DOI: 10.1007/s11425-005-0036-y

On the number of matchings of graphs formed by a graph operation

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Received October 08, 2005; accepted May 29, 2006

Abstract Let G be a simple graph. Define $R(G)$ to be the graph obtained from G by adding a new vertex e^* corresponding to each edge $e = (a, b)$ of G and by joining each new vertex e^* to the end vertices a and b of the edge e corresponding to it. In this paper, we prove that the number of matchings of $R(G)$ is completely determined by the degree sequence of vertices of G.

Keywords: matching, Hosoya index, matching polynomial, oriented incident matrix, skew adjacency matrix.

1 Introduction

Throughout this paper, we suppose that $G = (V(G), E(G))$ is a simple graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$, if not specified. For arbitrary $v_i \in V(G)$, we use $d_G(v_i)$ (or d_i) to denote the degree of v_i. Let $D(G)$ be the diagonal matrix of vertex degrees of G, that is, $D(G) = diag(d_1, d_2, \cdots, d_n)$. Let $A(G) = (a_{ij})_{n\times n}$ be the square matrix of order n, where a_{ij} equals one if (v_i, v_j) is an edge of G and zero otherwise. We call $A(G)$ to be the adjacency matrix of G. Suppose G^e is an orientation of G. Let $A(G^e) = (b_{ij})_{n\times n}$ be the matrix of order n defined as follows:

$$b_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an arc in } G^e, \\ -1 & \text{if } (v_j, v_i) \text{ is an arc in } G^e, \\ 0 & \text{otherwise.} \end{cases}$$

We call $A(G^e)$ to be the skew adjacency matrix of G^e (see for example ref. [1]). Obviously, $A(G^e)$ is a skew symmetric matrix of order n, that is, $(A(G^e))^T = -A(G^e)$. The oriented incident matrix of G^e is denoted by $M(G^e)$, which is defined as $M(G^e) = (c_{ij})_{n\times m}$, where

$$c_{ij} = \begin{cases} 1 & \text{if the arc } e_j \text{ is an out-arc of } v_i, \\ -1 & \text{if the arc } e_j \text{ is an in-arc of } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that $M(G^e)M(G^e)^T = D(G) - A(G)$ (see ref. [2]), which is called the Laplacian matrix of G.

Let $e = (u, v)$ be an edge of G (for the sake of convenience, sometimes we will use uv to denote edge (u, v)). Denote by $G - u - v$ (resp. $G - e = G - uv$) the induced subgraph of G by deleting vertices u and v (resp. edge e) from G. A set M of edges of G is a matching if every vertex of G is incident with at most one edge in M; it is a perfect matching if every vertex of G is incident with exactly one edge in M. Denote by $m(G, j)$ the number of matchings of G with j edges. We set $m(G, 0) = 1$ by convention. Thus $m(G, 1) = m$ is the number of edges in G, and if n is even then $m(G, \frac{n}{2})$ is the number of perfect matchings of G. Denote by $Z(G)$ the number of matchings of G, that is,

$$Z(G) = m(G, 0) + m(G, 1) + \cdots + m(G, [\frac{n}{2}]).$$

The number of matchings of G is also called the Hosoya index (see ref. [3]). The polynomial

$$m(G, x) = \sum_{j=0}^{[\frac{n}{2}]} (-1)^j m(G, j) x^{n-2j}$$

is called the matching polynomial of G. By the definitions of $Z(G)$ and $m(G, x)$, $Z(G) = \frac{m(G,x)|_{x=i}}{i^n}$, where $i^2 = -1$. Because of numerous applications to physics and chemistry, the number of matchings and the matching polynomial have been studied extensively not only by mathematicians but also by physicists and chemists (see for example refs. [4–15]). Note that the concept of a matching is a classical one in graph theory (see ref. [1]), it would not be unreasonable to expect that mathematical literature frequently. The first chemical application of $Z(G)$ was proposed in 1971 by a chemist Hosoya in ref. [3], which was used to describe the thermodynamic properties of saturated hydrocarbons. The computation of $Z(G)$ and $m(G, x)$ is NP-Complete (see ref. [16]). Some related work see for example refs. [17–21].

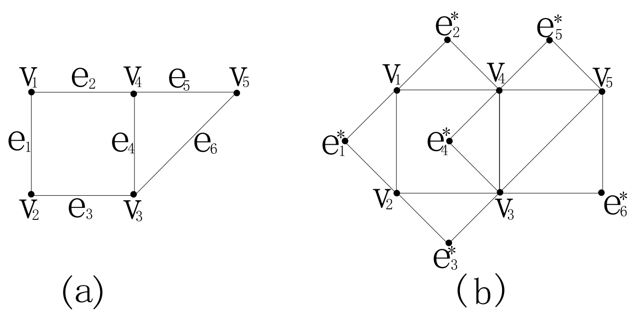

Figure 1: **(a)** The graph G. **(b)** The corresponding graph $R(G)$ of G.

Suppose $G = (V(G), E(G))$ is a simple graph. We define a new graph $R(G)$ (see the definition in page 63 in ref. [2]) as follows. Let $R(G) = (V(R(G)), E(R(G)))$ be the graph obtained from G by adding a new vertex e^* corresponding to each edge $e = (a, b) = ab$ of G and by joining each new vertex e^* to the end vertices a and b of the edge $e = (a, b)$ corresponding to it, that is, $R(G)$ is obtained from G by "changing each edge $e = ab$ of G into a triangle ae^*b". Hence $V(R(G)) = V(G) \cup \{e^*|e \in E(G)\}$ and $E(R(G)) = E(G) \cup \{(v_r, e^*), (v_s, e^*)|e = (v_r, v_s) \in E(G)\}$. Figure 1 (a) and (b) illustrate this procedure constructing $R(G)$ from G.

We have the following main result:

Theorem 1. Let $G = (V(G), E(G))$ be a simple graph with n vertices and $R(G)$ the graph defined above. Then the number of matchings of $R(G)$ can be given by

$$Z(R(G)) = (d_1 + 1)(d_2 + 1) \cdots (d_n + 1),$$

where $(d_1, d_2, \cdots, d_n)$ is the degree sequence of vertices of G.

We will give two different proofs, one inductive and the other bijective, for the above theorem.

2 Proofs

In order to prove Theorem 1, we need to introduce some lemmas.

Lemma 2[22]**.** Let G be a graph and $e = (a, b)$ an edge of G. Then

$$Z(G) = Z(G - ab) + Z(G - a - b),$$

where $G - ab$ and $G - a - b$ denote the induced subgraphs of G by deleting edge (a, b) and by deleting vertices a and b, respectively.

Let $G = (V(G), E(G))$ be a graph with n vertices and a a vertex of G. Construct a new graph $G' = (V(G'), E(G'))$ with $n + 2$ vertices such that $V(G') = V(G) \cup \{x, y\}, E(G') = E(G) \cup \{(a, x), (a, y), (x, y)\}$ (see Figure 2(a) and (b)).

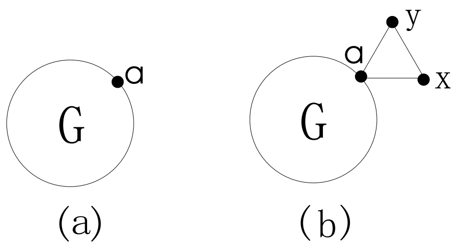

Figure 2: **(a)** The graph G. **(b)** The corresponding graph G'.

Lemma 3. Let G be a simple graph and G' the graph defined above. Then

$$Z(G') = 2Z(G' - x) = 2Z(G' - y).$$

Proof. By Lemma 2, we have

$$Z(G') = Z(G' - x - y) + Z(G' - xy) = Z(G) + Z(G' - xy),$$

$$Z(G' - xy) = Z(G - a) + Z(G' - x),\ Z(G' - x) = Z(G - a) + Z(G).$$

Hence we have

$$Z(G') = Z(G) + Z(G - a) + Z(G' - x) = 2Z(G' - x).$$

Similarly, we can prove $Z(G') = 2Z(G' - y)$. The lemma thus follows.

Lemma 4[5,12]**.** Let G be a simple graph with n vertices and with no cycle of even length and G^e arbitrary orientation of G. Then

$$Z(G) = \det(I_n + A(G^e)),$$

where I_n is a unit matrix of order n and $A(G^e)$ is the skew adjacency matrix of G^e.

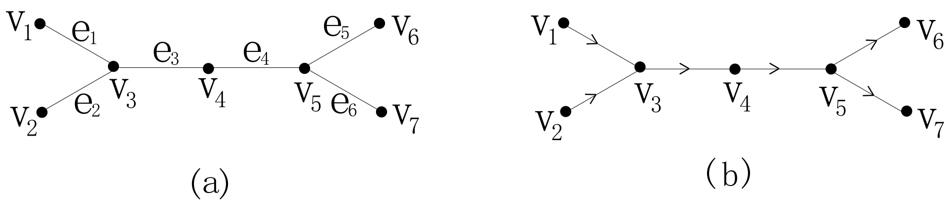

Figure 3: **(a)** The tree T in the proof of Lemma 5. **(b)** The orientation T^e of T.

Lemma 5. If T is a tree with n vertices and $R(T)$ is the graph defined above, then the number of matchings of $R(T)$ can be expressed by

$$Z(R(T)) = (d_1 + 1)(d_2 + 1)\cdots(d_n + 1),$$

where $(d_1, d_2, \cdots, d_n)$ is the degree sequence of vertices of T.

Proof. Suppose T^e is an orientation of T obtained as follows. For each edge (v_i, v_j) of T, we define its direction to be from v_i to v_j if $i < j$. For the tree T showed in Figure 3(a). The corresponding orientation T^e is illustrated in Figure 3(b).

Now we define an orientation $R(T)^e$ of $R(T)$ from the orientation T^e of T. Note that $R(T)$ has only two types of edges (see Figure 4(a)), one of which has the form (v_i, v_j) and the other of which has the form (v_k, e_l^*) (where we assume the end vertices of e_l in T are v_k and v_s). We define the direction of edges (v_i, v_j) of the first type to be the same as in T^e and that of edges (v_k, e_l^*) of the second type to be from v_k to e_l^* if $k < s$ and from e_l^* to v_k otherwise. The resulting orientation of $R(T)$ is denoted by $R(T)^e$. For the tree T shown in Figure 3(a), the corresponding orientation $R(T)^e$ is illustrated in Figure 4(b).

By a suitable labelling of vertices of $R(T)^e$, the skew adjacency matrix of $R(T)^e$ has the following form:

$$A(R(T)^e) = \begin{pmatrix} A(T^e) & M(T^e) \\ -M(T^e)^T & 0 \end{pmatrix},$$

where $A(T^e)$ and $M(T^e)$ denote the skew adjacency matrix of T^e and the oriented incident matrix of T^e, respectively.

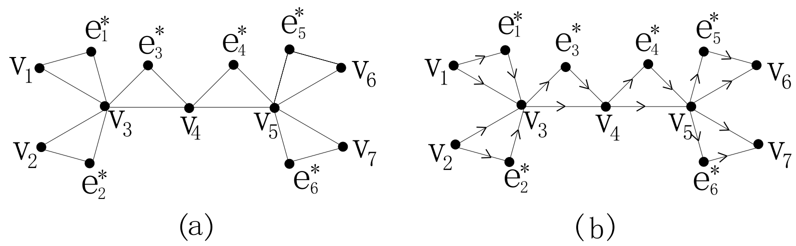

Figure 4: **(a)** The corresponding graph $R(T)$ of T in Figure 3(a). **(b)** The orientation $R(T)^e$ of $R(T)$.

Since T is a tree, by the definition of $R(T)$, $R(T)$ contains no cycle of even length. Using Lemma 4, we have

$$Z(R(T)) = \det(I_{2n-1} + A(R(T)^e)) = \det \begin{pmatrix} I_n + A(T^e) & M(T^e) \\ -M(T^e)^T & I_{n-1} \end{pmatrix}$$

$$= \det(I_n + A(T^e) + M(T^e)M(T^e)^T) = \det(I_n + A(T^e) + D(T) - A(T)),$$

where $D(T)$ and $A(T)$ are the diagonal matrix of vertex degrees of T and the adjacency matrix of T, respectively. By the definition of T^e, $A(T^e) - A(T)$ is a lower triangular matrix. Hence

$$Z(R(T)) = \det(I_n + D(T) + A(T^e) - A(T)) = (d_1 + 1)(d_2 + 1) \cdots (d_n + 1)$$

and the lemma follows.

The following corollary is immediate from Lemma 5.

Corollary 6. If G is a forest with n vertices and $R(G)$ is the graph defined above, then the number of matchings of $R(G)$ is given by

$$Z(R(G)) = (d_1 + 1)(d_2 + 1) \cdots (d_n + 1),$$

where $(d_1, d_2, \cdots, d_n)$ is the degree sequence of of vertices of G.

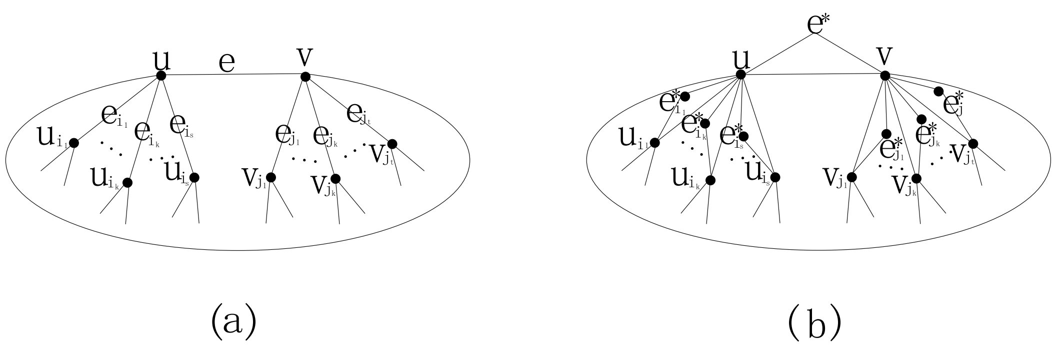

Figure 5: **(a)** The graph G. **(b)** The corresponding graph $R(G)$ of G.

The inductive proof of Theorem 1. We prove the theorem by induction on the cyclomatic number of G. If G contains no cycle, the theorem follows from Corollary 6. Now we suppose that G contains cycles and proceed by induction on the cyclomatic number of G. Let $e=(u,v)=uv$ be an edge on a cycle of G. Let $\{u_{i_1},u_{i_2},\cdots,u_{i_s},v\}$ and $\{v_{j_1},v_{j_2},\cdots,v_{j_t},u\}$ be the sets of vertices of G adjacent to u and v in G, respectively (see the graph shown in Figure 5(a)). For the sake of convenience, denote the edges (u,u_{i_k}) for $1\le k\le s$ and (v,v_{j_k}) for $1\le k\le t$ by e_{i_k} and e_{j_k}, respectively. Hence $d_G(u)=s+1, d_G(v)=t+1$. It is suffices to prove the following:

$$Z(R(G))=(s+2)(t+2)\prod_{w\in V(G),w\neq u,v}(d_G(w)+1).$$

Now we define three graphs G_u, G_v, and G_{uv} as follows:

(i) G_u is the graph with $n-1+s$ vertices obtained from G by deleting vertex u and adding s pendant edges $(u_{i_1},e'_{m_1}),(u_{i_2},e'_{m_2}),\cdots,(u_{i_s},e'_{m_s})$;

(ii) G_v is the graph with $n-1+t$ vertices obtained from G by deleting vertex v and adding t pendant edges $(v_{j_1},e'_{n_1}),(v_{j_2},e'_{n_2}),\cdots,(v_{j_t},e'_{n_t})$;

(iii) G_{uv} is the graph with $n-2+s+t$ vertices obtained from G by deleting vertex u and v and adding $s+t$ pendant edges $(u_{i_1},e'_{m_1}),(u_{i_2},e'_{m_2}),\cdots,(u_{i_s},e'_{m_s}),(v_{j_1},e'_{n_1}),$ $(v_{j_2},e'_{n_2}),\cdots,(v_{j_t},e'_{n_t})$.

Since $e=(u,v)$ is an edge on a cycle of G, the cyclomatic number of the four graphs $G-e,G_u,G_v$ and G_{uv} is smaller than that of G. Hence, by induction, we have

$$Z(R(G-e))=\prod_{w\in V(R(G-e))}(d_{R(G-e)}(w)+1)=(s+1)(t+1)\prod_{w\in V(G),w\neq u,v}(d_G(w)+1); \tag{1}$$

$$Z(R(G_u))=\prod_{w\in V(G_u)}(d_{G_u}(w)+1)=2^s(t+1)\prod_{w\in V(G),w\neq u,v}(d_G(w)+1); \tag{2}$$

$$Z(R(G_v))=\prod_{w\in V(G_v)}(d_{G_v}(w)+1)=2^t(s+1)\prod_{w\in V(G),w\neq u,v}(d_G(w)+1); \tag{3}$$

$$Z(R(G_{uv}))=\prod_{w\in V(G_{uv})}(d_{G_{uv}}(w)+1)=2^{s+t}\prod_{w\in V(G),w\neq u,v}(d_G(w)+1). \tag{4}$$

On the other hand, by repeated applications of Lemma 2 (see Figure 5(a) and (b)), we have

$$Z(R(G))=Z(R(G)-ue^*-ve^*-uv)+$$
$$Z(R(G)-u-e^*)+Z(R(G)-v-e^*)+Z(R(G)-u-v-e^*). \tag{5}$$

By the definition of $R(G)$, we have

$$R(G)-ue^*-ve^*-uv=R(G-uv)=R(G-e). \tag{6}$$

Note that $R(G)-u-e^*$ contains s pendant vertices $e^*_{i_1},e^*_{i_2},\cdots,e^*_{i_s}$, $R(G)-v-e^*$ contains t pendant vertices $e^*_{j_1},e^*_{j_2},\cdots,e^*_{j_t}$, and $R(G)-u-v-e^*$ contains $s+t$ pendant vertices $e^*_{i_1}, \cdots, e^*_{i_s}, e^*_{j_1}, \cdots, e^*_{j_t}$ (see Figure 5(b)). With repeated applications of Lemma 3, we can show that

$$Z(R(G_u)) = 2^s Z(R(G) - u - e^*); \tag{7}$$

$$Z(R(G_v)) = 2^t Z(R(G) - v - e^*); \tag{8}$$

$$Z(R(G_{uv})) = 2^{s+t} Z(R(G) - u - v - e^*). \tag{9}$$

Hence, by $(1-9)$, we have

$$\begin{aligned} Z(R(G)) &= Z(R(G) - ue^* - ve^* - uv) + Z(R(G) - u - e^*) + Z(R(G) - v - e^*) \\ &\quad + Z(R(G) - u - v - e^*) \\ &= Z(R(G-e)) + 2^{-s} Z(R(G_u)) + 2^{-t} Z(R(G_v)) + 2^{-s-t} Z(R(G_{uv})) \\ &= (s+1)(t+1) \prod_{w \in V(G), w \neq u,v} (d_G(w)+1) + (t+1) \prod_{w \in V(G), w \neq u,v} (d_G(w)+1) \\ &\quad + (s+1) \prod_{w \in V(G), w \neq u,v} (d_G(w)+1) + \prod_{w \in V(G), w \neq u,v} (d_G(w)+1) \\ &= (s+2)(t+2) \prod_{w \in V(G), w \neq u,v} (d_G(w)+1)). \end{aligned}$$

Note that $d_G(u) = s+1$ and $d_G(v) = t+1$. This implies that

$$(s+2)(t+2) \prod_{w \in V(G), w \neq u,v} (d_G(w)+1)) = (d_1+1)(d_2+1)\cdots(d_n+1).$$

Hence the theorem has been proved.

In order to give the bijective proof of Theorem 1, we need to introduce some notation as follows. Let $G = (V(G), E(G))$ be a simple graph with the vertex set $\{v_1, v_2, \cdots, v_n\}$ and the edge set $\{e_1, e_2, \cdots, e_m\}$ and let $R(G) = (V(R(G)), E(R(G)))$ be the graph defined above, where $V(R(G)) = V(G) \cup \{e^*_i | 1 \leq i \leq m\}$ and $E(R(G)) = E(G) \cup \{(v_i, e^*), (v_j, e^*) | e = (v_i, v_j) \in E(G)\}$. Hence $R(G)$ contains two types of vertices: v-type vertices and e-type vertices, and two types of edges: (v,v)-type edges and (v,e)-type edges. Assume that $\mathcal{M}(R(G))$ denotes the set of matchings of $R(G)$, where the empty set is regarded as a matching of $R(G)$ with zero edge. Let $M \in \mathcal{M}(R(G))$. Denote by $V(M^c)$ the set of v-type vertices of $R(G)$ which are not incident with the edges in M. Define $M^* = M \cup V(M^c)$ and $\mathcal{M}^*(R(G)) = \{M^* | M \in \mathcal{M}(R(G))\}$. By the definitions of $\mathcal{M}(R(G))$ and $\mathcal{M}^*(R(G))$, we have $|\mathcal{M}(R(G))| = |\mathcal{M}^*(R(G))|$ since $V(M^c)$ is uniquely determined by M. Let v_i be arbitrary v-type vertex of $R(G)$. Define E_i as the set of the (v,e)-type edges (v_i, e^*) of $R(G)$ incident with vertex v_i and set $X_i = \{v_i\} \cup E_i$. Obviously, $|X_i| = |E_i| + 1 = d_i + 1$, where d_i is the degree of v_i of G. Define $\mathcal{W}(R(T)) = X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \cdots, x_n) | x_i \in X_i\}$. Hence $|\mathcal{W}(R(G))| = (d_1+1)(d_2+1)\cdots(d_n+1)$.

For the graphs G and $R(G)$ showed in Figure 6(a) and (b), $E_1 = \{(1, e^*_1)\}, E_2 = \{(2, e^*_1), (2, e^*_2)\}, E_3 = \{(3, e^*_2)\}$, $X_1 = \{1, (1, e^*_1)\}, X_2 = \{2, (2, e^*_1), (2, e^*_2)\}, X_3 =$

$\{3, (3, e_2^*)\}$. $M = \{(1, e_1^*), (3, e_2^*)\}$ is a matching of $R(G)$ with two edges. By the definition of M^*, $M^* = \{2, (1, e_1^*), (3, e_2^*)\}$.

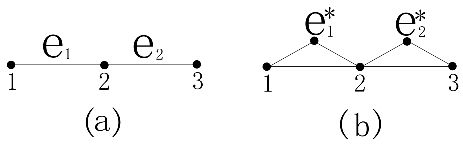

Figure 6: **(a)** The path G with three vertices. **(b)** The corresponding graph $R(G)$ of G.

The bijective proof of Theorem 1. Let $\mathcal{M}^*(R(G))$ and $\mathcal{W}(R(G))$ be two sets defined above. Hence $|\mathcal{M}^*(R(G))| = |\mathcal{M}(R(G))|$. We only need to construct a bijection $f : \mathcal{W}(R(G)) \longrightarrow \mathcal{M}^*(R(G))$. Given $W = (x_1, x_2, \cdots, x_n) \in \mathcal{W}(R(G))$, we need to define $f(W) \in \mathcal{M}^*(R(G))$. We consider the following three cases:

(i) if $x_i = v_i$, let $v_i \in f(W)$,

(ii) if there exists an e_{ij}^* such that $x_i = (v_i, e_{ij}^*)$ and $x_j = (v_j, e_{ij}^*)$, let $(v_i, v_j) \in f(W)$,

(iii) if there is an e_{ij}^* such that $x_i = (v_i, e_{ij}^*)$ and $x_j \neq (v_j, e_{ij}^*)$, let $(v_i, e_{ij}^*) \in f(W)$, where e_{ij} denotes the edge (v_i, v_j) in G.

All the elements in $f(W)$ are obtained only by the above procedure. Those elements in $f(W)$ obtained by the procedure (ii) and (iii) form a matching M in $R(G)$ and those in $f(W)$ obtained by the procedures (i) are exactly those v-type of vertices in $R(G)$ which are not incident with M. It is not difficult to see the above procedure constructing $f(W)$ is invertible. This implies that f is bijective (we will illustrate this bijection in the following example). Hence we have finished the bijective proof.

Example. Let G be a path with three vertices and $R(G)$ the corresponding graph of G (see Figure 6(a) and (b)). By the definition of $\mathcal{M}(R(G))$, $\mathcal{M}(R(G))$ denotes the set of matchings of $R(G)$. Hence $\mathcal{M}(R(G)) = \{M_i | 0 \leq i \leq 11\}$, where $M_0 = \Phi$, $M_1 = \{(1, 2)\}, M_2 = \{(2, 3)\}, M_3 = \{(1, e_1^*)\}, M_4 = \{(2, e_1^*)\}, M_5 = \{(2, e_2^*)\}, M_6 = \{(3, e_2^*)\}, M_7 = \{(1, e_1^*), (2, 3)\}, M_8 = \{(1, e_1^*), (3, e_2^*)\}, M_9 = \{(1, e_1^*), (2, e_2^*)\}$, $M_{10} = \{(1, 2), (3, e_2^*)\}, M_{11} = \{(2, e_1^*), (3, e_2^*)\}$. By the definition of M_i^* for $0 \leq i \leq 11$, we have $M_0^* = \{1, 2, 3\}, M_1^* = \{3, (1, 2)\}, M_2^* = \{1, (2, 3)\}, M_3^* = \{2, 3, (1, e_1^*)\}, M_4^* = \{1, 3, (2, e_1^*)\}, M_5^* = \{1, 3, (2, e_2^*)\}, M_6^* = \{1, 2, (3, e_2^*)\}, M_7^* = \{(1, e_1^*), (2, 3)\}, M_8^* = \{2, (1, e_1^*), (3, e_2^*)\}, M_9^* = \{3, (1, e_1^*), (2, e_2^*)\}, M_{10}^* = \{(1, 2), (3, e_2^*)\}, M_{11}^* = \{1, (2, e_1^*), (3, e_2^*)\}$; $E_1 = \{(1, e_1^*)\}, E_2 = \{(2, e_1^*), (2, e_2^*)\}, E_3 = \{(3, e_2^*)\}$; $X_1 = \{1, (1, e_1^*)\}, X_2 = \{2, (2, e_1^*), (2, e_2^*)\}, X_3 = \{3, (3, e_2^*)\}$. By the definition of W_i's, $W_0 = (1, 2, 3), W_1 = ((1, e_1^*), (2, e_1^*), 3), W_2 = (1, (2, e_2^*), (3, e_2^*)), W_3 = ((1, e_1^*), 2, 3), W_4 = (1, (2, e_1^*), 3), W_5 = (1, (2, e_2^*), 3), W_6 = (1, 2, (3, e_2^*)), W_7 = ((1, e_1^*), (2, e_2^*), (3, e_2^*)), W_8 = ((1, e_1^*), 2, (3, e_2^*)), W_9 = ((1, e_1^*), (2, e_2^*), 3), W_{10} = ((1, e_1^*), (2, e_1^*), (3, e_2^*)), W_{11} = (1, (2, e_1^*), (3, e_2^*))$. Obviously, $\mathcal{W}(R(G)) = X_1 \times$

$X_2 \times X_3 = \{W_i | 0 \leq i \leq 11\}$ and the mapping $f : W_i \longmapsto f(W_i) = M_i^*$ between $\mathcal{W}(R(G))$ and $\mathcal{M}^*(R(G))$ is bijective.

3 Conclusion

In this paper we show that the number of matchings of $R(G)$ can be expressed as a function of the degree sequence of vertices of G. It is natural to ask whether we can express the matchings polynomial of $R(G)$ by some parameters of G. It is also interesting to know whether we can express the matchings polynomial or the number of matchings of the total graph $T(G)$ and the line graph $L(G)$ of G by some parameters of G?

Acknowledgements We would like to thank the referees for providing some very helpful suggestions for revising this paper. This work was partially supported by FMSTF(2004J024), NSFF(E0540007), and NSC94-M001-017.

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