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Comparative statics without total differentiation of the first-order conditions

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Abstract

This paper shows that, for certain purposes, comparative statics may be done without the usual total differentiation of the first-order conditions. This is useful where the total differentiation of the first-order conditions gives complicated equations difficult to handle.

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1. Introduction

Much economic analysis consists of the comparative static analysis of equilibria of constrained optimization. It is well known that a comparative static analysis involves the total differentiation of the relevant first-order conditions for equilibrium or optimization. In fact, when one of us did a comparative static analysis without total differentiation of the first-order conditions, he was promptly accused by the referee of using an incorrect method.

This paper shows that, for certain purposes, comparative statics may be done without the total differentiation of the first-order conditions. For many problems, the total differentiation of the first-order conditions gives complicated equations difficult to handle. Our simplified method may thus be rather useful.

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2. A simple example

To show that, for certain purposes, comparative static analysis may be done without the total differentiation of the first-order conditions, we first consider a very simple and well-known example, the textbook income/leisure choice problem. An individual chooses the hours of work at given wage rate w to maximize her utility

$$U = U(c, x) \tag{1}$$

where c is consumption, and x is leisure, subject to

$$c = (1 - x)w \tag{2}$$

The first-order condition for this optimization problem is

$$U_x / U_c = w \tag{3}$$

where a subscript denotes partial differentiation, e.g. $U_x \equiv \partial U/\partial x$.

Now consider the comparative-static effects of an exogenous change in w. This may be derived by the total differentiation of both the constraint (2) and the first-order condition (3), which together define the equilibrium position. (Eq. (2) may also be regarded as one of the first-order conditions, being obtained from the partial differentiation of the relevant Lagrangean function with respect to the multiplier. In this paper, our terminology of 'first-order conditions' does not include the constraint equations.)

$$w\,\mathrm{d}x + \mathrm{d}c = (1 - x)\,\mathrm{d}w \tag{4}$$

$$(U_{xx} - wU_{cx})\,\mathrm{d}x + (U_{xc} - wU_{cc})\,\mathrm{d}c = U_c\,\mathrm{d}w \tag{5}$$

Rewriting (4) and (5) in matrix-vector form and solving by Cramer's rule gives

$$\mathrm{d}x/\mathrm{d}w = [U_c - (1 - x)(U_{xc} - wU_{cc})]/D \tag{6}$$

$$\mathrm{d}c/\mathrm{d}w = [(1 - x)(U_{xx} - wU_{cx}) - wU_c]/D \tag{7}$$

where $D \equiv U_{xx} - wU_{cx} - wU_{xc} + w^2 U_{cc}$ is negative from the second-order condition. This is the normal comparative statics manipulation which may be used to sign or evaluate the effects of a marginal change in the exogenous variables (w in the current example) on the endogenous variables (x and c). For such purposes, the total differentiation of the first-order conditions is necessary, as the resulting equations define the combined changes in the endogenous variables that continue to satisfy the first-order conditions, which remain satisfied before and after the change in the exogenous variables.

However, if the purpose of the exercise is not to evaluate the effect of w on c and x, but on U, the total differentiation of the first-order condition (3) is not needed. We may differentiate U with respect to w to obtain

$$\mathrm{d}U/\mathrm{d}w = U_c(\mathrm{d}c/\mathrm{d}w) + U_x(\mathrm{d}x/\mathrm{d}w) \tag{8}$$

which must be valid as U depends only on c and x and a change in preference is not involved. Next, the substitution of $\mathrm{d}c/\mathrm{d}w = 1 - x - w(\mathrm{d}x/\mathrm{d}w)$ from (4) into (8) gives, noting (3),

$$\mathrm{d}U/\mathrm{d}w = (1 - x)U_c \tag{9}$$

or in proportionate terms after multiplication with w/U,

$$\sigma^{Uw} = \eta^{Uc} \tag{9$'$}$$

where $\sigma^{Uw} \equiv (\mathrm{d}U/\mathrm{d}w)w/U$ and $\eta^{Uc} \equiv (\partial U/\partial c)c/U$.

To see that this method of obtaining equations like (9) or (9′) is valid, we may note that we will obtain exactly the same Eq. (9) if we use the conventional method of total differentiation of the first-order equations to obtain (6) and (7) and substitute them into (8).

3. A more general analysis

Consider a two-level maximization problem usually encountered in economics. The representative individual maximizes a (usually utility) function with respect to variables under her control $\boldsymbol{x} \equiv (x^1, \ldots, x^n)$. In general, this may be a function not only of $\boldsymbol{x}$, but also of $\boldsymbol{y} \equiv (y^1, \ldots, y^m)$ under the control of the government, and $\boldsymbol{z} \equiv (z^1, \ldots, z^r)$ which, though not under the direct control of the government, may be influenced by $\boldsymbol{y}$. In addition, there are exogenous variables $\boldsymbol{q} \equiv (q^1, \ldots, q^s)$. We have

$$\mathrm{Max}_{|\boldsymbol{x}|}\, U(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{q}) \tag{10}$$

where the subscript $\|$ indicates the variables the maximization is with respect to. The individual maximization is subject to the following general constraint

$$F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{q}) = 0 \tag{11}$$

The first-order conditions for this maximization problem are

$$U_{xi} = \lambda F_{xi} \tag{12}$$

where a subscript denotes partial differentiation, e.g. $U_{xi} \equiv \partial U/\partial x^i$ and λ is the relevant Lagrangean multiplier.

Given that $\boldsymbol{x}$ is chosen in accordance with the above maximization problem, the government maximizes a function (in many economic problems, this is the same as the utility function above)

$$G(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{q}) = 0 \tag{13}$$

subject to

$$H(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{q}) = 0 \tag{14}$$

Now consider the effect of a change in an exogenous variable q^i on U. Since $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{q}) = 0$ from (11), we have

$$\begin{aligned} \mathrm{d}U(.)/\mathrm{d}q^i &= d[U(.) - \lambda F(.)]/\mathrm{d}q^i \\ &= \Sigma_k\,(U_{xk} - \lambda F_{xk})(\mathrm{d}x^k/\mathrm{d}q^i) + \Sigma_k\,(U_{yk} - \lambda F_{yk})(\mathrm{d}y^k/\mathrm{d}q^i) \\ &\quad + \Sigma_k\,(U_{zk} - \lambda F_{zk})(\mathrm{d}z^k/\mathrm{d}q^i) + U_{qi} - \lambda F_{qi} \\ &= \Sigma_k\,(U_{yk} - F_{yk}U_{xj}/F_{xj})(\mathrm{d}y^k/\mathrm{d}q^i) + \Sigma_k\,(U_{zk} - F_{zk}U_{xj}/F_{xj})(\mathrm{d}z^k/\mathrm{d}q^i) \\ &\quad + U_{qi} - F_{qi}U_{xj}/F_{xj} \end{aligned} \tag{15}$$

where the last equation follows from the substitution of λ from (12) and the x^j in the last line of (15) can be any $j = 1, \ldots, n$.

We may first apply the result above to the simple case discussed in the previous section. In the simple case, we have only one-level maximization, thus $\boldsymbol{y}$ and $\boldsymbol{z}$ do not exist, making the first two terms (under the summation signs) of the last line of (15) vanish. Also, the exogenous variable w does not enter the utility function (1), making the third term U_{qi} vanish, leaving only the last term. The relevant exogenous variable q^i is w, and since the x^j can be any x^j under the control of the individual, we may choose it to be c. Substituting $F_{qi} = -(1 - x)$ [from the value of $\partial F/\partial w$, after writing (2) as $F = c - (1 - x)w = 0$], and $U_{xj} = U_c$, $F_{xj} = 1$ (from the value of $\partial F/\partial c$) into (15), we have (9) exactly. This shows the consistency of the more general analysis with that of the simple example above.

For more complicated cases where the first two terms of the last line of (15) do not vanish, it is true that, without the total differentiation of the first-order conditions to evaluate (or at least sign) the effects of an exogenous change on the relevant variables, the evaluation of the effect on the objective function U may be incomplete, as $\mathrm{d}U(.)/\mathrm{d}q^i$ depends on the un-evaluated $(\mathrm{d}y^k/\mathrm{d}q^i)$ and $\mathrm{d}z^k/\mathrm{d}q^i$. However, in many cases, this inadequacy is more apparent than real. This is so because, even if we go through the traditional total differentiation of the first-order conditions, the resulting values of $(\mathrm{d}y^k/\mathrm{d}q^i)$ and $\mathrm{d}z^k/\mathrm{d}q^i$ may still be unsignable, such as being dependent on the balance of the substitution and income effects. Thus, we lose nothing by just using the simple method represented by (15). Appendix A provides an example to illustrate this point.

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Appendix A

Consider a case where the representative individual takes the aggregate/average variables as given and chooses her own work/leisure to maximize

$$U = U(c, x, R, E, G); \quad U_c, U_x, U_R, U_E, U_G > 0 \tag{A.1}$$

where U is utility, c is consumption, x is leisure, $R = y/Y$ is relative income, y is income of the individual, Y is average income of the economy, E is environmental quality, and G is government spending on public goods. (Note that the upper case stands for the average value of the relevant individual variable for the whole economy. For a general change throughout the whole economy, we then have $dw = dW$, etc., and $y = Y$, $w = W$, etc. from the representative nature of the representative individual. These equations are needed for the derivation of Eq. (A.7) below.) The maximization is subject to the budget constraint

$$c = (1-t)y = (1-t)(1-x)w \tag{A.2}$$

where t is income tax rate (taken as given by the individual), and w is wage-rate, determined exogenously. The first-order income/leisure choice condition is

$$U_x = (1-t)wU_c + (w/Y)U_R \tag{A.3}$$

Subject to the maximization choice of individuals, the government chooses tax-rate t and the proportion of tax revenue used for the abatement (A) of environmental disruption α to maximize (with apology to the public choice school) the utility of the representative individual (A.1). Subject to

$$G = N(1-\alpha)tY \tag{A.4}$$

$$E = E(A, Y); \quad E_A > 0,\ E_Y < 0 \tag{A.5}$$

$$A = \alpha tYN \tag{A.6}$$

where G is government spending on public goods G, N is the given number of individuals, and E is environmental quality. That $E_Y < 0$ captures the environmental disruption effect of most production and consumption.

In the model above, using the simplified method (without total differentiation of all first-order conditions) to evaluate the comparative static effect of a change in the exogenous variable W on U, we may derive

$$\sigma^{UW}_{|t^*,\alpha^*|} = [1/(1-t)]\eta^{Uc} + \eta^{UE}\eta^{EY} + [(x/(1-x))[\sigma^{XW} - \sigma^{Xt}(1+\sigma^{tW})][\eta^{UR} - \eta^{UG} - \eta^{UE}(\eta^{EA} + \eta^{EY})] \tag{A.7}$$

where $\sigma^{ab} \equiv (da/db)b/a$, $\eta^{ab} \equiv (\partial a/\partial b)b/a$ for any a, b and $|t^*,\alpha^*|$ indicates that the tax rate and the proportion of revenue used for abatement are being optimized.

Since the right hand side of (A.7) depends not only on the endogenous response elasticities (the various η's) but also on how the amount of work/leisure responds to the exogenous variable W (both directly and through the change in t with respect to W), the analysis may be said to be incomplete. However, even if we go through the traditional method of the total differentiation of all the relevant first-order conditions to evaluate the responses to the exogenous change in W, we will still be unable in general to sign the changes in work/leisure, which depend on the balance between the substitution and the income effects. Thus, nothing has really been lost in using the simplified method to obtain (A.7). This equation says that an exogenous general increase in real wages/productivity in the whole economy has: (1) a positive effect on the utility of the representative individual through the intrinsic consumption effect η^{Uc}, (2) a negative environmental disruption effect $\eta^{UE}\eta^{EY}$ due to higher production, and (3) an ambiguous indirect effect if work/leisure is affected. A change in work/leisure generates indirect effects despite the optimization with respect to its choice at the individual level which trades off only consumption (from after-tax income) with leisure, ignoring the indirect effects through the relative-income effect, the financing of public spending (through higher tax revenue), and the unabated disruption on the environment. If $(\eta^{EA} + \eta^{EY})$ is negative/positive (additional disruption of higher production is not fully/more than fully abated), more work and production has negative/positive effects through changes in environmental disruption. This explains the right hand side of (A.7).