A CLASS OF LYM ORDERS IN DIVISOR LATTICES

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. We present a new class of LYM orders, which generalizes Lih’s result and is a common generalization of Griggs’ result and a result of West, Harper and Daykin.

1. INTRODUCTION

A partially ordered set (or poset) is a set equipped with a reflexive, antisymmetric, and transitive relation. A poset $P$ is ranked if there is a rank function $r : P \to \mathbb{N}$ such that $r(x) = 0$ if $x$ is a minimal element of $P$ and $r(z) = r(y) + 1$ if $z$ covers $y$ in $P$. We call $r(x)$ the rank of $x$. The rank of $P$ is the maximum value of $r(x)$ taken over all $x \in P$. Let $P_i$ denote the set of elements of rank $i$ in $P$. Its cardinality $|P_i|$ is called the $i$th Whitney number of $P$. We say that $P$ is LC if the Whitney numbers of $P$ form a log-concave sequence, that is,

$$|P_i|^2 \geq |P_{i-1}| \cdot |P_{i+1}|$$

for all $i > 0$. An antichain is a subset of pairwise incomparable elements of $P$. We say that $P$ has the Sperner property if the maximum size of an antichain in $P$ equals the largest Whitney number of $P$. We say that $P$ has the LYM property if

$$\sum_i |A \cap P_i|/|P_i| \leq 1$$

for every antichain $A$ of $P$. It is well known that the LYM property implies the Sperner property ([5]).

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The subset lattice or the Boolean lattice $B_n$ is the poset of all subsets of an $n$-element set, ordered by inclusion. In 1928, Sperner [13] showed, in current terminology, that the subset lattice has the Sperner property. In 1967, Rota [12] made a famous conjecture that the partition lattice has the Sperner property. Although the conjecture was shown to be invalid in general by Canfield [1] in 1978, efforts to prove analogues of Sperner’s theorem for other posets have led to the emergence of an entire theory (see [4] for details). In 1980, Lih [11] discovered a generalization of Sperner’s theorem. Let $X = \{1, 2, \ldots, n\}$ be an $n$-element set and $Y$ a subset of $k$ elements of $X$ where $k \leq n$. Let $C(n, k)$ be the collection of all subsets of $X$ which intersect $Y$, ordered by inclusion. Lih showed that $C(n, k)$ has the Sperner property. Griggs [6] further showed, among other things, that $C(n, k)$ has the LYM property before long. He also generalized this result as follows.

**Theorem 1.** ([6]). Let $X = \{1, 2, \ldots, n\}$ be partitioned into parts $X_1, X_2, \ldots, X_r$. Suppose that $I_i \subseteq \{0, 1, \ldots, |X_i|\}$ is an arithmetic progression for each $i$. Then

$$P = \{Z \subseteq X : |Z \cap X_i| \in I_i, 1 \leq i \leq r\},$$

ordered by inclusion, is LYM and LC.

On the other hand, West, Harper and Daykin [16] gave a different generalization of Lih’s result.

**Theorem 2.** ([16]). Let $C_1 \subset C_2 \subset \cdots \subset C_s$ be a chain of subsets of $X = \{1, 2, \ldots, n\}$. Suppose that $\{a_i\}$ and $\{b_i\}$ are two nondecreasing sequences with $a_i \leq b_i$ for $1 \leq i \leq s$. Then

$$P = \{Z \subseteq X : a_i \leq |Z \cap C_i| \leq b_i, 1 \leq i \leq s\},$$

ordered by inclusion, is LYM and LC.

They also hoped to find out a common generalization of their result and that of Griggs. Indeed, there are similarities in the statements and the proofs of Theorem 1 and Theorem 2. In this note we broaden these results to the divisor lattice and give a common generalization.

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ be a positive integer, where the $p_i$ are distinct primes and $e_i \in \mathbb{N}$. The divisor lattice $D(n)$ is the poset of all (positive) divisors of $n$, ordered by divisibility. As usual, let $\sigma(n) = e_1 + e_2 + \cdots + e_t$ denote the number of prime divisors of $n$ counted according to multiplicity. Then $D(n)$ is a ranked poset with the rank function $\sigma$. Clearly, $D(n)$ reduces to $B_n$ when $n$ is square-free. Denote by $(m, k)$ the largest common divisor of two positive integers $m, k$ and replace $\sigma((m, k))$ by $\sigma(m, k)$. Given two nonnegative integers $a \leq b$, denote $[a, b] = \{i \in \mathbb{N} : a \leq i \leq b\}$. Our main result is the following.
Theorem 3. Let $n = n_1n_2 \cdots n_r$ where $n_i$ are positive integers of pairwise coprime. Suppose that $I_i \subseteq [0, \sigma(n_i)]$ is an arithmetic progression and $J_i = [a_i, b_i]$ where $a_i \leq b_i$ for each $i$. Then

$$P = \{ m \in D(n) : \sigma(m, n_j) \in I_j \text{ and } \sum_{i=1}^{j} \sigma(m, n_i) \in J_j \text{ for } 1 \leq j \leq r \},$$

ordered by divisibility, is LYM and LC.

When $n$ is square-free, the corresponding result is the following.

Corollary 1. Suppose that $X = \{1, 2, \ldots, n\}$ is partitioned into parts $X_1, X_2, \ldots, X_r$. Let $I_i \subseteq [0, |X_i|]$ be an arithmetic progression and $J_i = [a_i, b_i]$ where $a_i \leq b_i$ for each $i$. Then

$$P = \{ Z \subseteq X : |Z \cap X_j| \in I_j \text{ and } \sum_{i=1}^{j} |Z \cap X_i| \in J_j \text{ for } 1 \leq j \leq r \},$$

ordered by inclusion, is LYM and LC.

It is not difficult to see that Theorem 1 and 2 follow immediately from Corollary 1. In fact, we can obtain Theorem 1 by putting each $J_i = [0, n]$ in Corollary 1. On the other hand, suppose that $C_1 \subset C_2 \subset \cdots \subset C_s$ is a chain of subsets of $X$. Let

$$X_1 = C_1, X_2 = C_2 \setminus C_1, \ldots, X_s = C_s \setminus C_{s-1}, X_{s+1} = X \setminus C_s.$$

Then $X_1, X_2, \ldots, X_s, X_{s+1}$ is a partition of $X$ and $C_i = X_1 \cup X_2 \cup \cdots \cup X_i$ (1 $\leq i \leq s$). We obtain Theorem 2 by putting $J_{s+1} = [0, n]$ and $J_i = [0, |X_i|]$ (1 $\leq i \leq s + 1$) in Corollary 1.

2. Proof of Theorem 3

We use the product theorem for LYM posets to prove Theorem 3. The (direct) product $Q_1 \times Q_2$ of two posets $Q_1$ and $Q_2$ is defined to be the set of all pairs $(q_1, q_2), q_1 \in Q_1, q_2 \in Q_2$, with the order given by $(q_1, q_2) \leq (q'_1, q'_2)$ if and only if $q_1 \leq q'_1$ in $Q_1$ and $q_2 \leq q'_2$ in $Q_2$. Furthermore, the product of two ranked posets $Q_1$ and $Q_2$ is defined to be the poset together with the rank function $r$ given by $r(q_1, q_2) = r_1(q_1) + r_2(q_2)$, where $r_1$ and $r_2$ are the rank functions of $Q_1$ and $Q_2$, respectively. The product of two LYM posets $P$ and $Q$ may not be LYM in general, but it will be true if $P$ and $Q$ are LC also. The following result is discovered by Harper [7] and later independently by Hsieh and Kleitman [10].
**Product Theorem.** If two posets $Q_1, Q_2$ are both LYM and LC, then so is their product poset $Q_1 \times Q_2$.

A subposet of a poset $Q$ is a subset of $Q$ whose elements are ordered as in $Q$. Let $Q = \bigcup_{i=0}^{n} Q_i$ be a poset of rank $n$. Given a subset $I$ of $[0, n]$, let $Q_I = \bigcup_{i \in I} Q_i$ be the subposet of $Q$ induced by $I$. Clearly, an antichain of $Q_I$ is also antichain of $Q$. It follows that if the poset $Q$ is LYM, then so is the subposet $Q_I$.

Let $\{W_i\}_{i=0}^{n}$ be a log-concave sequence of positive numbers. Then the sequence $\left\{W_i/W_{i-1}\right\}_{i=1}^{n}$ is nonincreasing. Thus $W_j/W_{j-1} \geq W_k/W_{k-1}$ for $j \leq k$, or equivalently, $W_jW_{k-1} \geq W_{j-1}W_k$. It follows that

\[W_i^2 \geq W_{i-1}W_{i+1} \geq W_{i-2}W_{i+2} \geq \cdots \geq W_{i-d}W_{i+d}.\]

Let $I = \{a, a+d, a+2d, \ldots, a+md\}$ be an arithmetic progression in the closed interval $[0, n]$. Then the inequality (2.1) implies that the subsequence $\{W_i\}_{i \in I}$ is log-concave.

From the above discussion, we can conclude the following.

**Lemma 1.** Let $Q$ be a ranked poset of rank $n$ and let $I$ be an arithmetic progression in the closed interval $[0, n]$. If $Q$ is LYM and LC, then so is the subposet $Q_I$ induced by $I$.

We now prove Theorem 3.

**Proof of Theorem 3.** We proceed by induction on $r$. If $r = 1$, then

\[P = \{m \in D(n) : \sigma(m, n) \in I \cap J\},\]

where $I \subseteq [1, \sigma(n)]$ is an arithmetic progression and $J = [a, b]$. Clearly, $I \cap J$ is still an arithmetic progression. Note that $P$ consists of those elements of $D(n)$ with rank in $I \cap J$ and it is also well known that $D(n)$ is LYM and LC ([3]). Hence the subposet $P$ of $D(n)$ is LYM and LC by Lemma 1.

Suppose next that $r > 1$. Consider the following two posets:

\[P_1 = \{m \in D(n_r) : \sigma(m, n_r) \in I_r\}\]

and

\[P_2 = \{m \in D(n_1 \cdots n_{r-1}) : \sigma(m, n_j) \in I_j \text{ and } \sum_{i=1}^{j} \sigma(m, n_i) \in J_j \text{ for } 1 \leq j \leq r-1\}\]

By the induction hypotheses and Lemma 1, both $P_1$ and $P_2$ are LYM and LC. So $P_1 \times P_2$ is also LYM and LC by the Product Theorem. Note that $P_1 \times P_2$ is
isomorphic to the subposet of $D(n)$

$$Q = \{ m \in D(n) : \sigma(m, n_j) \in I_j \text{ for } 1 \leq j \leq r \text{ and } \sum_{i=1}^{j} \sigma(m, n_i) \in J_j \text{ for } 1 \leq j \leq r-1 \}$$

and that $P$ is the subposet $Q_{J_r}$ of $Q$ induced by $J_r$. Hence $P$ is LYM and LC by Lemma 1. This completes the proof of Theorem 3.


\section{Remarks}

Let $F$ be a collection of $t$-subsets of $X = \{1, \cdots, n\}$. Consider the filter generated by $F$:

$$P(F) = \{ Y \subseteq X : Y \supseteq A \text{ for some } A \in F \},$$

which is a subposet of the Boolean lattice $B_n$. Lih [11] conjectured that $P(F)$ has the Sperner property. The case $t = 0$ is just the classical Sperner theorem and the case $t = 1$ is Lih’s result about $C(n, k)$. However, Zhu [18] found counterexamples to the conjecture with $t > n/2$. Griggs [6] showed that the conjecture fails for $t = 4$ and Zha [17] constructed counterexamples for all $t \geq 4$ and $n \geq 2t - 1$. Horrocks [8, 9] gave a graph-theoretical interpretation for the $t = 2$ conjecture and left 116 exceptional graphs in his proof. Cheng and Lih [2] carried on further with Horrocks’s reduction method to reduce the number of exceptional graphs and gave a complete proof for the $t = 2$ conjecture. The conjecture remains open for $t = 3$. An interesting problem is to consider analogue of Lih’s conjecture for the divisor lattices and other posets. We also refer the reader to [14, 15] for a subspace lattice analogue of Lih’s poset $C(n, k)$.

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