On the Matching Polynomials of Graphs with Small Number of Cycles of Even Length

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ABSTRACT: Suppose that G is a simple graph. We prove that if G contains a small number of cycles of even length then the matching polynomial of G can be expressed in terms of the characteristic polynomials of the skew adjacency matrix $A(G^e)$ of an arbitrary orientation G^e of G and the minors of $A(G^e)$. In addition to a formula previously discovered by Godsil and Gutman, we obtain a different formula for the matching polynomial of a general graph. Int J Quantum Chem 105: 124–130, 2005

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1. Introduction

In this article, we suppose that $G = (V(G), E(G))$ is a finite simple graph with the vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge-set $E(G) = \{e_1, e_2, \ldots, e_\varepsilon\}$. Denote by $G \times H$ the Cartesian product of two graphs G and H and by I_n the unit matrix of order n, respectively. We refer to Lovász and Plummer [23] for terminology and notation not defined here.

Suppose that G is a simple graph. Let $\Phi(G)$ be the set of all orientations of G. For an arbitrary orientation G^e ($\in \Phi(G)$) of G, the skew adjacency matrix of G^e, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (a_{ij})_{n\times n}, \quad a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G^e), \\ -1 & \text{if } (v_j, v_i) \in E(G^e), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the skew adjacency matrix $A(G^e)$ is a skew symmetric matrix.

A set M of edges in G is a matching if every vertex of G is incident with at most one edge in M;

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it is a perfect matching if every vertex of G is incident with exactly one edge in M. Denote by $\phi_k(G)$ the number of k-element matchings in G. We set $\phi_0(G) = 1$ by convention. Thus $\phi_1(G) = \varepsilon$ is the number of edges in G, and if n is even then $\phi_{n/2}(G)$ is the number of perfect matchings of G and will be denoted by $\phi(G)$. The number of matchings of G is called the Hosoya index (see Ref. [18]) and will be denoted by $Z(G)$, that is, $Z(G) = \sum_{k\geq o} \phi_k(G)$. The following two polynomials $m(G, x)$ and $g(G, x)$ are called the matching polynomial and matching generating polynomial of G, respectively (see Ref. [23]):

$$m(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \phi_k(G) x^{n-2k},$$

$$g(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \phi_k(G) x^k.$$

Then $m(G, x) = x^n g(G, -x^{-2})$ and $Z(G) = g(G, 1)$.

The matching polynomial is a crucial concept in the topological theory of aromaticity [1, 15, 30, 31] and is also named the acyclic polynomial [15, 30, 31]. Sometimes the matching polynomial is also referred to as the reference polynomial [1]. It takes no great imagination to consider the idea of representing a molecule by a graph, with the atoms as vertices and the bonds as edges, but it is surprising that there can be an excellent correlation between the chemical properties of the molecule and suitably chosen parameters of the associated graph. One such parameter is the sum of the absolute values of the zeros of the matching polynomial of the graph, and this has been related to properties of aromatic hydrocarbons. The roots of the matching polynomial correspond to energy levels of a molecule and a reference structure. A new application in Hückel theory was discussed in detail by Trinajstić [30]. Hosoya [18] used the matching polynomial in chemical thermodynamics. For further applications of the matching polynomial in chemistry, see, e.g., Trinajstić's book [31].

The matching polynomial $m(G, x)$ has also been independently discovered in statistical physics and in a mathematical context. In statistical physics it was first considered by Heilman and Lieb in 1970 [16] and soon after by Kunz [21] and Gruber and Kunz [13]. Finally, Farrell [5] introduced what he called the matching polynomial of a graph, this time in a mathematical context. This is a polynomial in two variables (say x and y) which results if, in the definition of $m(G, x)$, the term $(-1)^k$ is replaced by y^k. Godsil and Gutman [7] reported some properties of matching polynomial of a graph. Beezer and Farrell [3] considered the matching polynomial of the distance-regular graph.

The matching polynomial of a tree T equals its characteristic polynomial $\det(xI - A)$, where A is the adjacency matrix of T. A related question has been posed in Ref. [2]: For any graph G, is it possible to construct a Hermitian matrix $H(G)$ such that the matching polynomial of G is the characteristic polynomial of $H(G)$? The matrix $H(G)$ can be constructed for certain graphs including unicyclic graphs and cacti (the progress in this field can be found in Refs. [9–12, 14, 15, 17, 26, 27, 29]). But, in general, $H(G)$ does not exist [14]. The matching polynomial of an outerplanar graph can be determined in polynomial time (see Refs. [6, 23]). But the computation of the matching polynomial of general graphs is NP-complete [23]. Let G be a graph with adjacency matrix $A = (a_{ij})_{n\times n}$. If S is a subset of $E(G)$, let A_S be the matrix such that

$$(A_S)_{ij} = \begin{cases} a_{ij} & \text{if } (v_i, v_j) \notin S, \\ -a_{ij} & \text{if } (v_i, v_j) \in S. \end{cases}$$

Godsil and Gutman [8] showed that

$$m(G, x) = \frac{1}{2^{-\varepsilon}} \sum_{S\subseteq E(G)} \det(xI - A_S)$$

and used this to deduce that if G is connected then the spectral radius of A is at least as large as the largest zero of $m(G, x)$, and that equality holds if and only if G is a tree. Furthermore, Godsil and Gutman [7] proved an interesting result as follows: Let $u = (u_1, u_2, \ldots, u_\varepsilon) \in \{-1, +1\}^\varepsilon$. Let $e_1, e_2, \ldots, e_\varepsilon$ be the edges of G and G_u be the weighted graph obtained from G by associating the weight u_i with the edge e_i $(i = 1, 2, \ldots, \varepsilon)$. Further let $A(G_u)$ be the adjacency matrix of G_u. Then

$$m(G, x) = 2^{-\varepsilon} \sum_{u} \det(xI - A(G_u)), \tag{1}$$

where the summation ranges over all 2^ε distinct ε-tuples u.

So far, there exist no valid methods to calculate the matching polynomial of a graph except the outerplanar graph G, the matching polynomial of which can be determined in polynomial time (see Exercise 8.5.4 in Ref. [23]).

In next section, we prove that if G contains only small number of cycles of even length then the matching polynomial of G can be expressed in terms of the characteristic polynomials of the skew adjacency matrix $A(G^e)$ of an arbitrary orientation G^e of G and the minors of $A(G^e)$. In our case, we reduce the running time of (1) by a factor of about $1/2^{\varepsilon-k}$ (where k is the number of cycles of even length in G), because we only need to calculate 2^k determinants. In addition to formula (1) previously discovered by Godsil and Gutman, we obtain a different formula for the matching polynomial of a general graph. In Section 3, some further problems are posed.

2. Main Results

Proposition 1 ([6, 23]). Let G be a graph with ε edges. The number of perfect matchings of G

$$\phi(G) = \frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(A(G^e)),$$

where the sum is over all orientations of G.

Proposition 1 is equivalent to the following:

Proposition 2 ([23]). Let G be a graph and G^e a random orientation of G, obtained by orienting each edge in G independently of the others with probability 1/2 in either direction. Let $A(G^e)$ be the skew adjacency matrix of G. Then the expected value of det $A(G^e)$ is $\phi(G)$.

Proposition 3 ([6]). Let $e = uv$ be an edge of a graph G. Then

$$m(G, x) = m(G - e, x) - m(G - u - v, x),$$

where $G - e$ (resp. $G - u - v$) denotes the subgraph of G by deleting edge e (resp. vertices u and v and the incident edges).

To give a valid method to calculate the matching polynomial, the matching generating polynomial and the Hosoya index for the graph containing small number of cycles of even length, we need to introduce the following terminology.

Let G be a graph. If M is a perfect matching of G, an M-alternating cycle in G is a cycle whose edges are alternately in $E(G)\backslash M$ and M. We say that a cycle C of G is nice if $G - C$ contains a perfect matching, where $G - C$ denotes the induced subgraph of G obtained from G by deleting the vertices of C and incident edges.

If G^e is an orientation of a simple graph G and C is a cycle of even length, we say that C is oddly oriented in G^e if C contains odd number of edges that are directed in G^e in the direction of each orientation of C. We say that G^e is a Pfaffian orientation of G if every nice cycle of even length of G is oddly oriented in G^e.

Proposition 4 ([23]). Let G^e be a Pfaffian orientation of a graph G. Then

$$[\phi(G)]^2 = \det A(G^e),$$

where $A(G^e)$ is the skew adjacency matrix of G^e.

The method to enumerate perfect matchings of plane graphs in Proposition 4 was found by the physicist Kasteleyn [19, 20], and is called the Pfaffian method. This method was generalized by Little [22]. McCuaig [24], McCuaig et al. [25], and Robertson et al. [28] found a polynomial-time algorithm to show whether a bipartite graph has a Pfaffian orientation. Yan and Zhang [32, 33] used the Pfaffian method to obtain some formulas of the number of perfect matchings and the permanental polynomials for a type of symmetric graphs.

Proposition 5 ([32]). Let T be a tree with n vertices, and T^e be an arbitrary orientation. Then θ_1, $\theta_2, \ldots, \theta_n$ are eigenvalues of $A(T)$ if and only if $i\theta_1$, $i\theta_2, \ldots, i\theta_n$ are eigenvalues of $A(T^e)$, where $A(T)$ and $A(T^e)$ are the adjacency matrix of T and the skew adjacency matrix of T^e, respectively, and $i^2 = -1$.

Proposition 6 ([32]). Suppose G is a graph with n vertices. If G admits an orientation G^e under which every cycle of even length is oddly oriented in G^e then $\phi(G \times K_2) = \det(I_n + A(G^e))$, where I_n is a unit matrix of order n.

Lemma 7. Suppose that G is a simple graph with even number of vertices having no cycles of even length and G^e is an arbitrary orientation of G. Then

$$\det(A(G^e)) = \begin{cases} 1, & \text{if } G \text{ has perfect matchings,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that G contains no cycles of even length. The number of perfect matchings of G equals one or zero. Otherwise, suppose that G contains two perfect matchings F_1 and F_2. Then the symmetric difference of F_1 and F_2 has at least one alternating cycle, a contradiction. By the definition of the Pfaffian orientation, G^e is a Pfaffian orientation of G. Hence, by Proposition 4, the square of the number of perfect matchings of G equals det($A(G^e)$). So the lemma has been proved.

Theorem 8. Suppose that G is a simple graph with n vertices having no cycles of even length and G^e is an arbitrary orientation of G. Then

1. $m(G, x) = \det(xI_n + iA(G^e))$,
2. $g(G, x) = \det(I_n + \sqrt{x}A(G^e))$,
3. $Z(G) = \det(I_n + A(G^e))$,

where $i^2 = -1$, and $A(G^e)$ is the skew adjacency matrix of G^e.

Proof. For an arbitrary orientation G^e $(\in \Phi(G))$ of G, we have

$$\det(xI_n + A(G^e)) = \sum_{k=0}^{n}\left[\sum_{H^e}\det(A(H^e))\right]x^{n-k},$$

where the second sum is over all induced subdigraphs with k vertices of G^e. Note that if k is odd then $\det(A(H^e)) = 0$, hence

$$\det(xI + A(G^e)) = \sum_{j=0}^{\lfloor n/2\rfloor}\left[\sum_{H^e}\det(A(H^e))\right]x^{n-2j}, \quad (2)$$

where the second sum ranges over all induced subdigraphs with $2j$ vertices of G^e. Because there exist no cycles of even length in G, the underlying graph of H^e contains no cycles of even length. Hence, by Lemma 7,

$$\det(A(H^e)) = \begin{cases} 1, & \text{if } H \text{ has perfect matchings,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{H^e}\det(A(H^e))$ in Eq. (2) equals the number of induced subgraphs with $2j$ vertices of G, each of which has perfect matchings.

Let Ψ_1 and Ψ_2 denote the set of j-element matchings of G and that of induced subgraphs with $2j$ vertices of G which have perfect matchings, respectively. Suppose $\theta{:}\Psi_1 \to \Psi_2$ is a map such that $\theta(M_j) = G[M_j]$ for an arbitrary j-element matching M_j of G, where $G[M_j]$ is the induced subgraph of G, the vertex-set of which is $V(M_j)$. It is easy to see that θ is a surjection. Suppose $M_j^1, M_j^2 \in \Psi_1$ and $M_j^1 \neq M_j^2$. Note that G contain no cycles of even length. It is not difficult to see that $G[M_j^1] \neq G[M_j^2]$. This shows that θ is a injection. So we have proved that the number $\phi_j(G)$ of j-element matchings of G equals the number of induced subgraphs with $2j$ vertices of G, each of which has perfect matchings. Hence we have

$$\det(xI_n + A(G^e)) = \sum_{j=0}^{\lfloor n/2\rfloor}\left[\sum_{H^e}\det(A(H^e))\right]x^{n-2j}$$
$$= \sum_{j=0}^{\lfloor n/2\rfloor}\phi_j(G)x^{n-2j} = x^n g(G, x^{-2}).$$

Using the above result, we can show that

$$m(G, x) = \det(xI + iA(G^e)),$$
$$g(G, x) = \det(I + \sqrt{x}A(G^e)), \quad Z(G) = \det(I + A(G^e)).$$

Hence the theorem has been proved.

Remark 9. By Exercise 3.2.3 in Bondy and Murty [4], a graph G contains no cycles of even length if and only if every block of G is either a cycle of odd length or a K_2.

By Propositions 5 and 6 and Theorem 8, we have the following:

Corollary 10. If G is a graph with n vertices containing no cycles of even length then $\phi(G \times K_2) = Z(G) = \det(I_n + A(G^e))$ for any orientation G^e of G, where $A(G^e)$ is the skew adjacency matrix of G^e.

Corollary 11. If T is a tree with n vertices, then the matching polynomial of T equals the characteristic polynomial of T.

Corollary 12. Suppose G is a graph with n vertices, which has exactly one cycle C of even length. Let $e' = v_s v_t$ be an edge in C and G^e an arbitrary orientation of G, and let $A(G^e) = (a_{ij})_{n\times n}$ be the skew adjacency matrix of G^e. Then

$$m(G, x) = \det(xI_n + iB_1) - \det(xI_{n-2} + iB_2),$$
$$g(G, x) = \det(I_n + \sqrt{x}B_1) + x\det(I_{n-2} + \sqrt{x}B_2),$$
$$Z(G) = \det(I_n + B_1) + \det(I_{n-2} + B_2),$$

where $B_1 = (b_{ij})_{n\times n}$, b_{ij} equals zero if $(i, j) = (s, t)$ or (t, s) and $a_{i,j}$ otherwise, B_2 is the minor of $A(G^e)$ by deleting the sth and tth rows and columns.

Proof. Note that both of graphs $G - e'$ and $G - v_s - v_t$ contain no cycles of even length. Hence, by Theorem 8,

$$m(G - e', x) = \det(xI_n + iB_1),$$
$$m(G - v_s - v_t, x) = \det(xI_{n-2} + iB_2).$$

By Proposition 3, $m(G, x) = m(G - e', x) - m(G - v_s - v_t)$. Hence

$$m(G, x) = \det(xI_n + iB_1) - \det(xI_{n-2} + iB_2).$$

Since $m(G, x) = x^n g(G, -x^{-2})$ and $Z(G) = g(G, 1)$, the corollary follows.

Similarly, we can prove the following.

Corollary 13. Suppose G is a graph with n vertices, which has exactly two cycles C_1 and C_2 of even length. Let $e' = v_s v_t$ and $e'' = v_l v_m$ be two edges in C_1 and C_2, respectively. Let G^e be an arbitrary orientation of G and $A(G^e) = (a_{ij})_{n\times n}$ be the skew adjacency matrix of G^e. Then

$$M(G, x) = \det(xI_n + iB_1) + \det(xI_{n-4} + iB_2) - \det(xI_{n-2} + iB_3) - \det(xI_{n-2} + iB_4),$$

$$g(G, x) = \det(I_n + \sqrt{x}B_1) + x^2\det(I_{n-4} + \sqrt{x}B_2) + x\det(I_{n-2} + \sqrt{x}B_3) + x\det(I_{n-2} + \sqrt{x}B_4),$$

$$Z(G) = \det(I_n + B_1) + \det(I_{n-4} + B_2) + \det(I_{n-2} + B_3) + \det(I_{n-2} + B_4),$$

where B_1 is the skew adjacency matrix of $G^e - e' - e''$, B_2 the skew adjacency matrix of $G^e - v_s - v_t - v_l - v_m$, B_3 the skew adjacency matrix of $G^e - e' - e_l - v_m$ and B_4 the skew adjacency matrix of $G^e - e'' - v_s - v_t$.

Remark 14. Similarly to that in Corollaries 12 and 13, by deleting one edge from each of the k cycles of even length of G, we can express the matching polynomial of G with k ($k > 2$, k is small) cycles of even length in terms of characteristic polynomials of 2^k skew adjacency matrices, which are either the skew adjacency matrix $A(G^e)$ of an orientation G^e of G or the minors of $A(G^e)$. Because k is small, this method calculating the matching polynomial is valid.

Theorem 15. Suppose that G is a simple graph with n vertices and ε edges. Then the matching polynomial of G

$$m(G, x) = \frac{1}{2^\varepsilon}\sum_{G^e\in\Phi(G)} \det(xI_n + iA(G^e)),$$

where the sum ranges over all orientations G^e of G and $i^2 = -1$.

Proof. For an arbitrary orientation G^e ($\in \Phi(G)$) of G, we have

$$\det(xI_n + A(G^e)) = \sum_{k=0}^{n}\left[\sum_{H^e}\det(A(H^e))\right]x^{n-k},$$

where the second sum is over all induced subdigraphs with k vertices of G^e. Note that if k is odd then $\det(A(H^e)) = 0$. Hence

$$\det(xI_n + A(G^e)) = \sum_{j=0}^{\lfloor n/2\rfloor}\left[\sum_{H^e}\det(A(H^e))\right]x^{n-2j},$$

where the second sum is over all induced subdigraphs with $2j$ vertices of G^e. Thus we have

$$\frac{1}{2^\varepsilon}\sum_{G^e\in\Phi(G)}\det(xI_n + A(G^e)) = \frac{1}{2^\varepsilon}\sum_{G^e\in\Phi(G)}\sum_{j=0}^{\lfloor n/2\rfloor}\left[\sum_{H^e}\det(A(H^e))\right]x^{n-2j}.$$

Suppose that the underlying graph of H^e is H and the number of edges in H is ε'. Then the subgraph $G - E(H)$ of G has $2^{\varepsilon-\varepsilon'}$ orientations. Denote the set of these $2^{\varepsilon-\varepsilon'}$ orientations by $\Phi(G - E(H)) = \{M'_k\}_{k=1}^{2^{\varepsilon-\varepsilon'}}$. For an arbitrary orientation M'_k in $\Phi(G - E(H))$, then H has $2^{\varepsilon'}$ orientations. Hence we may partition the set $\Phi(G)$ of all orientations of G into $\Phi(G) = \cup_{k=1}^{2^{\varepsilon-\varepsilon'}} M_k^*$, where M_k^* is the set of those $2^{\varepsilon'}$ orientations of G in which H has $2^{\varepsilon'}$ orientations and the orientation of the subgraph $G - E(H)$ is M'_k. Hence, by Proposition 1, we have

$$\frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \sum_{j=0}^{\lfloor n/2 \rfloor} \left[\sum_{H^e} \det(A(H^e)) \right] x^{n-2j}$$
$$= \frac{1}{2^{\varepsilon}} \sum_{k=1}^{2^{\varepsilon-\varepsilon'}} \sum_{H^e \in M_k^*} \sum_{j=0}^{\lfloor n/2 \rfloor} \left[\sum_{H^e} \det(A(H^e)) \right] x^{n-2j}$$
$$= \frac{1}{2^{\varepsilon}} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{H^e} \sum_{k=1}^{2^{\varepsilon-\varepsilon'}} \left[\sum_{H^e \in M_k^*} \det(A(H^e)) \right] x^{n-2j}$$
$$= \frac{1}{2^{\varepsilon}} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{H^e} \sum_{k=1}^{2^{\varepsilon-\varepsilon'}} [2^{\varepsilon'} \phi(H)] x^{n-2j}$$
$$= \frac{1}{2^{\varepsilon}} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{H^e} 2^{\varepsilon-\varepsilon'} [2^{\varepsilon'} \phi(H)] x^{n-2j} = \sum_{j=0}^{\lfloor n/2 \rfloor} \left[\sum_{H^e} \phi(H) \right] x^{n-2j},$$

where $\phi(H)$ denotes the number of perfect matchings of H. Now we prove the following claim.

Claim. $\sum_{H^e} \phi(H)$ ($= \sum_H \phi(H)$) equals $\phi_j(G)$, the number of j-element matchings in G.

Let Ψ be the set of j-matchings of G and Υ equals $\cup M(H)$, where $M(H)$ denotes the set of perfect matchings of H and the union $\cup$ ranges over all induced subgraphs with $2j$ vertices of G. Note that $M(H_1) \cap M(H_2) = \emptyset$ for any two induced subgraphs H_1 and H_2 with $2j$ vertices of G such that $H_1 \neq H_2$. It is not difficult to see that there exists a bijection between Ψ and Υ. The claim thus follows and then

$$\frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(xI_n + A(G^e))$$
$$= \frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \sum_{j=0}^{\lfloor n/2 \rfloor} \left[\sum_{H^e} \det(A(H^e)) \right] x^{n-2j}$$
$$= \sum_{j=0}^{\lfloor n/2 \rfloor} \left[\sum_{H^e} \phi(H) \right] x^{n-2j} = \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j(G) x^{n-2j}$$
$$= x^n \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j(G)(x^{-2})^j = x^n g(G, x^{-2}),$$

that is,

$$\frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(xI_n + A(G^e)) = x^n g(G, x^{-2}).$$

Because

$$\frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(ixI_n + A(G^e))$$
$$= (ix)^n g(G, (ix)^{-2}), \quad m(G, x) = x^n g(G, -x^2)$$
$$= \frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(xI_n - iA(G^e))$$
$$= \frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(xI_n + iA(G^e)).$$

Thus we have finished the proof of the theorem.

By Theorem 15, we can prove the following.

Corollary 16. Suppose that G is a simple graph with n vertices and ε edges. Then

$$g(G, x) = \frac{1}{2^{\varepsilon}} \sum_{G^e \in \Phi(G)} \det(I_n + \sqrt{x} A(G^e)),$$
$$Z(\mathrm{G}) = \frac{1}{2^{\varepsilon}} \sum_{\mathrm{G}^e \in \Phi(\mathrm{G})} \det(I_n + A(G^e)),$$

where $g(G, x)$ and $Z(G)$ denote the matching generating polynomial and the Hosoya index of G, respectively.

Remark 17. Proposition 1 is a special case of Theorem 15 (the case $x = 0$). Hence Theorem 15 generalizes the result of Proposition 1.

Remark 18. Though we obtain algebraic expressions for the matching polynomial by Theorem 15 and for the Hosoya index by Corollary 16, the computation of these two formulae is not in polynomial time.

3. Conclusion

Theorem 15 and Corollary 16 assert that the average values of $\det(xI_n + iA(G^e))$, $\det(I_n + \sqrt{x}A(G^e))$, and $\det(I_n + A(G^e))$ over the orientations of G are equal to $m(G, x)$, $g(G, x)$, and $Z(G)$, respectively. Note that, by Propositions 1 and 2, the average value of $\det(A(G^e))$ over the orientations of G is the number of perfect matchings of G. On the other hand, it is well known (see p. 121 in Godsil [6]) that $\det(A(G^e))$ is bounded above by the square of the number of perfect matchings of G. Hence $\max\{\det(A(G^e)) - \phi(G)\} = \phi^2(G) - \phi(G)$. Based on this it is natural to ask what is the upper bound for

$\det(I_n + A(G^e)) - Z(G)$, that is, what is $\max\{\det(I_n + A(G^e)) - Z(G)\}$? For a graph with no cycles of even length, by Theorem 8 all determinants $\det(I_n + A(G^e))$ over the orientations of G are equal (to the Hosoya index $Z(G)$). Hence $\max\{\det(I_n + A(G^e)) - Z(G)\} = \min\{\det(I_n + A(G^e)) - Z(G)\} = 0$.

Note that the matching polynomial $m(T, x)$ of a tree T equals the characteristic polynomial of T, that is, $m(T, x)$ can be expressed by the characteristic polynomial of the adjacency matrix of T. As that stated in the introduction of this article, the matching polynomials of certain graphs including unicyclic graphs and cacti can be expressed in terms of the characteristic polynomials of certain Hermitian matrices [9–12, 14, 15, 17, 26, 29]. In this article, we express the matching polynomial of a graph G with small number of cycles of even length in terms of the characteristic polynomials of the skew adjacency matrix of some orientation of G and its minors. Yan et al. [33] use the characteristic polynomial of the skew adjacency matrix of some orientation to express the permanental polynomial of a type of graphs. It is also interesting to find graphs whose matching polynomial can be expressed in terms of the characteristic polynomial of some matrices.

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