that a contractible graph transformation can be decomposed into a sequence of elementary regular cell expansions and elementary regular cell collapses on certain associated simplicial complexes, whereby the graph homotopy is reduced to a special type of simple-homotopy. Since simple-homotopy equivalence preserves "homotopy groups", and, of course, it also preserves "homology groups", all results in [4,5] can be derived from the present work.

The graph homotopy and simple-homotopy are closely related to the Graham reduction for hypergraphs, which was originally introduced to define acyclic hypergraphs and acyclic database schemes, see [12]. The importance of acyclic database schemes lies in the existence of their information-lossless decomposition, see [9,10]. Recall that the second operation in the definition of the Graham reduction is to remove the hyperedges contained in another hyperedge. For this reason, we associate a simplicial complex to each hypergraph by including all nonempty subsets of any hyperedge. With this association, the first operation in the Graham reduction corresponds to a special type of simple-homotopy on the associated simplicial complexes; we call this special type of simple-homotopy the *Graham homotopy*. In the second half of this paper, we present a topological interpretation for the Graham reduction and derive a local formula for counting the number of cycles of a hypergraph in terms of the associated simplicial complex.

The acyclic hypergraphs and relational database schemes were introduced easily by the Graham reduction. However, the concepts of cycles and independent cycles for hypergraphs and relational database schemes are still unclear. We suspect that for some combinatorial optimization problems, it is the number of cycles of certain associated hypergraphs that determines the computational complexity. For instance, the satisfiability problem may be transformed into a problem on an induced hypergraph and its computational complexity will be reduced to the cycle structures of the induced hypergraph. The detailed exposition of cycle structures for hypergraphs will be given elsewhere.

2. Graph homotopy and simple-homotopy

Let $G=(V,E)$ be a simple graph, i.e., a graph without loops and multiple edges, where V is the vertex set and E the edge set; we always assume that V is finite. For each vertex v of G, let $N(v,G)$ denote the set of vertices of G that are adjacent to v; the *graph link* of v in G, denoted $L(v,G)$, is the subgraph of G induced by the vertex set $N(v,G)$. For a subgraph G' of G, the *joint graph link* of G' in G, denoted $L(G',G)$, is the subgraph of G induced by the vertex set

$$N(G',G) = \bigcap_{v \in V(G')} N(v,G).$$

Please notice the difference between the graph link we defined for graphs and the topological link defined for simplicial complexes in combinatorial topology. We now define contractible graphs inductively by gluing and deleting vertices and edges as follows: (1) the graph of a single vertex is *contractible*; (2) a graph is called *contractible* if it can be obtained from a contractible graph by a sequence of the following graph operations.

(GO1) Deleting a vertex: A vertex v of a graph G can be deleted if its graph link $L(v,G)$ is contractible;

(GO2) Gluing a vertex: If G' is a contractible subgraph of G, then a vertex v not in G can be glued to G to produce a new graph G'' so that the graph link $L(v,G'')$ is G';

(GO3) Deleting an edge: An edge uv of G can be deleted if the joint graph link $L(uv,G) = L(u,G) \cap L(v,G)$ is contractible;

(GO4) Gluing an edge: For two non-adjacent vertices u and v of G, the edge uv can be glued to G if the joint graph link $L(uv,G)$ is contractible.

By definition, all complete graphs are contractible. It is easy to check that chordal graphs are contractible, see [16]. Two graphs are said to be *graph homotopy equivalent* if one can be obtained from the other by a sequence of graph operations (GO1–GO4). It is clear that the graph homotopy defines an equivalence relation on the class of graphs; therefore, graphs are classified into graph homotopy classes. This classification of graphs may be related to the classification of topological spaces by certain topological transformations. To see this relationship, we associate with each graph a simplicial complex.

Let us recall that an abstract *simplicial complex* over a finite set V is a collection K of nonempty subsets of V, called (*open*) *simplices* or *cells*, such that $V = \bigcup_{\sigma \in K} \sigma$, and for each $\sigma \in K$, if $\rho \subset \sigma$ and $\rho \neq \emptyset$, then $\rho \in K$. An open simplex σ is called a *face* of an open simplex τ if $\sigma \subset \tau$, denoted $\sigma \leqslant \tau$ or $\tau \geqslant \sigma$; σ is called a *facet* of τ if $\tau = \sigma \cup \{v\}$ for some $v \notin \sigma$. The geometric realization $|K|$ is the metric space of nonnegative real-valued functions f on V such that there is an open simplex σ so that $\sum_{v \in \sigma} f(v) = 1$ and $f(v) > 0$ if and only if $v \in \sigma$; and the metric is given by

$$d(f,g) = \left[\sum_{v \in V} (f(v) - g(v))^2 \right]^{1/2},$$

see [14]. A *simplicial map* from a simplicial complex K_1 over V_1 to a simplicial complex K_2 over V_2 is a map $f : V_1 \to V_2$ such that if σ is an open simplex of K_1, then $f(\sigma)$ is an open simplex of K_2. A *graph homomorphism* from a graph G_1 to a graph G_2 is a map $f : V(G_1) \to V(G_2)$ such that for each edge uv of G_1, $f(u)f(v)$

is an edge of G_2. Given a graph G, let $\Delta(G)$ denote the collection of all complete subgraphs of G; it is clear that $\Delta(G)$ is a simplicial complex, called the *clique complex* of G (for complete subgraphs are also called *cliques* in graph theory).

Let us denote by $\mathscr{G}$ the category of graphs with graph homomorphisms and by $\mathscr{K}$ the category of simplicial complexes with simplicial maps. If $f: G_1 \to G_2$ is a graph homomorphism, then for any complete subgraph K_i of i vertices in G_1, its image $f(K_i)$ is a complete subgraph in G_2; so f induces a simplicial map $\Delta_f: \Delta(G_1) \to \Delta(G_2)$, given by $\Delta_f(K_i) = f(K_i)$. Therefore Δ defines a functor from $\mathscr{G}$ to $\mathscr{K}$.

Conversely, given a simplicial complex K. Let $\mathrm{sk}^i(K)$ denote the i-dimensional skeleton of K, i.e., $\mathrm{sk}^i(K) = \{\sigma \in K\colon \sigma \text{ has at most } i+1 \text{ elements}\}$, $i \geqslant 0$. The 0-skeleton together with the 1-skeleton give rise to a graph $\mathrm{sk}(K) = (\mathrm{sk}^0(K), \mathrm{sk}^1(K) - \mathrm{sk}^0(K))$, where the vertex set is the 0-skeleton and the edge set is the pure 1-skeleton. Then sk defines a functor from $\mathscr{K}$ to $\mathscr{G}$. For each graph G, it is clear that $\mathrm{sk}\Delta(G) = G$. However, we do not have $\Delta\mathrm{sk}(K) = K$ for every simplicial complex K. For instance, if K is the boundary of a tetrahedron, i.e., K consists of all nonempty subsets of V, but not V itself, then $\mathrm{sk}(K)$ is the complete graph K_4; so $\Delta(K_4)$ represents a solid tetrahedron, including the largest cell V. Of course, $\Delta\mathrm{sk}(K) = \Delta(K_4)$ is different from K. Nevertheless, if we take the first barycentric subdivision $\mathrm{sd}K$ of K, it is easy to see that $\Delta\mathrm{sk}(\mathrm{sd}K) = \mathrm{sd}K$. We state this as the following proposition.

Proposition 2.1. *The map Δ is a functor from the category $\mathscr{G}$ of graphs to the category $\mathscr{K}$ of simplicial complexes, and the map* sk *is a functor from $\mathscr{K}$ to $\mathscr{G}$. Moreover, for any graph G and any simplicial complex K,* $\mathrm{sk}\Delta(G) = G$ *and* $\Delta\mathrm{sk}(\mathrm{sd}K) = \mathrm{sd}K$.

In order to present a topological interpretation for graph homotopy, we need the concept of simple-homotopy for simplicial complexes. Let K be a simplicial complex. A *face pair* is an ordered pair (σ,τ) of cells such that σ is a facet of τ; (σ,τ) is called *simplicially collapsible* in K if both σ and τ are cells of K and σ cannot be a face of any cell of K other than τ; and (σ,τ) is called *simplicially expandable* to K if σ and τ are not cells of K, but all faces of τ other than σ are contained in K. If a simplicial complex L can be obtained from K by a sequence of elementary simplicial collapses and elementary simplicial expansions (see below), we say that K is *simplicially simple-homotopy equivalent to L*.

(SC) Elementary simplicial collapse: A face pair (σ,τ) collapsible in K can be deleted;
(SE) Elementary simplicial expansion: A face pair (σ,τ) expandable to K can be added to K.

We say that K *collapses simplicially to L* or L *expands simplicially to K*, written $K \searrow L$ or $L \nearrow K$, if L can be obtained from K by a sequence of only elementary simplicial collapses, see Fig. 1. It is easy to see that every simplicial cone collapses to a point.

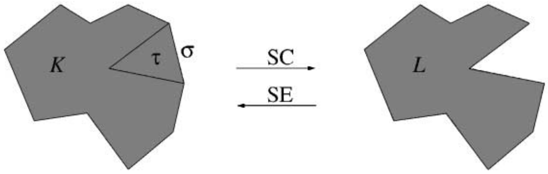


Fig. 1. K collapses to L and L expands to K.

As pointed out by Whitehead [2], it is technically difficult to treat simple-homotopy in the context of simplicial complexes. The difficulty might be the inflexibility of constructing new cells without introducing new vertices in the category of simplicial complexes. For this reason, we need cell complexes and regular cell complexes. A *cell complex* K is a finite collection of disjoint open cells of a compact topological space X, such that $X = \bigcup_{\sigma \in K} \sigma$ and for any two cells σ and τ, if $\sigma \cap \bar{\tau} \neq \emptyset$, then $\sigma \subset \bar{\tau}$; and in this case, we say that σ is *face* of τ, written $\sigma \leqslant \tau$ or $\tau \geqslant \sigma$. A face σ of τ is called a *facet* of τ if $\dim \sigma = \dim \tau - 1$; and in this case (σ, τ) is called a *face pair*. A face pair (σ, τ) in K is called *collapsible* if σ is not a proper face of any cell in K other than τ. For a collapsible face pair (σ, τ) in K, let L denote the cell subcomplex of K without the cells σ and τ; we say that K *collapses to* L by removing (σ, τ) from K and L *expands to* K by adding (σ, τ) to L, written $K(\sigma, \tau)^{-1} = L$ or $K = L(\sigma, \tau)$; removing (σ, τ) from K is called an *elementary cell expansion* and adding (σ, τ) to L is called an *elementary cell collapse*. Two cell complexes K and L are said to be *simple-homotopy equivalent* if one can be obtained from the other by a sequence of elementary cell expansions and elementary cell collapses, written $K \cong L$.

A cell complex K is called *regular* if every closed cell $\hat{\sigma}$ of K is homeomorphic to a closed simplex and the homeomorphism induces an isomorphism between their face orderings. An elementary cell expansion (collapse) (σ, τ) is called an *elementary regular cell expansion* (*collapse*) if the closed cell $\hat{\tau}$ is homeomorphic to a closed simplex and the homeomorphism induces an isomorphism between their face orderings. Similarly, *regular cell expansions*, *regular cell collapses*, and *regular simple-homotopy equivalence* are defined in an obvious way for regular cell complexes.

It is clear that simple-homotopy for cell complexes is a special type of topological homotopy. In particular, elementary expansion (collapse) preserves topological invariants. The following lemmas and propositions will be needed in the next section.

Lemma 2.2. *Let K and L be* (*regular*) *cell complexes such that $K(\sigma_1, \tau_1)^{-1}(\sigma_2, \tau_2) = L$, i.e.*, *$(\sigma_1, \tau_1)$ is* (*regularly*) *collapsible in K and (σ_1, τ_2) is* (*regularly*) *expandable to L.*

(a) *If $(\sigma_1, \tau_1) = (\sigma_2, \tau_2)$, then $K = L$.*
(b) *If $(\sigma_1, \tau_1) \neq (\sigma_2, \tau_2)$, then $K(\sigma_2', \tau_2')(\sigma_1, \tau_1)^{-1} \cong L$, where (σ_2', τ_2') is a face pair* (*regularly*) *expandable to K and the faces of τ_2' other than σ_2' are the same faces of τ_2 other than σ_2.*

Proof. The cells $\sigma_1, \tau_1, \sigma_2, \tau_2$ satisfy the properties: (i) σ_1 and τ_1 are cells of K; (ii) σ_1 is not a face of τ_2 unless $\sigma_1 = \sigma_2$ or $\sigma_1 = \tau_2$, τ_1 is not face of τ_2 unless $\tau_1 = \sigma_1$ or $\tau_1 = \tau_2$; and (iii) σ_2 is not a cell of K unless $\sigma_2 = \sigma_1$ or $\sigma_2 = \tau_1$, τ_2 is not a cell of K unless $\tau_2 = \sigma_1$ or $\tau_2 = \tau_1$. We divide our argument into the following cases.

Case 1: τ_1 is a face of τ_2. Obviously, σ_1 is a proper face of τ_2. If $\tau_1 = \tau_2$, then by (ii), $\sigma_1 = \sigma_2$; hence $K = L$. If τ_1 is a proper face τ_2, then by (ii), both σ_1 and τ_1 must be the same as σ_2, a contradiction. This means that this latter subcase does not exist.

Case 2: both τ_1 and σ_1 are not faces of τ_2. Then by (iii), σ_2 and τ_2 are not cells of K; and by the expansion property of (σ_2, τ_2) to $K(\sigma_1, \tau_1)^{-1}$, the faces of τ_2 other than σ_2 are cells of K. Thus (σ_2, τ_2) is expandable to K. Similarly, σ_1 and τ_1 are cells of $K(\sigma_2, \tau_2)$ by (i); and by collapsibility of (σ_1, τ_1) in K, the cell σ_1 is not a proper face of any cell of K except τ_1; so σ_1 is not a proper face of any cell of $K(\sigma_2, \tau_2)$, but τ_1. Therefore (σ_1, τ_1) is collapsible in $K(\sigma_2, \tau_2)$. Thus $K(\sigma_2, \tau_2)(\sigma_1, \tau_1)^{-1} = L$.

Case 3: τ_1 is not a face of τ_2, but σ_1 is a face of τ_2. It follows from (ii) that σ_1 is either σ_2 or τ_2. Assume $\sigma_1 = \tau_2$, then τ_2 is a cell of K and σ_2 is a facet of σ_1. However, by (iii), σ_2 must be either σ_1 or τ_1, contrary to $\sigma_2 < \sigma_1$. This means that we must have $\sigma_1 = \sigma_2$. Of course, in this case, $\tau_2 \neq \sigma_1$ and $\tau_2 \neq \tau_1$. Then by (iii), τ_2 is not a cell of K. Note that $\sigma_1 (= \sigma_2)$ is a facet of both τ_1 and τ_2. Let (σ_2', τ_2') be a new copy of (σ_2, τ_2), such that the faces of τ_2' other than σ_2' are the same faces of τ_2 other than σ_2; σ_2' and τ_2' are not cells of K and distinct from τ_2 and σ_2, respectively. Now it is clear that (σ_2', τ_2') is expandable to K, the pair (σ_1, τ_1) is collapsible from $K(\sigma_2', \tau_2')$, and $K(\sigma_2', \tau_2')(\sigma_1, \tau_1)^{-1} \cong L$. □

Proposition 2.3. *Let K and L be* (*regular*) *cell complexes. If K and L are* (*regularly*) *simple-homotopy equivalent*, *then L can be obtained from K by doing all expansions before collapses*, *i.e.*,

$$L = K(\sigma_1, \tau_1) \cdots (\sigma_m, \tau_m)(\sigma_1', \tau_1')^{-1} \cdots (\sigma_n', \tau_n')^{-1}. \quad (1)$$

Moreover, $\#(L) = \#(K) + 2m - 2n$.

Proof. Let $L = K(\sigma_1'', \tau_1'')^{\varepsilon_1} \cdots (\sigma_k'', \tau_k'')^{\varepsilon_k}$, where $\varepsilon_i = \pm 1$. Whenever there are two consecutive face pairs such that $(\sigma_i'', \tau_i'')^{-1}(\sigma_{i+1}'', \tau_{i+1}'')$ appears in the sequence of expansions and collapses, we apply Lemma 2.2 to switch the order of the two pairs and rename the cells if necessary. Expression (1) for L can be attained by switching the order of such pairs finitely many times. □

Let K and L be cell complexes. If K and L are simple-homotopy equivalent, we define

$$\ell(K, L) = \min\{m : L = K(\sigma_1, \tau_1) \cdots (\sigma_m, \tau_m)(\sigma_1', \tau_1')^{-1} \cdots (\sigma_n', \tau_n')^{-1}\},$$

called the *simple-homotopy distance* from K to L; otherwise we define $\ell(K, L) = \infty$. It follows from Proposition 2.3 that

$$\ell(L, K) = \ell(K, L) + \frac{\#(K) - \#(L)}{2}.$$

Similarly, if K and L are regular cell complexes and are regularly simple-homotopy equivalent, we define $\ell_r(K,L)$ to be the minimum number of elementary regular cell expansions in all sequences of elementary regular cell expansions and elementary regular cell collapses from K to L; otherwise define $\ell_r(K,L)=\infty$.

Proposition 2.4. *The set $\mathscr{C}$ of all finite cell complexes is a metric space with the metric $d(K,L)=\ell(K,L)+\ell(L,K)$; and the set $\mathscr{C}_r$ of all finite regular cell complexes is a metric space with the metric $d_r(K,L)=\ell_r(K,L)+\ell_r(L,K)$.*

Proof. It is obvious that $d(K,K)=0$ and $d(K,L)=d(L,K)$ for $K,L\in\mathscr{C}$. Now it suffices to show that $\ell(K,M)\leqslant\ell(K,L)+\ell(L,M)$ for cell complexes K, L and M. If K is not simple-homotopy equivalent to L or L is not simple-homotopy equivalent to M, then the inequality $\ell(K,M)\leqslant\ell(K,L)+\ell(L,M)$ holds automatically. Assume $L=K\prod_{i=1}^{m}(\sigma_i,\tau_i)\prod_{j=1}^{n}(\sigma_j',\tau_j')^{-1}$ and $M=L\prod_{i=1}^{p}(\alpha_i,\beta_i)\prod_{j=1}^{q}(\alpha_j',\beta_j')^{-1}$, then

$$M=K\prod_{i=1}^{m+p}(\sigma_i,\tau_i)\prod_{j=1}^{n+q}(\sigma_j',\tau_j')^{-1},$$

where $(\sigma_i,\tau_i)\cong(\alpha_{i-m},\beta_{i-m})$ for $i>m$ and $(\sigma_j',\tau_j')=(\alpha_{j-n}',\beta_{j-n}')$ for $j>n$. It is clear that $\ell(K,M)\leqslant m+p=\ell(K,L)+\ell(L,M)$. The proof for $(\mathscr{C}_r,d_r)$ to be a metric space is similar. $\square$

3. Graph homotopy reduction

We shall show in this section that the graph homotopy can be reduced to simple-homotopy on regular cell complexes. To this end, we first state and prove a result about simple-homotopy; then we show that the edge deletion and gluing can be realized by a vertex gluing and a vertex deletion; and finally we reduce the vertex gluing (deletion) as a sequence of elementary regular cell expansions and elementary regular cell collapses.

Let σ be a cell of a cell complex K and v a vertex not in K. If σ is a singleton $\{v_0\}$, the *join* of v_0 and v is an open segment between v_0 and v, denoted v_0*v. If σ is not a singleton, the *join* of σ and v, denoted $\sigma*v$, is inductively defined to be an open cell whose proper faces are the joint cells $\rho*v$ ($p<\sigma$), the singleton $\{v\}$, and all faces of σ.

Lemma 3.1. *Let K be a* (*regular*) *cell complex and L a subcomplex of K. If*

$$L=\{v_0\}(\sigma_1,\tau_1)\cdots(\sigma_m,\tau_m),$$

where v_0 is a vertex of L, then for a vertex v not in K, $K\cup L*v$ is (*regularly*) *simple-homotopy equivalent to K. Moreover,*

$$K\cup L*v=K(v,v_0*v)(\sigma_1*v,\tau_1*v)\cdots(\sigma_m*v,\tau_m*v).$$

Proof. Set $L_i=\{v_0\}(\sigma_1,\tau_1)\cdots(\sigma_i,\tau_i)$, where $0\leqslant i\leqslant m$ and $L_0=\{v_0\}$. We proceed induction on m. For $m=0$, we have $L=\{v_0\}$; it is obvious that $K\cup L*v=$

$K(v, v_0 * v)$. For $m > 0$, by the induction hypothesis, we have $K \cup L_{m-1} * v = K(\sigma_1 * v, \tau_1 * v) \cdots (\sigma_{m-1} * v, \tau_{m-1} * v)$. Let σ' be any face of $\tau_m * v$ other than $\sigma_m * v$. Then $\sigma' = \sigma * v$ for some face σ of τ_m other than σ_m. Since (σ_m, τ_m) is expandable to L_{m-1}, the cell σ must be contained in L_{m-1}; so $\sigma'(= \sigma * v)$ is a cell of $K \cup L_{m-1} * v$. On the other hand, σ_m is not a cell of L_{m-1}; so $\sigma_m * v$ is not a cell of $K \cup L_{m-1} * v$. This means that $(\sigma_m * v, \tau_m * v)$ is expandable to $K \cup L_{m-1} * v$. Clearly, $K \cup L * v = (K \cup L_{m-1} * v)(\sigma_m * v, \tau_m * v)$. □

Theorem 3.2. *Let K be a (regular) cell complex and L a subcomplex of K. If*

$$L = \{v_0\}(\sigma_1, \tau_1) \cdots (\sigma_m, \tau_m)(\sigma'_1, \tau'_1)^{-1} \cdots (\sigma'_n, \tau'_n)^{-1},$$

where v_0 is a vertex of L, then for any vertex v not in K, $K \cup L * v$ is (regularly) simple-homotopy equivalent to K. Moreover,

$$\begin{aligned} K \cup L * v = K&(\sigma'_n, \tau'_n) \cdots (\sigma'_1, \tau'_1)(v, v_0 * v) \\ &(\sigma_1 * v, \tau_1 * v) \cdots (\sigma_m * v, \tau_m * v) \\ &(\sigma'_1 * v, \tau'_1 * v)^{-1}(\sigma'_1, \tau'_1)^{-1} \cdots \\ &(\sigma'_n * v, \tau'_n * v)^{-1}(\sigma'_n, \tau'_n)^{-1}. \end{aligned}$$

Proof. Note that $L(\sigma'_n, \tau'_n) \cdots (\sigma'_1, \tau'_1) = \{v_0\}(\sigma_1, \tau_1) \cdots (\sigma_m, \tau_m)$. We may assume that the cells $\sigma'_1, \ldots, \sigma'_n, \tau'_1, \ldots, \tau'_n$ are not contained in K; this can be done by renaming the cells if necessary. We proceed induction on n. For $n = 0$, there is nothing to prove; it is the case of Lemma 3.1. For $n > 0$, set $L' = L(\sigma'_n, \tau'_n)$ and $K' = K \cup L'$. Then

$$L' = \{v_0\}(\sigma_1, \tau_1) \cdots (\sigma_m, \tau_m)(\sigma'_1, \tau'_1)^{-1} \cdots (\sigma'_{n-1}, \tau'_{n-1})^{-1}.$$

By the induction hypothesis, we have

$$\begin{aligned} K' \cup L' * v = K'&(\sigma'_{n-1}, \tau'_{n-1}) \cdots (\sigma'_1, \tau'_1)(v, v_0 * v) \\ &(\sigma_1 * v, \tau_1 * v) \cdots (\sigma_m * v, \tau_m * v) \\ &(\sigma'_1 * v, \tau'_1 * v)^{-1}(\sigma'_1, \tau'_1)^{-1} \cdots \\ &(\sigma'_{n-1} * v, \tau'_{n-1} * v)^{-1}(\sigma'_{n-1}, \tau'_{n-1})^{-1}. \end{aligned}$$

Notice that $K' = K(\sigma'_n, \tau'_n)$ and $K' \cup L' * v = (K \cup L * v)(\sigma'_n, \tau'_n)(\sigma'_n * v, \tau'_n * v)$. The required result follows by substitution. □

Next, we show that the edge deletion and the edge gluing can be realized by a vertex gluing and a vertex deletion. This is done by the following two lemmas.

Lemma 3.3. *For any graph G and a vertex v not in G, the* cone graph *$G * v$ is contractible, where $V(G * v) = V(G) \cup \{v\}$ and $E(G * v) = E(G) \cup \{uv\colon u \in V(G)\}$.*

Proof. We proceed induction on the number of vertices of G. It is obviously true when G has only one vertex. Suppose it is true for any graph with k or less vertices. Now

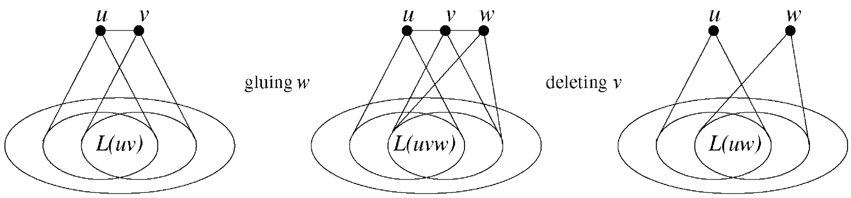


Fig. 2. Edge deletion is realized by vertex gluing and vertex deletion.

consider an arbitrary graph G with $k+1$ vertices. Let u be a vertex of G; the cone graphs $L(u,G)*v$ and $(G-u)*v$ are contractible by the induction hypothesis. Notice that $L(u,G)*v$ is contained in $(G-u)*v$. Then $G*v$ can be obtained by gluing u to $L(u,G)*v$ in $(G-u)*v$. Thus $G*v$ is contractible by definition of vertex gluing. □

Lemma 3.4. *The edge deletion* (*gluing*) *can be realized by the composition of a vertex gluing* (*deletion*) *and a vertex deletion* (*gluing*).

Proof. Let G be a graph with adjacent vertices u and v. If the joint graph link $L(uv,G)$ is contractible, we need to find some vertex deletion and gluing to have the edge uv removed from G. Let w be a vertex not in G. Since $(L(v,G)-u)*v$ is a contractible subgraph of G, we can glue w to $(L(v,G)-u)*v$ in G. Notice that both $L(uv,G)$ and $(L(v,G)-u)*w$ are contractible and $L(uv,G)$ is contained in $(L(v,G)-u)*w$. Then $L(uv,G)*u\cup(L(v,G)-u)*w$ is contractible, because it can be obtained by gluing u to $L(uv,G)$ in $(L(v,G)-u)*w$. It is clear that the graph link of v in $G\cup((L(v,G)-u)*v)*w$ is $L(uv,G)*u\cup(L(v,G)-u)*w$, which has been shown to be contractible. Thus v can be removed by a vertex deletion. Rename the vertex w as v. We have removed the only edge uv by a vertex gluing and a vertex deletion, see Fig. 2. □

The edge gluing is similar to the edge deletion, just reversing the procedure.

Theorem 3.5. *Let G be a graph and let G' be a subgraph of G. If $\Delta(G')$ is regularly simple-homotopy equivalent to a point, then for a vertex v not in G, the simplicial complex $\Delta(G)$ is regularly simple-homotopy equivalent to $\Delta(G\cup G'*v)$.*

Proof. Let $K=\Delta(G)$ and $L=\Delta(G')$. Then $K\cup L*v=\Delta(G\cup G'*v)$. By Theorem 3.2, $\Delta(G\cup G'*v)$ is regularly simple-homotopy equivalent to $\Delta(G)$. □

Corollary 3.6. *If G is a contractible graph, then $\Delta(G)$ is regularly simple-homotopy equivalent to a point.*

Proof. Let $T_i^{\varepsilon_i}$ $(1\leqslant i\leqslant n)$ denote the vertex gluing or vertex deletion such that $\{v\}T_1^{\varepsilon_1}\cdots T_n^{\varepsilon_n}=G$, where v is a vertex (may or may not be in G), $\varepsilon_i=\pm 1$, $T_i^{\varepsilon_i}$ is a vertex gluing for $\varepsilon_i = +1$ and a vertex deletion for $\varepsilon_i = -1$. Obviously, the single point $\Delta(\{v\})$ is simple-homotopy equivalent to a point. By Theorem 3.5, $\Delta(\{v\})$ is regularly simple-homotopy equivalent to $\Delta(\{v\}T_1^{\varepsilon_1})$; again, $\Delta(\{v\}T^{\varepsilon_1})$ is regularly simple-homotopy equivalent to $\Delta(\{v\}T_1^{\varepsilon_1}T_2^{\varepsilon_2})$; and so on. By transitivity, $\Delta(\{v\}T_1^{\varepsilon_1}\cdots T_n^{\varepsilon_n})$ is regularly simple-homotopy equivalent to a point. □

The following theorem follows immediately from Theorem 3.5 and Corollary 3.6.

Theorem 3.7. *Let G be a graph and G' a contractible subgraph. Then for a vertex v not in G, the simplicial complex $\Delta(G)$ is regularly simple-homotopy equivalent to $\Delta(G \cup G' * v)$.*

We have shown that homotopy groups, as well as homology groups, are invariant under graph transformations. Of course, the Euler characteristic is unchanged under graph transformations. In particular, the main results in [4–8,16] are consequences of Theorem 3.7. It should be pointed out that Lemma 3.4 first resulted from a discussion with Chang [1]. Our original reduction of graph homotopy to simple-homotopy is to express both vertex gluing and edge gluing by sequences of elementary expansions and elementary collapses, respectively.

4. Graham homotopy

Hypergraphs are useful structures to study relational databases, see [12]. Acyclic hypergraphs, which are the extension of trees in graph theory, correspond to acyclic database schemes. The Graham reduction for defining acyclic hypergraphs can be viewed as a new type of combinatorial homotopy, which is much stronger than simple-homotopy and graph homotopy. The acyclic database schemes come from the work of many people, see the bibliography and comments in [12, pp. 482–484]. The database scheme problems were first formulated by Namibar [13] in terms of hypergraphs. The algorithm (Graham reduction) to test acyclicity for hypergraphs was first introduced by Graham [3]; Yu and Ozsoyoglu [17,18] also independently formulated the algorithm in terms of "join graphs". In this section, we introduce Graham homotopy and an invariant measure, namely cycle rank, to describe cycle structures of hypergraphs.

A *hypergraph* $H = (V, E)$ consists of a finite nonempty set V, whose elements are called *vertices*, and a collection $E = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of nonempty subsets of V, called *hyperedges* or *cells*, such that $V = \bigcup_{i=1}^n \sigma_i$. A hypergraph is called *reduced* if there is no hyperedge properly contained in another hyperedge. A vertex v of a hypergraph H is called *isolated* if v belongs to one and only one hyperedge of H. The Graham reduction for hypergraphs are the following two operations:

(GR1) Deleting an isolated vertex;
(GR2) Deleting σ_i if $\sigma_i \subset \sigma_j$ for some $j \neq i$.

A hypergraph is called *acyclic* if it can be reduced to have no hyperedges by the Graham reduction; otherwise, it is called *cyclic*.

Associated to a hypergraph $H=(V,E)$ is the simplicial complex

$$\Delta(H)=\{\sigma\subset V:\ \sigma\neq\emptyset \text{ and } \sigma\subset\sigma_i \text{ for some } \sigma_i\in E\}.$$

It is clear that the reduced hypergraphs over a finite set V are in one-to-one correspondence with the simplicial complexes over V. We shall give a topological interpretation of the Graham reduction on simplicial complexes. Let us first combine (GR1) and (GR2) together as one operation.

(GR) A vertex v can be deleted from V and from all hyperedges σ_i if v belongs to exactly one maximal hyperedge.

It is clear that a hypergraph is acyclic if and only if it can be reduced to empty by (GR). If a vertex v belongs to exactly one maximal hyperedge of H, say σ_1, then $\Delta(H)$ can be obtained by gluing v to the closed simplex $\Delta(\sigma_1-\{v\})$. Obviously, this gluing can be obtained by a sequence of elementary simplicial expansions, since every simplex can be grown from a vertex only by elementary simplicial expansions. This simple-homotopy property is the key idea to define Graham homotopy.

Let K be a simplicial complex and $\hat{\sigma}$ a closed simplex of K. By Theorem 3.2, the simplicial complex $L=K\cup\hat{\sigma}*v$ is simplicially simple-homotopy equivalent to K. We say that K is *Graham expandable to* L and L is *Graham collapsible to* K; removing the cells $\hat{\sigma}*v$ from L is called an *elementary Graham collapse* and adding the cells $\hat{\sigma}*v$ to K is called an *elementary Graham expansion*. Two simplicial complexes are said to be *Graham homotopy equivalent* if one can be obtained from the other by a sequence of elementary Graham expansions and elementary Graham collapses.

Theorem 4.1. *The application of Graham reduction* (GR) *to a hypergraph H corresponds to an elementary Graham collapse on the associated simplicial complex* $\Delta(H)$. *Moreover, if* $\Delta(H)=\Delta(\mathrm{sk}(\Delta(H)))$, *the application of* (GR) *to H corresponds to a graph homotopy on* $\mathrm{sk}(\Delta(H))$.

Let σ be a cell of $\Delta(H)$. The *link* of σ in $\Delta(H)$ is the simplicial complex

$$\mathrm{lk}(\sigma,\Delta(H))=\{\tau\in\Delta(H):\ \sigma\cap\tau=\emptyset,\ \sigma\cup\tau\in\Delta(H)\}.$$

Notice that the link $\mathrm{lk}(\sigma,\Delta(H))$ is empty if σ is a maximal cell. Let $c(H)$ denote the number of connected components of $\Delta(H)$ and $c(\sigma,H)$ the number of connected components of $\mathrm{lk}(\sigma,\Delta(H))$. The cycle rank $r(H)$ was introduced by Lee (in a slightly different way) in [9–11] to generalize the ordinary cycle rank of a graph.

Definition 4.2. The *cycle rank* of a hypergraph H is the integer

$$r(H)=c(H)+\sum_{\sigma\in\Delta(H)}[c(\sigma,H)-1].$$

Theorem 4.3. *The cycle rank of a hypergraph is invariant under Graham reduction* (GR).

Proof. Let v be a vertex of a maximal cell, say τ, such that v is not a vertex of any other maximal cells. Let H' denote the hypergraph on $V-\{v\}$, obtained from H by removing v. For simplicity, we write $\tau - v$ instead of $\tau-\{v\}$. For any cell σ of $\Delta(H)$, we have the following four cases.

Case 1: $v\notin\sigma$ and $\sigma\neq\tau-v$. Then $c(\sigma,H')=c(\sigma,H)$;

Case 2: $v\notin\sigma$ and $\sigma=\tau-v$. Then $c(\tau-v,H')=c(\tau-v,H)-1$;

Case 3: $v\in\sigma$ and $\sigma\neq\tau$. Then $c(\sigma,H)=1$;

Case 4: $v\in\sigma$ and $\sigma=\tau$. Then $c(\tau,H)=0$.

Notice that the cell σ in Case 3 and Case 4 will vanish from $\Delta(H')$ when v is removed from H. Thus, we have

$$\begin{aligned} r(H) &= c(H)+[c(\tau-v,H)-1]+[c(\tau,H)-1] \\ &\quad + \sum_{\substack{v\notin\sigma \\ \sigma\neq\tau-v}}[c(\sigma,H)-1]+\sum_{\substack{v\in\sigma\\ \sigma\neq\tau}}[c(\sigma,H)-1] \\ &= C(H)+[c(\tau-v,H')-1]+\sum_{\substack{v\notin\sigma \\ \sigma\neq\tau-v}}[c(\sigma,H')-1] \\ &= c(H')+\sum_{\sigma\in\Delta(H')}[c(\sigma,H')-1] \\ &= r(H'). \qquad\Box \end{aligned}$$

Theorem 4.4. *A hypergraph H is acyclic if and only if $r(H)=0$. Moreover, the cycle rank is always nonnegative.*

Proof. It follows from Theorem 4.3 that if H is acyclic, then $r(H)=0$. Now assume that $r(H)=0$. We want to prove that H is acyclic. Suppose H is cyclic. Apply (GR) to reduce H to a hypergraph H' until it cannot be further reduced. Then the maximal hyperedges $\tau_1,\ldots,\tau_m$ of H' must satisfy the following properties:

$$\tau_i\subset\bigcup_{j\neq i}\tau_j,\quad \tau_i\not\subseteq\tau_j\quad\text{for } i\neq j\quad\text{and}\quad(\tau_i-\tau_j)\cap\tau_k\neq\emptyset\quad\text{for some } k\neq i. \tag{2}$$

Notice that $c(\sigma,H')\geqslant 1$ for any $\sigma\in\Delta(H')$, except for maximal cells $\tau_1,\ldots,\tau_m$; while $c(\tau_i,H')=0$, $1\leqslant i\leqslant m$. Denote $I(H')=\{\tau_i\cap\tau_j\neq\emptyset\colon 1\leqslant i<j\leqslant m\}$. Since $C(H')\geqslant 1$, it suffices to show that

$$\sum_{\sigma\in I(H')}[c(\sigma,H')-1]\geqslant m.$$

For each maximal cell of $\Delta(H')$, say τ_1, there exists a maximal cell, say τ_2, such that $\tau_1 \cap \tau_2 \neq \emptyset$. In view of (2), $\tau_1 - \tau_2$ must intersect another maximal cell, say τ_3, i.e., $(\tau_1 - \tau_2) \cap \tau_3 \neq \emptyset$. Of course, $\tau_1 \cap \tau_2 \neq \tau_1 \cap \tau_3$. This shows that τ_1 contributes a component $\tau_1 - \tau_2$ in $\mathrm{lk}(\tau_1 \cap \tau_2, \Delta(H'))$ and a component $\tau_1 - \tau_3$ in $\mathrm{lk}(\tau_1 \cap \tau_3, \Delta(H))$. This means that for each cell $\tau_i \cap \tau_j \neq \emptyset$ with $i \neq j$, $\mathrm{lk}(\tau_i \cap \tau_j, \Delta(H'))$ contains at least two components $\tau_i - \tau_j$ and $\tau_j - \tau_i$. Consider the bipartite graph with the vertex set $I(H') \cup \{\tau_1, \ldots, \tau_m\}$ and the edge set $\{(\tau_i \cap \tau_j, \tau_i) : \tau_i \cap \tau_j \neq \emptyset, i \neq j\}$, then $\sum_{\sigma \in I(H')} c(\sigma, H')$ should be the number of edges of the bipartite graph. Thus

$$\begin{aligned} \sum_{\sigma \in I(H')} c(\sigma, H') &= \frac{1}{2}\left(\sum_{i=1}^{m} \deg(\tau_i) + \sum_{\sigma \in I(H')} \deg(\sigma)\right) \\ &\geqslant \tfrac{1}{2}(2m + 2|I(H')|) \\ &= m + |I(H')|. \end{aligned}$$

We therefore have a contradiction: $r(H) = r(H') \geqslant c(H') \geqslant 1$. The nonnegativity of the cycle rank follows from the same arguments. □

The nonnegativity of the cycle rank $r(H)$ for a hypergraph H automatically gives rise to an inequality about the number of components of $\Delta(H)$ and the number of components of the links $\mathrm{lk}(\sigma, \Delta(H))$ at cells σ.

Corollary 4.5. *Let K be a simplicial complex with $\#(K)$ cells. Let $c(K)$ denote the number of connected components of $|K|$ and $c(\sigma)$ the number of components of $\mathrm{lk}(\sigma, K)$). Then*

$$\sum_{\sigma \in K} c(\sigma) \geqslant \#(K) - c(K).$$

The present proof of Theorem 4.3 has a clear topological interpretation, i.e., (GR2) is unnecessary from a topological point of view; this is why we can ignore (GR2) and modify (GR1) to (GR). It should be pointed out that Theorem 4.3 was first proved in [15] with respect to "ear removal", which can be viewed as composition of a sequence of consecutive operations of (GR1) and (GR2). Another proof of Theorem 4.3 with respect to (GR1) and (GR2) is given in [11]. We have known that Graham homotopy implies simple-homotopy. However, the converse is not true. The following is such an counterexample.

Example 4.6. The hypergraph H_s, whose vertex set V is $\{1,2,3,4\}$ and edges set E consists of all nonempty proper subsets of V, is simplicially simple-homotopy equivalent to a point, but it cannot be Graham homotopy equivalent to a point.

The reduced hypergraph of H_s consists of the hyperedges $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,3,4\}$. It is easy to check that $r(H_s) = 1$. The geometric realization of H_s is the boundary of a tetrahedron with one open facet removed. Obviously, H_s is simple-homotopy equivalent to a point. This example also shows that the converse of Theorem 4.7 is not true, because $\mathrm{sk}(H_s)$ is chordal.

Theorem 4.7. *If H is an acyclic hypergraph, then $G = \mathrm{sk}\Delta(H)$ is chordal.*

Proof. Suppose G is not chordal for some acyclic hypergraph H. Then there is a cycle $v_1v_2\cdots v_nv_1$ in G without chords, $n \geqslant 4$. If $v_{i-1}v_i \in \tau_i$ and $v_iv_{i+1} \in \tau_{i+1}$ for some maximal hyperedges τ_i and τ_{i+1}, then $v_{i-1} \notin \tau_{i+1}$ and $v_{i+1} \notin \tau_i$, for in another case $v_{i-1}v_{i+1}$ will be a chord. Hence $\tau_i \neq \tau_{i+1}$ and each v_i belongs to two maximal hyperedges. Thus, no vertex v_i can be removed by (CR). Notice that when some vertices other than those v_i are removed by (GR) in the reduction process, the nonremovable property of v_i still holds. By definition of acyclicity, H is cyclic, a contradiction. □

Corollary 4.8. *If a hypergraph H is Graham homotopy equivalent to a point, then H can be reduced to a point only by elementary Graham collapses.*

Proof. By Theorem 4.3, the cycle rank $r(H)$ is invariant under Graham reduction (GR). Since the cycle rank of a point is zero, then $r(H) = 0$. By Theorem 4.4, H is acyclic. Thus H can be contracted to a point only by elementary Graham collapses. □

5. The graph of Bing's house

In this section we use Bing's house to show that the graph homotopy (as well as simple-homotopy) is significantly different from the Graham homotopy. Bing's house is a closed topological surface (not a 2-manifold); see Fig. 3. It is well known that Bing's house is simple-homotopy equivalent to a point, but *cannot* be contracted to a point only by elementary collapses, see [2], because there is no collapsible face pair. Bing's house can be realized by a graph G_b, whose vertices and edges are shown in Fig. 4. Like the situation of simple-homotopy, we shall see that G_b is graph homotopy equivalent to a point, but it cannot be contracted to a point only by vertex deletion. The latter can be checked directly by computing the graph links of the vertices of G_b, all are not contractible, as follows:

$L(u_1)$ is the union of two cycles $u_2u_7v_7v_1u_2$ and $u_4u_5v_5v_1u_4$;
$L(u_2)$ is the cycle $u_1u_7u_6u_3v_3v_2v_1u_1$;
$L(u_3)$ is the cycle $u_2u_6u_4v_3u_2$;
$L(u_5)$ is the cycle $u_1u_4u_6v_5u_1$;
$L(u_6)$ is the cycle $u_2u_3u_4u_5v_5v_6v_7u_7u_2$;
$L(v_1)$ is the union of three cycles $u_1u_2v_2v_7u_1$, $u_1u_4v_4v_5u_1$, and $u_1v_5w_1v_7u_1$;
$L(v_2)$ is the cycle $u_2v_1v_7v_6w_2v_3u_2$;
$L(v_3)$ is the cycle $u_2u_3u_4v_4w_4w_3w_2v_2u_2$;
$L(v_5)$ is the union of two cycles $u_1u_5u_6v_6v_4v_1u_1$ and $v_1v_4v_6w_6w_5w_1v_1$.

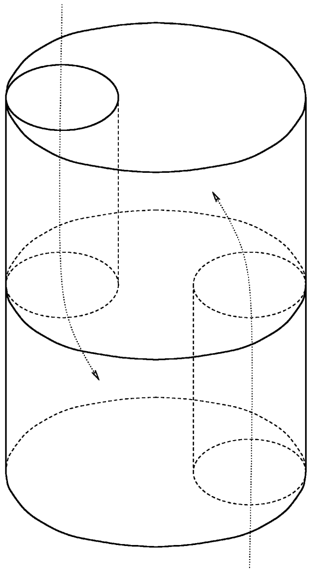


Fig. 3. Bing's house.

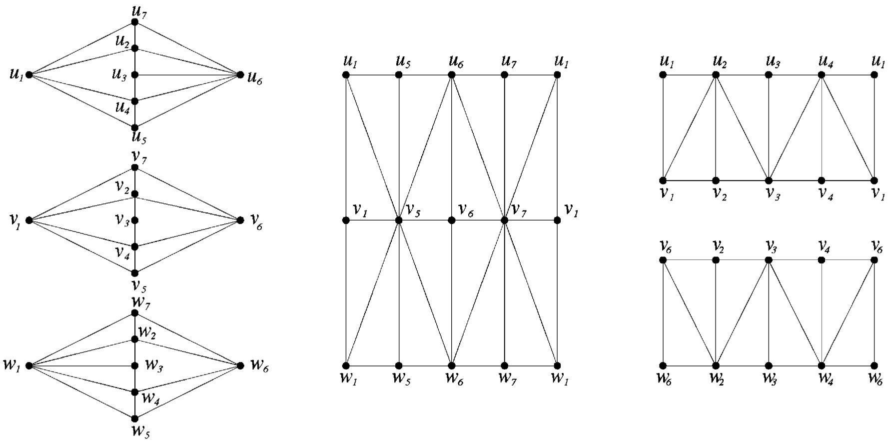
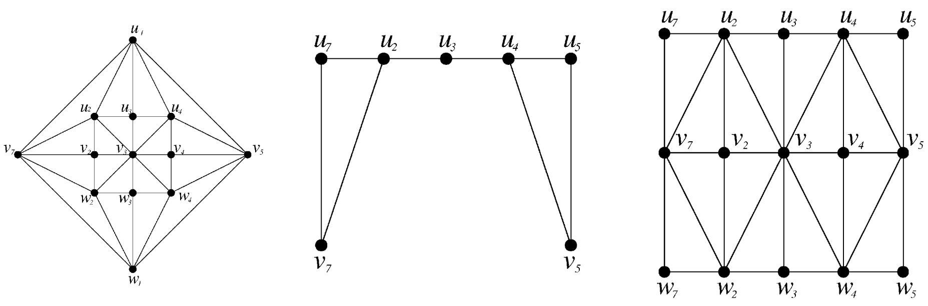


Fig. 4. The graph G_b of Bing's house.

By symmetry, $L(u_4)$ and $L(u_7)$ are isomorphic to $L(u_2)$ and $L(u_5)$, respectively; $L(v_4)$, $L(v_6)$, and $L(v_7)$ are isomorphic to $L(v_2)$, $L(v_1)$, and $L(v_5)$, respectively; $L(w_i)$ is isomorphic to $L(u_i)$ for $i=2,3,4,5,7$; $L(w_1)$ and $L(w_6)$ are isomorphic to $L(u_6)$ and $L(u_1)$, respectively.

To see that G_b is graph homotopy equivalent to a point, we can glue the edges v_7w_2, v_5w_4, v_1w_2, v_1w_4, v_1w_3, v_1v_3, u_3v_1, and u_1u_3 consecutively to G_b to fill up the



(a) $L(v_1, G')$ (b) $L(u_1, G'')$ (c) G'''

Fig. 5. The graph contractibility of G_b.

"upper room" of $\Delta(G_b)$, because the joint graph links of these edges (listed below) are all contractible right before they are glued.

$L(v_7w_2)$ is the path $v_2v_6w_6w_7w_1$;
$L(v_5w_4)$ is the path $v_4v_6w_6w_5w_1$, isomorphic to $L(v_7w_2)$;
$L(v_1w_2)$ is the path $v_2v_7w_1$;
$L(v_1w_4)$ is the path $v_4v_5w_1$, isomorphic to $L(v_1w_2)$;
$L(v_1w_3)$ is the path $w_2w_1w_4$;
$L(v_1v_3)$ is the path $u_2v_2w_2w_3w_4v_4u_4$;
$L(u_3v_1)$ is the path $u_2v_3u_4$;
$L(u_1u_3)$ is the path $u_2v_1u_4$.

Similarly, we can glue the edges u_2v_7, u_4v_5, u_2v_6, u_4v_6, u_3v_6, v_3v_6, v_6w_3, and w_3w_6 consecutively to fill up the "lower room" of $\Delta(G_b)$, because the joint graph links of these edges right before they are glued are isomorphic to those joint graph links of the edges for filling up the "upper room". With all those edges glued for filling up the two "rooms", the graph G_b is expanded to a graph G' and $\Delta(G')$ is a triangulated solid cube. Notice that $L(v_1, G')$ and $L(v_6, G')$ are isomorphic and contractible, see Part (a) of Fig. 5. Then v_1 and v_6 can be deleted from G' to obtain a graph G''. Now the graph links of u_1, u_6, w_1, and w_6 in G'' are all isomorphic and contractible, see Part (b) of Fig. 5. By removing the vertices u_1, u_6, w_1, w_6 from G'', we obtain a contractible graph G''', see Part (c) of Fig. 5. We thus have proved that G_b is graph homotopy equivalent to a point.

The above example shows that for graph homotopy, as well as for simple-homotopy, there is no straightforward algorithm to test whether a simplicial complex is contractible. However, for Graham homotopy, the situation is quite different. If a simplicial complex is Graham homotopy equivalent to a point, then it can always be reduced to a point only by elementary Graham collapses. This special property of Graham homotopy makes it useful to the theory of algorithms of theoretical computer science. This hints that, if we compare elementary collapses to the forward steps in a computer algorithm and elementary expansions to backtracks, then an algorithm with backtracks is essentially different from one without backtracks (possibly the difference is between polynomial and exponential). Though we cannot yet formulate precise idea on the comparison of algorithms and homotopies, we still believe that our viewpoint is useful and worth exploration. The Graham homotopy also suggests that some stronger homotopies need be studied even for contractible spaces. The numerical characterizations for simple-homotopy and graph homotopy of simplicial complexes are particularly important. Before drawing this paper to a close, we pose two questions here. We posit that the answers to both questions are no.

Question 1. Is there any algorithm to compute the simple-homotopy distance $\ell(K,L)$ between two arbitrary simplicial complexes K and L? (The special case that K is a point is also interesting and important.)

Question 2. If a simplicial complex K is regularly simple-homotopy equivalent to a point, can K be expanded to a simplex without adding any new vertex?

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