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Hamiltonicity of hypercubes with a constraint of required and faulty edges

Lih-Hsing Hsu · Shu-Chung Liu · Yeong-Nan Yeh

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Abstract Let R and F be two disjoint edge sets in an n-dimensional hypercube Q_n. We give two constructing methods to build a Hamiltonian cycle or path that includes all the edges of R but excludes all of F. Besides, considering every vertex of Q_n incident to at most $n-2$ edges of F, we show that a Hamiltonian cycle exists if (A) $|R|+2|F|\le 2n-3$ when $|R|\ge 2$, or (B) $|R|+2|F|\le 4n-9$ when $|R|\le 1$. Both bounds are tight. The analogous property for Hamiltonian paths is also given.

Keywords Hamiltonian cycles and paths · Edge-fault-tolerance · Required edge · Hypercubes

1 Introduction

Hamiltonian property is a classical problem in graph theory and computer science. For Hamiltonian property on a general graph, the reader can refer (Häggkvist 1979; Kronk 1969). In this paper, we consider the problem of embedding a ring or a path in

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L.-H. Hsu
Department of Computer Science and Information Engineering, Providence University, Taichung County, Taiwan

S.-C. Liu (✉)
General Education Center, China University of Technology, Hsinchu, Taiwan
e-mail: liularry@cute.edu.tw

Y.-N. Yeh
Institute of Mathematics, Academia Sinica, Taipei, Taiwan

a hypercube, Q_n, under the condition that some required edges must be passed and some faulty edges must not. Let R and F denote the sets of required edges and faulty edges, respectively. The pair (R, F) is called a *constraint* and it must be *reasonable*, that means, $R \cap F = \emptyset$ and R is an edge set of independent paths, i.e., the subgraph induced by R contains no cycle and no *branching point* (a vertex of degree ≥ 3). We abbreviate Hamiltonian cycle to HC and Hamiltonian path to HP. Furthermore, "a Hamiltonian cycle of Q_n *under* (or *satisfying*) constraint (R, F)" is abbreviated to "an HC$(Q_n; R, F)$", and we call such a cycle an (R, F)-*Hamiltonian cycle of* Q_n. Briefly, we write HC$(Q_n; R)$ for HC$(Q_n; R, \emptyset)$. Similarly, we define the analogous abbreviations for HP.

Let $r = |R|$ and $f = |F|$. It is known that an HC$(Q_n; \emptyset, F)$ exist if $f \leq n - 2$ (Latifi et al. 1992; Tseng 1996). In a recent paper, the authors generalized this previous result by the following theorem.

Theorem 1.1 (Hsu et al. 2001) *Let $n \geq 2$ and (R, F) be a reasonable constraint. Then*

(HC) *an HC$(Q_n; R, F)$ exists if $r + 2f \leq 2n - 3$;*
(HP) *an HP$(Q_n; R, F)$ exists if $r + 2f \leq 2n - 1$.*

These two bound are tight since there are counterexamples for every pair (r, f) exceeding the bounds (see Hsu et al. 2001). For instance, no Hamiltonian cycle exists if $n - 1$ faulty edges are incident to a particular vertex. In this paper, we avoid this trivial non-Hamiltonian condition and consider the Hamiltonian property under the restriction on incidency as follows.

[RI] Every vertex is incident to at most $n - 2$ faulty edges.

A. Sengupta proved that, under [RI] restriction, an HC$(Q_n; \emptyset, F)$ exists if $n \geq 4$ and $f \leq n - 1$ (Sengupta 1998). In Sect. 3, we will derive a generalized result: for $n \geq 3$, an HC$(Q_n; R, F)$ exists if (A) $r + 2f \leq 2n - 3$ when $r \geq 2$, or (B) $r + 2f \leq 4n - 9$ when $r \leq 1$. Notice that the inequality in (A) is as same as the one in Theorem 1.1(HC), that means, [RI] restriction does not relax the old boundary when $r \geq 2$. When $n \geq 3$, both bounds in (A) and (B) are tight due to the following two counterexamples as the inequalities fail.

Example 1.2 Let $n \geq 3$. Given a vertex u in Q_n. First we choose a 2-path joining two neighbors of u without passing through u, and ask the two edges of this path required. For the other $n - 2$ neighbors b of u, we make either ub faulty or b incident to two required edges, none of which is ub. This (R, F) around u satisfies [RI], and it has $r \geq 2$ and $r + 2f = 2n - 2$. Clearly, no HC$(Q_n; R, F)$ exists.

Example 1.3 Let $n \geq 3$. Suppose that $\langle v_1, v_2, v_3, v_4, v_1 \rangle$ is a 4-cycle of Q_n. Let both v_1 and v_3 be incident to $n - 2$ faulty edges, none of which is an edge of the 4-cycle. This (R, F) satisfies [RI], and it has $r = 0$ and $f = 2n - 4$; so $r + 2f = 4n - 8$. No HC$(Q_n; R, F)$ exists because both v_1 and v_3 have only one way in and out, and following that way we obtain the given 4-cycle.

Considering that every vertex is incident to at most $n-1$ faulty edges, we have an analogous result for Hamiltonian paths: with $n \geq 2$, an HP(Q_n; R, F) exists if (A) $r \geq 2$ and $r + 2f \leq 2n - 1$ (as same as in Theorem 1.1) or (B) $r \leq 1$ and $r + f \leq 2n - 3$ (not $r + 2f$). Again, both bounds are tight when $n \geq 3$.

Example 1.4 Let $n \geq 3$. Some counterexamples for (A) can be found in (Hsu et al. 2001). As for (B), let $\langle v_1, v_2, v_3, v_4, v_1 \rangle$ is a 4-cycle of Q_n. Let all edges incident to v_1 and v_2 be faulty except v_1v_4 and v_2v_3. So there are $2n-3$ faulty edges which includes v_1v_2. Also let v_3v_4 be a required edge. So $r + f = 2n - 2$ and no Hamiltonian path exists for both v_1 and v_2 have only one way out, and together with the required edge v_3v_4 we obtain a 3-path. We get another counterexample for $(r, f) = (0, 2n-2)$ by setting no edge required and all edges incident to v_1 and v_2 faulty except v_1v_2.

In practical, divide-and-conquer is a highly efficient algorithm to deal with problems involving a symmetric structure. In the next section we will introduce two kinds of algorithm using the method of divide-and-conquer. In Sect. 3, we give the main results for constructing HC's and HP's.

2 Algorithms for constructing Hamiltonian cycles and paths

First of all, let us summarize some notation and definitions. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. In a hypercube Q_n, a vertex u can be represented by an n-bit string $\mathrm{u}_1\mathrm{u}_2\cdots\mathrm{u}_n$. To avoid confusion, the bits of a string are printed by the same alphabet without slant. An edge of Q_n is called an ith *dimensional edge* if its two end vertices differ only in the ith bit. For the recursive structure, Q_n can be bisected into two $(n-1)$-hypercubes, denoted by $Q^0_{n,i}$ and $Q^1_{n,i}$, by removing all ith dimensional edges. The subgraph $Q^j_{n,i}$ is actually induced by the vertex set $\{\mathrm{u}_1\mathrm{u}_2\cdots\mathrm{u}_n \in V(Q_n) \mid u_i = j\}$ for $j = 0, 1$. For symmetry, we will always bisect Q_n into $Q^0_{n,n}$ and $Q^1_{n,n}$, and briefly we use Q^0 and Q^1 to denote them, respectively. For any vertex $u \in V(Q_n)$, let u' denote the vertex such that uu' forms an nth dimensional edge. Obviously, $(u')' = u$. Similarly, for an edge $e = uv$ belongs to either $E(Q^0)$ or $E(Q^1)$, we define $e' = u'v'$. For an edge subset X without any nth-dimensional edge, we define its *dual copy* $X' = \{e' \mid e \in X\}$.

Given a constraint (R, F), let $R^0 = R \cap E(Q^0)$ and $r^0 = |R^0|$. And define R^1, r^1, F^0, f^0, F^1, and f^1 similarly. Also let $R^* = R - R^0 - R^1$ and $r^* = |R^*|$, and similarly for F^* and f^*, i.e., r^* and f^* count the required and faulty nth dimensional edges, respectively.

We say an edge $e \notin R$ is *free with respect to* (R, F) if the constraint $(R \cup \{e\}, F)$ is still reasonable; otherwise it is *unfree*. In other words, e is unfree w.r.t. (R, F) if and only if either $e \in R \cup F$ or $R \cup \{e\}$ contains a branching point or a cycle. We briefly say e is *free w.r.t.* R (or R-free) if it is free w.r.t. $(R, \emptyset)$, while F is ignored and not necessarily an empty set.

In the following two subsections, we assume $r^* = 0$ and introduce two kinds of algorithms for constructing HC's or HP's by the method of divide-and-conquer.

2.1 Dealing one division and then the other

Algorithm 2.1 Let Q_n be bisected into Q^0 and Q^1 with $r^* = 0$. If

(H1) an HC(Q^0; R^0, F^0) exists, say C_0, and
(H2) there exists $xy \in E(C_0) - R^0$ such that
 (a) $x'y'$ is free w.r.t. R^1 (also w.r.t. R for $r^* = 0$) and $xx', yy' \notin F^*$;
 (b) an HC(Q^1; $R^1 \cup \{x'y'\}$, $F^1 - \{x'y'\}$) exists, say C_1,

then there exists an HC(Q_n; R, F), namely $C_0 \cup C_1 \cup \{xx', yy'\} - \{xy, x'y'\}$.

Notice that in this algorithm one shall obtain C_0 first. Also notice that it is possible that $x'y'$ is faulty for the hypothesis (H2a) only requests $x'y'$ free w.r.t. R^1.

Let us derive a sufficient condition for the hypothesis (H2a). Given an edge set A of independent paths or cycles in Q^0 and a *nonempty* required edge set R^1 in Q^1, let us define $\bar{\mathcal{F}}_A(R^1) = \{e \in E(A) \mid e' \text{ is unfree w.r.t. } R^1\}$, where e' is the dual edge of e in Q^1. First we shall simply consider $R^1 = E(P_b^a)$, a path joining a and b. If there is no confusion, we will abuse the notation P_b^a as the edge set $E(P_b^a)$. An edge $e \in \bar{\mathcal{F}}_A(P_b^a)$ can be either of the two types: (a) $e = a'b' \in E(A)$, while $P_b^a \cup \{e'\}$ is a cycle or $E(P_b^a) = \{e'\}$; (b) e' is incident to an internal vertex of P_b^a, while either $e' \in P_b^a$ or $P_b^a \cup \{e'\}$ yields one branch point or two. Thus

$$|\bar{\mathcal{F}}_A(P_b^a)| = B(a'b' \in E(A)) + \sum_{xy \in A} B(x' \text{or } y' \in V_{\text{int}}(P_b^a)), \tag{1}$$

where $B(\cdot)$ is the Boolean function and $V_{\text{int}}(P_b^a)$ is the set of internal vertices of P_b^a. In particular, when $A = C_0$ is a Hamiltonian cycle in Q^0, we have

$$|\bar{\mathcal{F}}_{C_0}(P_b^a)| \le B(a'b' \in E(C_0)) + 2(|E(P_b^a)| - 1) - |E((C_0)') \cap E_{\text{int}}(P_b^a)|, \tag{2}$$

where $E_{\text{int}}(P_b^a)$ is the set of internal edges of P_b^a. The second term is due to that, for every internal vertex u of P_b^a, the dual vertex u' is incident to exactly two edges of C_0. The last term is the lower estimate for the number of those e' who yield two branch points, i.e., each of them is incident to two internal vertices of P_b^a. This lower estimate also claims that the strict inequality occurs when there are $x, y \in V_{\text{int}}(P_b^a)$ such that $xy \in E((C_0)') - E_{\text{int}}(P_b^a)$.

In general, R^1 contains k independent paths, say $P_{b_1}^{a_1}, \ldots, P_{b_k}^{a_k}$. We are actually interested to know an upper bound of $|R^0 \cup \bar{\mathcal{F}}_{C_0}(R^1)| = |R^0| + |\bar{\mathcal{F}}_{C_0}(R^1)| - |R^0 \cap \bar{\mathcal{F}}_{C_0}(R^1)|$. For $R^0 \subseteq E(C_0)$, we know $R^0 \cap \bar{\mathcal{F}}_{C_0}(R^1) = \bar{\mathcal{F}}_{R^0}(R^1)$, and then we derive the following by using (2) applying on C_0 and R^0 and (1) in part:

$$\begin{aligned} |R^0 \cup \bar{\mathcal{F}}_{C_0}(R^1)| &\le r^0 + \sum_{i=1}^{k} B(a_i b_i \in (E(C_0) - R^0)') + 2(r^1 - k) \\ &\quad - |E((C_0)') \cap E_{\text{int}}(R^1)| - \sum_{xy \in R^0} B(x' \text{ or } y' \in V_{\text{int}}(R^1)) \quad (3) \\ &\le r^0 + 2r^1 - k - \sum_{i=1}^{k} B(P_{b_i}^{a_i} \text{ is of even length}). \quad (4) \end{aligned}$$

The last upper bound is obtained by ignoring the two subtractions and considering every Boolean function applying on those $P_{b_i}^{a_i}$ of odd length in the second term always 1, while those $P_{b_i}^{a_i}$ of even length always make their Boolean function 0. Of course, when R^1 contains a single path, (4) shall be $r^0 + 2r^1 - 1 - B(r^1$ is even). Notice that if R^1 is empty, then $|R^0 \cup \bar{\mathcal{F}}_{C_0}(R^1)| = r^0$; so the bound in (4) does not work. To fixed this exceptional case, let us define

$$m(x) := \begin{cases} 0, & \text{if } x = 0; \\ 2x - 2, & \text{if } x \text{ is nonzero and even;} \\ 2x - 1, & \text{if } x \text{ is nonzero and odd.} \end{cases}$$

Thus the bound $r^0 + m(r^1)$ works for any r^1.

The last value we shall count is number of edges in $E(C_0) - R^0 \cup \bar{\mathcal{F}}_{C_0}(R^1)$ that are incident to an edge of F^*. The exact number is the subtraction of the two terms: (i) the number of such edges in $E(C_0)$ and (ii) the number of such edges in $R^0 \cup \bar{\mathcal{F}}_{C_0}(R^1)$. The number in (i) is

$$2f^* - \sum_{xy \in E(C_0)} B(x, y \in V(F^*)), \tag{5}$$

while the number in (ii), which can be evaluated by referring (3), is too tedious to given here. We just simply use the number in (i) or even $2f^*$ as an upper bound. Let us define three different bounds, B_1, B_2 and B_3, as follows:

$$B_1 = \text{the sum of the values in (3) and (5)},$$

$$B_2 = r^0 + \sum_{i=1}^{k} m(|E(P_{b_i}^{a_i})|) + 2f^*,$$

$$B_3 = r^0 + m(r^1) + 2f^*.$$

The last one is obtained from B_2 by assuming $k = 1$. Clearly, $B_1 \le B_2 \le B_3$. Now we conclude with the following lemma.

Lemma 2.2 *In Algorithm 2.1, there exists an edge satisfying hypothesis* (H2a) *if $B_i < 2^{n-1}$ for some i.*

Following Theorem 1.1, one can assume $r^0 + 2f^0 \le 2n - 5$ to ensure hypothesis (H1) and $r^1 + 2f^1 \le 2n - 6$ to ensure (H2b). So we have the next theorem as a conclusion.

Theorem 2.3 *Suppose $n \ge 3$ and $r^* = 0$. Then an $HC(Q_n; R, F)$ exists if an $HC(Q^0; R^0, F^0)$ exists, $B_i < 2^{n-1}$ for some i, and $r^1 + 2f^1 \le 2n - 6$.*

We can even use Theorem 3.2 given latter to guarantee (H1) and (H2b).

To construct a Hamiltonian path in Q_n, we can mimic Algorithm 2.1 by only modifying hypothesis (H1) to be (H1′), in which “HC” and C_0 are replaced with“HP” and P_0, respectively. By the same idea, we can only modify hypothesis (H2) to be (H2′) by replacing “HC” and C_1 with “HP” and P_1. In these two new algorithms, the argument above still works. We remind that when using the algorithm adopting (H1′) and (H2), the value 2^{n-1} in Lemma 2.2 and Theorem 2.3 shall be changed to be $2^{n-1}-1$. As for the algorithm adopting (H1) and (H2′), the value $2n-6$ shall be changed to be $2n-5$.

2.2 Dealing two divisions at the same time

Suppose $e \in E(Q^0)$. A pair of edges (e, e') is $R^0 \cup R^1$*-dual-free* if e is R^0-free and e' is R^1-free. If (e, e') is $R^0 \cup R^1$-dual-free, we can simply say e (or e') is dual-free.

Algorithm 2.4 Let Q_n be bisected into Q^0 and Q^1 with $r^* = 0$. If

(H3) there exists an $R^0 \cup R^1$-dual-free edge $x'y' \in E(Q^1)$ and $xx', yy' \notin F^*$, and
(H4) there exist an HC$(Q^0; R^0 \cup \{xy\}, F^0 - \{xy\})$, say C_0, and an HC$(Q^1; R^1 \cup \{x'y'\}, F^1 - \{x'y'\})$, say C_1.

Then there exists an HC$(Q_n; R, F)$, namely $C_0 \cup C_1 \cup \{xx', yy'\} - \{xy, x'y'\}$.

To find a sufficient condition for (H3), we shall first investigate a new defined edge set $\bar{\mathcal{F}}(R^0, R^1) = \{e \in E(Q^0) \mid e$ is R^0-unfree or e' is R^1-unfree$\}$, just like we did for $\bar{\mathcal{F}}_{C^0}(R^1)$ in the last subsection. One can follow our previous argument to get a sufficient condition for (H3). In practical, there are plenty of $R^0 \cup R^1$-dual-free edges, for the searching is among $E(Q^0)$; so we just find a suitable edge for (H3) directly.

Remark We can easily adjust these two algorithms when $r^* = 1$. Given $R^* = \{xx'\}$ in this case, the two corresponding algorithms are searching for a free xy while x is fixed. As for $r^* \geq 2$, we need more skill to deal with the construction. Since we only use the case when $r^* = 0$, the two cases above are not concerned in this paper.

3 Main results

Given an edge set A and a vertex v, we define $\deg_A(v)$ to be the number of edges in A that are incident to u. We need one more lemma, then we are ready to prove our main results.

Lemma 3.1 (Simmons 1978) *In Q_n, there is an HP joining any two vertices u and v with odd distance.*

Theorem 3.2 *Suppose $n \geq 3$ and F satisfying* (*RI*) *restriction, i.e.,* $\deg_F(x) \leq n-2$ *for every $x \in V(Q_n)$. There exists an HC$(Q_n; R, F)$ provided that* (*A*) *$r \geq 2$ and $r + 2f \leq 2n-3$, or* (*B*) *$r \leq 1$ and $r + 2f \leq 4n-9$.*

Remark We can also write (B) as $r \leq 1$ and $f \leq 2n-5$.

Proof Referring Theorem 1.1, we realize that (RI) restriction is redundant for part (A); so, nothing to prove for this part. As for (B), we need only deal with the extreme values $(r, f) = (1, 2n-5)$, because the inequality of (B) provides $f \leq 2n-5$

when $r=0$ or $r=1$. For $r=1$, let uv be the only required edge. By Theorem 1.1, the statement is true when $n=3$. Now we assume $n\geq 4$ and consider following three cases.

(1) Suppose there is a vertex w with $\deg_F(w)=n-2$. We can make the remained two nonfaulty edges incident to w required, and then the $n-2$ faulty edges incident to w are redundant. So (R,F) can be replaced by a new *reasonable* constraint which contains $n-3$ faulty edges, and three or two (when uv is incident to w) required edges. By Theorem 1.1, we can find an HC satisfying this new constraint and this HC is also an HC$(Q_n;R,F)$.

For the remained two cases, we assume that $\deg_F(x)\leq n-3$ for every $x\in V(Q_n)$.

(2) Suppose all faulty edges and uv are nth dimensional edges. Let us bisect Q_n into Q^0 and Q^1, and say $u\in V(Q^0)$; so, there are 2^{n-2} vertices in Q^0 with odd distance from u. Since $f^*=2n-5<2^{n-2}$, we can find a nonfaulty nth dimensional edge xx' such that $x\in V(Q^0)$ and the distance between x and u is odd. Clearly, the distance between x' and v is also odd. By Lemma 3.1, let us find a u–x HP in Q^0 and a v–x' HP in Q^1, then we get an HC in $(Q_n;uv,F)$ by connecting these two paths with uv and xx'.

(3) Suppose xx' is a faulty edge in different dimension compared with uv. By symmetry, we can assume that xx' is an nth dimensional edge with $x\in V(Q^0)$, and also assume $f^0\geq f^1$. Clearly, $f^0+f^1\leq 2n-6$, $f^1\leq n-3$, and both $\deg_{F^0}(y)$, $\deg_{F^1}(y')\leq n-3$ for every $y\in V(Q^0)$. Also note that $\{r^0,r^1\}=\{0,1\}$, and then $r^0+m(r^1)+2f^*\leq 1+2(2n-5)<2^{n-1}$ for $n\geq 4$. Now we consider following three subcases:

(a) $f^0\leq n-4$. This subcase is done by applying Theorem 2.3, since both r^0+2f^0 and r^1+2f^1 are less than or equal to $2n-7$.

(b) $n-3\leq f^0\leq 2n-7$. By induction, there exists an HC$(Q^0;R^0,F^0)$. When $f^1\leq n-4$, we have $r^1+2f^1\leq 2n-7$. When $f^1=n-3$ (so $f^0=n-3$ too), for symmetry we can assume $uv\in E(Q^0)$ and then $r^1+2f^1=2n-6$. Both situations satisfy the hypothesis of Theorem 2.3 and the proof follows.

(c) $f^0=2n-6$ (so $F^*=\{xx'\}$ and $f^1=0$). Since $\deg_F(x)\leq n-3$ and xx' is faulty, we know $\deg_{F^0}(x)\leq n-4\leq f^0-2$; so, there are at least two edges in F^0 which are not incident to x. Choose any of them, say yz; but when $uv\in E(Q^1)$, we need to ask $yz\neq u'v'$. By induction, there exists an HC$(Q^0;R^0,F^0-\{yz\})$, say C_1. If $yz\notin E(C_1)$, we can apply Theorem 2.3, i.e., using Algorithm 2.1. If $yz\in E(C)$, we need to find an HC$(Q^1;R^1\cup\{y'z'\})$, say C_2, and then $C_1\cup C_2\cup\{yy',zz'\}-\{yz,y'z'\}$ is a desired HC. □

In the next theorem, we show an analogous property for Hamiltonian paths. Notice that we use a weaker restriction rather than (RI) restriction here.

Theorem 3.3 *Suppose* $n\geq 2$ *and* $\deg_F(x)\leq n-1$ *for every* $x\in V(Q_n)$. *There exists a Hamiltonian path in* Q_n *provided that* (*A*) $r\geq 2$ *and* $r+2f\leq 2n-1$, *or* (*B*) $r\leq 1$ *and* $r+f\leq 2n-3$.

Proof We need only prove part (B) and deal with two cases: (I) $(r,f)=(0,2n-3)$ and (II) $(r,f)=(1,2n-4)$. The statement is derived directly from Theorem 1.1(HP) when $n = 2$ for both cases or when $n = 3$ for case (II). As for case (I) when $n = 3$, one can check by brute force. Now we assume $n \geq 4$ in the following discussion.

Case (I): (1) Suppose $\deg_F(x) \leq 1$ for every $x \in V(Q_n)$. Since $2n - 3 > n$, we can assume that at least two nth dimensional edges are faulty. In addition, we assume $f^0 \geq f^1$, and then $f^0 \leq 2n - 5$ and $f^1 \leq n - 3$. It is trivial when $f^0 = 0$. When $f^0 > 0$, let us choose a edge $xy \in F^0$. By assumption xx' and yy' are nonfaulty. There exists an $\mathrm{HP}(Q^0; \{xy\}, F^0 - \{xy\})$), say P, by induction and an $\mathrm{HC}(Q^1; \{x'y'\}, F^1 - \{x'y'\})$, say C, by Theorem 1.1. Clearly, $P \cup C \cup \{xx', yy'\} - \{xy, x'y'\}$ is a desired HP.

(2) Suppose $\max\{\deg_F(x) \mid x \in V(Q_n)\} \geq 2$ and w is a vertex reaching the maximum. Let $\bar{F}$ contain those faulty edges not incident to w, and edge xw be a nonfaulty edge incident to w. If $\deg_F(w) = n - 1$, we apply Theorem 1.1 to find an $\mathrm{HC}(Q_n; \{xw\}, \bar{F})$, say C. If $\deg_F(w) \leq n - 2$ (so $\deg_F(x) \leq n - 2$ for every vertex x), we can find an $\mathrm{HC}(Q_n; \{xw\}, \bar{F})$, say C again, by Theorem 3.2. Clearly, C consists at most one faulty edge, which must be incident to w. Thus an $\mathrm{HP}(Q_n; \emptyset, F)$ can be obtained by removing a proper edge of C.

Case (II): (1) Suppose there is a vertex w with $\deg_F(w) = n - 1$. Again, we let $\bar{F}$ contain those faulty edges not incident to w, and edge xw be the only nonfaulty edge incident to w. By Theorem 1.1, there is an $\mathrm{HC}(Q_n; R \cup \{xw\}, \bar{F})$, say C. We are done by following the same argument about C in last paragraph.

(2) Suppose $\deg_F(x) \leq n - 2$ for every $x \in V(Q_n)$. Let $xy \in F$ and we can find an $\mathrm{HC}(Q_n; R, F - \{xy\})$ by Theorem 3.2. Again, this cycle contains at most one faulty edge, namely xy, so the proof follows. □

References

Latifi S, Zheng S, Bagherzadeh N (1992) Optimal ring embedding in hypercubes with faulty links. In: Fault-tolerant computing symposium, pp 178–184

Häggkvist R (1979) On F-Hamiltonian graphs. In: Bondy, Murty (eds) Graph theory and related topics. Academic, New York, pp 219–231

Hsu LH, Liu SC, Yeh YN (2001) Edge-required and edge-fault-tolerant Hamiltonicity of hypercubes. In: Second pacific rim conference on mathematics, Taipei

Kronk HV (1969) Variations on a theorem of Posa. In: Chartrand G, Kapoor SF (eds) The many facets of graph theory. Springer, Berlin, pp 193–197

Sengupta A (1998) On ring embedding in hypercubes with faulty nodes and links. Inf Process Lett 68(4):207–214

Simmons G (1978) Almost all n-dimensional rectangular lattices are Hamilton-laceable. Congr Numer 21:649–661

Tseng YC (1996) Embedding a ring in a hypercube with both faulty links and faulty nodes. Inf Process Lett 59:217–222