Dyck Paths with Peak- and Valley-Avoiding Sets

By Shu-Chung Liu, Jun Ma, and Yeong-Nan Yeh

In this paper, we focus on Dyck paths with peaks and valleys, avoiding an arbitrary set of heights. The generating functions of such types of Dyck paths can be represented by continued fractions. We also discuss a special case that requires all peak and valley heights to avoid congruence classes modulo k. We study the shift equivalence on sequences, which in turn induces an equivalence relation on avoiding sets.

1. Introduction

An *n-Dyck path* is a lattice path in the first quadrant, starting at (0, 0) and ending at ($2n$, 0), with only two kinds of steps—*rise step:* $U = (1, 1)$ and *fall step:* $D = (1, -1)$. We call n the *semilength* because there are $2n$ steps and use $\|P\|$ to denote the semilength of a Dyck path P. We can also consider an n-Dyck path P as a word of $2n$ letters, using only U and D. In this word, let P_k denote the kth ($1 \leq k \leq 2n$) letter from the left.

Let $\mathcal{D}_n$ denote the set of all n-Dyck paths and $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$, which consists of all Dyck paths. It is well known that $|\mathcal{D}_n|$, the cardinality of $\mathcal{D}_n$, equals the nth Catalan number, $c_n = \frac{1}{n+1}\binom{2n}{n}$, and the generating function $C(z) := \sum_{n\geq 0} c_n z^n$ satisfies the functional equation $C(z) = 1 + zC(z)^2$ and $C(z) = \frac{1-\sqrt{1-4z}}{2z}$, explicitly.

Address for correspondence: Jun Ma, Institute of Mathematics, Academia Sinica, Taipei, Taiwan; e-mail: majun@math.sinica.edu.tw

In view of vectors, let $node(P, k) = \sum_{i=1}^{k} P_i$, which is exactly the coordinates of the ending node after moving the first k steps of P. The second coordinate is also called the *height* of this node. For short, let $h(P, k)$ be the height of this node. If a joint node is formed by a rise step followed by a fall step, then this node is called a peak; if a joint node is formed by a fall step followed by a rise step, then this node is called a valley.

Let $\mathbb{P}(\mathbb{N})$ be the set of positive (nonnegative) integers. Also let $[m, n] := \{m, m+1, \ldots, n\}$ $(m \leq n)$ and $[n] := [1, n]$, briefly. Let $\Lambda_P(V_P)$ be the set of heights of all peaks (valleys) of P. Clearly, $\Lambda_P \subseteq \mathbb{P}$ and $V_P \subseteq \mathbb{N}$. Given a pair (A, B) of integer sets, let $\mathcal{D}_{n,k,\bar{A},\bar{B}}$ consist of n-Dyck paths P with exactly k peaks and satisfying $\Lambda_P \cap A = \emptyset$ and $V_P \cap B = \emptyset$, and let $d_{n,k,\bar{A},\bar{B}} = |\mathcal{D}_{n,k,\bar{A},\bar{B}}|$. The over-line notation $(\bar{A})$ is adopted form [2]. Because peak-heights are not less than 1 and valley-heights are never less than 0, we have $\mathcal{D}_{n,k,\bar{A},\bar{B}} = \mathcal{D}_{n,k,\overline{A\cap\mathbb{P}},\overline{B\cap\mathbb{N}}}$. Most of the times, we just require $A \subset \mathbb{P}$ and $B \subset \mathbb{N}$. The set A (B) is called a *peak-avoiding set* (*valley-avoiding set*) of P and (A, B) is an *avoiding pair*. It is unnecessary to count the number of valleys because it must be $k-1$. Moreover, let $\mathcal{D}_{n,\bar{A},\bar{B}} = \bigcup_{k\geq 0} \mathcal{D}_{n,k,\bar{A},\bar{B}}$, which is the set of all n-Dyck path with an avoiding pair (A, B), and $d_{n,\bar{A},\bar{B}} = |\mathcal{D}_{n,\bar{A},\bar{B}}|$. Also let $\mathcal{D}_{\bar{A},\bar{B}} = \bigcup_{n\geq 0} \mathcal{D}_{n,\bar{A},\bar{B}}$.

There were several previous results for Dyck paths focusing on peak-avoiding sets. Deutsch [1] showed that $d_{n,\overline{\{1\}},\bar{\emptyset}}$ is exactly the nth Fine number. Peart and Woan [3] considered the case $d_{n,\overline{\{h\}},\bar{\emptyset}}$ for any $h \in \mathbb{P}$ and proved that $d_{n,\overline{\{2\}},\bar{\emptyset}}$ equals the $(n-1)$th Catalan number. Recently, Eu et al. [2] studied any peak-avoiding set in general and concluded that the generating function of $d_{n,\bar{A},\bar{\emptyset}}$ can be represented by continued fractions. Among many sequences involved, they introduced the *shift equivalence* of two sequences, $\langle a_n \rangle \equiv_s \langle b_n \rangle$, holding the property that there exist non-negative integers p and q such that $a_{p+n} = b_{q+n}$ for all $n \in \mathbb{N}$.

In this paper, more general cases as $B \neq \emptyset$ are considered. In section 2, we consider any avoiding pair (A, B) and derive a reduction formula and then a closed-form formula (as a continued fraction) for the generating function

$$D_{\bar{A},\bar{B}}(y, z) := \sum_{n,k\geq 0} d_{n,k,\bar{A},\bar{B}} y^k z^n.$$

Besides, we develop a new approach, which is different from [2], to obtain the results in its closed form. This new approach is based on a 2×2 matrix, depicting the recurrence relations between $D_{\bar{A},\bar{B}}(y, z)$ and $D_{\overline{A-1},\overline{B-1}}(y, z)$. Several interesting examples are given in section 3 by using this approach.

Use $md_{n,*,\bar{A},\bar{B}}$ and $vd_{n,*,\bar{A},\bar{B}}$ to denote the mean and the variance of peaks of all Dyck paths in $\mathcal{D}_{n,\bar{A},\bar{B}}$, respectively. Then the following equations are obtained,

$$md_{n,*,\bar{A},\bar{B}} = \frac{\sum_{k\geq 0} k \cdot d_{n,k,\bar{A},\bar{B}}}{\sum_{k\geq 0} d_{n,k,\bar{A},\bar{B}}} - \text{as}\frac{\sum_{k\geq 0} k \cdot d_{n,k,\bar{A},\bar{B}}}{\sum_{k\geq 0} d_{n,k,\bar{A},\bar{B}}}$$

and

$$vd_{n,*,\bar{A},\bar{B}} = \frac{\sum_{k\geq 0} k^2 \cdot d_{n,k,\bar{A},\bar{B}}}{d_{n,\bar{A},\bar{B}}} - (md_{n,*,\bar{A},\bar{B}})^2.$$

For any Dyck path P, we use z, y and x to mark the semilength, the number of peaks and the number of valleys of P, respectively. In general, one use $D_{\bar{A},\bar{B}}(x, y, z)$ to denote the generating functions of the sequence which is formed by the number of Dyck path with semilength n, k peaks and l valleys. Note that the number of valleys of P equals the number of peaks of P minus 1 for any n-Dyck path. Thus, we may ignore the variable x, let $D_{\bar{A},\bar{B}}(y, z)$ and $D_{\bar{A},\bar{B}}(z)$ be the generating functions of $d_{n,k,\bar{A},\bar{B}}$ and $d_{n,\bar{A},\bar{B}}$, respectively. Clearly, $D_{\bar{A},\bar{B}}(1, z)$ is the generating function of $d_{n,\bar{A},\bar{B}}$, i.e., $D_{\bar{A},\bar{B}}(z) = D_{\bar{A},\bar{B}}(1, z)$.

For any the generating function $f(z)$, let $[z^i]f(z)$ denote the coefficient of z^i in $f(z)$, then

$$md_{n,*,\bar{A},\bar{B}} = \frac{[z^n] \left.\frac{\partial D_{\bar{A},\bar{B}}(y, z)}{\partial y}\right|_{y=1}}{[z^n]D_{\bar{A},\bar{B}}(1, z)}$$

and

$$vd_{n,*,\bar{A},\bar{B}} = \frac{[z^n] \left.\frac{\partial^2 D_{\bar{A},\bar{B}}(y, z)}{\partial y^2}\right|_{y=1}}{[z^n]D_{\bar{A},\bar{B}}(1, z)} + md_{n,*,\bar{A},\bar{B}} - (md_{n,*,\bar{A},\bar{B}})^2.$$

In Section 4, we compute the mean and the variance of peaks of all n-Dyck paths in $\mathcal{D}_{n,\bar{A},\bar{B}}$ when $(A, B) \in \{(\emptyset, [0, m]), (\bar{\mathbb{O}}, \bar{\mathbb{O}}), (\bar{\mathbb{E}}, \bar{\mathbb{E}})\}$.

In Section 5, we study an equivalence relation among avoiding pairs according to the shift equivalence between sequences arising from these pairs.

2. The heights of peaks and valleys avoiding to any (*A*, *B*)

Let $A \subset \mathbb{P}$ and $B \subset \mathbb{N}$ in this section. We discuss the generating function $D_{\bar{A},\bar{B}}(y, z) := \sum_{n,k\geq 0} d_{n,k,\bar{A},\bar{B}} y^k z^n$. Given an integer set S and a positive integer i, let $S - i = \{n - i \mid n \in S\}$ and $S + i = \{n + i \mid n \in S\}$. The reader should always be aware that $\mathcal{D}_{n,k,\overline{A-i},\overline{B-j}} = \mathcal{D}_{n,k,\overline{(A-i)\cap\mathbb{P}},\overline{(B-j)\cap\mathbb{N}}}$.

For any Dyck path $P \in \mathcal{D}_{\bar{A},\bar{B}}$ with $\|P\| \geq 1$, we can decompose P into UP_1DP_2, where D is the first fall step returning to height 0 and P_1, P_2 are two subwords; thus, P_1, P_2 are still Dyck paths. Clearly,

$$P_1 \in \begin{cases} \mathcal{D}_{\overline{A-1},\overline{B-1}} & \text{if} \quad 1 \notin A \\ \mathcal{D}_{\overline{A-1},\overline{B-1}} \setminus \{\emptyset\} & \text{if} \quad 1 \in A \end{cases} \quad \text{and} \quad P_2 \in \begin{cases} \mathcal{D}_{\bar{A},\bar{B}} & \text{if} \quad 0 \notin B \\ \{\emptyset\} & \text{if} \quad 0 \in B. \end{cases}$$

Let χ be the Boolean function such that $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Then the functions $D_{\bar{A},\bar{B}}(y,z)$ satisfies the following recurrence relation:

$$D_{\bar{A},\bar{B}}(y,z) = 1 + z(D_{\overline{A-1},\overline{B-1}}(y,z) + \chi(1 \notin A)y - 1)D_{\bar{A},\bar{B}}(y,z)^{\chi(0 \notin B)}. \quad (1)$$

Particularly, when $1 \notin A$ and $P_1 = \emptyset$, we get a peak at *node*$(P, 1)$. This is why $\chi(1 \notin A)y$ appears in (1). Precisely, we have

$$D_{\bar{A},\bar{B}}(y,z) = \begin{cases} 1 - z + \chi(1 \notin A)yz + zD_{\overline{A-1},\overline{B-1}} & \text{if} \quad 0 \in B; \\ \dfrac{1}{1 + z - \chi(1 \notin A)yz - zD_{\overline{A-1},\overline{B-1}}} & \text{if} \quad 0 \notin B. \end{cases} \quad (2)$$

Given a continued fraction with $k + 1$ layers, which is made by k (mixed) fractions, let us display it in the following general form:

$$M_1 + \cfrac{N_1}{M_2 + \cfrac{N_2}{\ddots \, M_{k-1} + \cfrac{}{M_k + \cfrac{N_k}{M_{k+1} + Q}}}}.$$

We call M_i mixed parts, N_i numerators, and $M_{k+1} + Q$ the "final" denominator (Actually, it is not a denominator when $k = 0$). Let $m = \max A \cup (B + 1)$. By iterating Equation (2) n times ($n \leq m$), we obtain a representation of $D_{\bar{A},\bar{B}}(y,z)$ as a continued fraction with $|[0, n-1] - B| + 1$ layers because a new fraction is created due to an element in $[0, n-1] - B$. We describe the detail of this continued fraction by the following theorem:

THEOREM 1.

(a) Given $A \subset \mathbb{P}$, $B \subset \mathbb{N}$ *and* $n \leq \max A \cup (B+1)$, *suppose* $\hat{B}_n := [0, n-1] - B = \{\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_k\}$ *listed increasingly, where* $k = |[0, n-1] - B|$. *In addition, let* $\hat{b}_0 = 0$ *and* $\hat{b}_{k+1} = n$. *Then* $D_{\bar{A},\bar{B}}(y,z)$ *equals a continued fraction with* $k + 1$ *layers such that*

$$M_1 = \chi(0 \in B)\left(1 - z^{\hat{b}_1} + y \sum_{\substack{1 \leq i \leq \hat{b}_1 \\ i \notin A}} z^i\right),$$

$$M_j = 1 + z^{\hat{b}_j - \hat{b}_{j-1}} - y \sum_{\substack{\hat{b}_{j-1}+1 \leq i \leq \hat{b}_j \\ i \notin A}} z^{i - \hat{b}_{j-1}}, \quad \textit{for } 2 \leq j \leq k+1;$$

$$N_j = (-1)^{\chi(j\neq 1)} z^{\hat{b}_j - \hat{b}_{j-1}} \quad \textit{for } 1 \leq j \leq k+1;$$

$$Q = (-1)^{\chi(k\neq 0)} z^{\hat{b}_{k+1} - \hat{b}_k} D_{\overline{A-n},\overline{B-n}}(y,z).$$

(b) Extremely, if we let $n = \max A \cup (B+1)$ *and it is finite, then* $Q = (-1)^{\chi(k\neq 1)} z^{\hat{b}_{k+1} - \hat{b}_k} D_{\bar{\emptyset},\bar{\emptyset}}(y,z)$. *In general,*

$$D_{\bar{A},\bar{B}}(y,z) = M_1 + \frac{P_1 + P_2\, D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{Q_1 + Q_2\, D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}, \tag{3}$$

for some polynomials $M_1, P_1, P_2, Q_1, Q_2 \in \mathbb{Z}(y,z)$, *where* M_1 *is same as in part (a).*

Proof: Because part (b) comes directly form (a), we only prove part (a). When $n = 1$, the formulas in the theorem are just a translation of Equation (2). Now suppose $n \geq 2$. If $\hat{B}_n = \emptyset$, then there is no fraction and

$$\begin{aligned} D_{\bar{A},\bar{B}}(y,z) &= 1 - z + \chi(1 \notin A)yz + z(1 - z + \chi(2 \notin A)yz \\ &\quad + z(\cdots + (1 - z + \chi(n \notin A) + zD_{\overline{A-n},\overline{B-n}})\cdots)) \\ &= \left(1 - z^n + y \sum_{\substack{1\leq i\leq n\\ i\notin A}} z^i\right) + z^n D_{\overline{A-n},\overline{B-n}}(y,z). \end{aligned}$$

Provided that $0 \in B$, $k = 0$, $\hat{b}_0 = 0$ and $\hat{b}_1 = n$, we see the first term above is M_1 and the second term is Q. The case with $\hat{B}_n = \{n-1\}$ can be proved in a similar way. So, we leave the check to the reader. (Especially, why only Q with $k = 0$ and N_1 are leading positively?)

In addition, we suppose $\hat{B}_n \neq \emptyset$ and $\hat{B}_n \neq \{n-1\}$. To apply induction, we use $\hat{b}'_1, \hat{b}'_2, \ldots$ to denote the corresponding $\hat{b}_i$ for $\hat{B}_{n-1}$. Now let us see the case when $n-1 \in B$. Since $0 \in B - (n-1)$, we have $D_{\overline{A-(n-1)},\overline{B-(n-1)}}(y,z) = 1 - z + \chi(1 \notin A-(n-1))yz + zD_{\overline{A-n},\overline{B-n}}$; thus, applying Equation (2) for the nth time does not create a new fraction. This agrees with the fact $\hat{B}_n = \hat{B}_{n-1}$, for their cardinalities count the numbers of fractions. By induction, nothing changes but the new denominator $M_{k+1} + Q$ is equal to

$$\begin{aligned} & M'_{k+1} + Q' \quad \text{(by applying Equation (2) } n-1 \text{ times)} \\ &= 1 + z^{\hat{b}'_{k+1} - \hat{b}'_k} - y \sum_{\substack{\hat{b}'_k + 1 \leq i \leq \hat{b}'_{k+1}\\ i\notin A}} z^{i - \hat{b}'_k} - z^{\hat{b}'_{k+1} - \hat{b}'_k} \\ &\qquad \times (1 - z + \chi(1 \notin A-(n-1))yz + zD_{\overline{A-n},\overline{B-n}}) \\ &= \left(1 + z^{\hat{b}_{k+1} - \hat{b}_k} - y \sum_{\substack{\hat{b}_k + 1 \leq i \leq \hat{b}_{k+1}\\ i\notin A}} z^{i - \hat{b}_k}\right) - z^{\hat{b}_{k+1} - \hat{b}_k} D_{\overline{A-n},\overline{B-n}}. \end{aligned}$$

Notice that $\hat{b}_{k+1} = n$ and $\hat{b}'_{k+1} = n-1$; though $\hat{b}_i = \hat{b}'_i$ for $0 \leq i \leq k$ (due to $\hat{B}_n = \hat{B}_{n-1}$).

As for the case when $n-1 \notin B$, we have $k = |\hat{B}_n| = |\hat{B}_{n-1}| + 1$ and $\hat{b}_k = n-1 \in \hat{B}_n - \hat{B}_{n-1}$, and also $\hat{b}_{k+1} = n$ and $\hat{b}_i = \hat{b}'_i$ for $0 \le i \le k$. Here $k \ge 2$ because we assume $\hat{B}_n \neq \{n-1\}$. Since $0 \notin B - (n-1)$, applying Equation (2) for the nth time creates a new fraction. We need only observe the denominator in the previous stage by induction:

$$
\begin{aligned}
&M'_k + Q' \text{ (by applying Equation (2) } n-1 \text{ times)}\\
&= 1 + z^{\hat{b}'_k - \hat{b}'_{k-1}} - y \sum_{\substack{\hat{b}'_{k-1}+1 \le i \le \hat{b}'_k \\ i \notin A}} z^{i - \hat{b}'_{k-1}}\\
&\quad + \frac{-z^{\hat{b}'_k - \hat{b}'_{k-1}}}{1 + z - \chi(1 \notin A - (n-1))yz - zD_{\overline{A-n},\overline{B-n}}}\\
&= 1 + z^{\hat{b}_k - \hat{b}_{k-1}} - y \sum_{\substack{\hat{b}_{k-1}+1 \le i \le \hat{b}_k \\ i \notin A}} z^{i - \hat{b}_{k-1}}\\
&\quad + \frac{-z^{\hat{b}_k - \hat{b}_{k-1}}}{1 + z^{\hat{b}_{k+1} - \hat{b}_k} - \chi(n \notin A)yz^{\hat{b}_{k+1} - \hat{b}_k} - z^{\hat{b}_{k+1} - \hat{b}_k} D_{\overline{A-n},\overline{B-n}}}\\
&= M_k + \frac{N_k}{M_{k+1} + Q} \quad \text{(by applying Equation (2) } n \text{ times).} \qquad \blacksquare
\end{aligned}
$$

As a demonstration of Theorem 1(a), a simple computation yields

$$
D_{\overline{\{1,3\}},\overline{\{0,3\}}}(y,z) = 1 - z + \cfrac{z}{1 + z - yz - \cfrac{z}{1 + z^2 - yz^2 - z^2 D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}}.
$$

The continued fraction above does have $1 + 4 - 2 = 3$ layers.

Let $(A, B) = (\emptyset, \emptyset)$. Directly from Equation (1), we get well-studied formulas:

$$
D_{\bar{\emptyset},\bar{\emptyset}}(y,z) = \frac{1 + z - zy - \sqrt{(1 + z - zy)^2 - 4z}}{2z} \quad \text{and}
$$

$$
D_{\bar{\emptyset},\bar{\emptyset}}(1,z) = \frac{1 - \sqrt{1 - 4z}}{2z} = C(z).
$$

The second one is the generating function of the Catalan numbers. If we only assign $B = \emptyset$, then we obtain

$$
D_{\bar{A},\bar{\emptyset}}(y,z) = \frac{1}{1 + z - \chi(1 \notin A)yz - zD_{\overline{A-1},\bar{\emptyset}}(y,z)}. \tag{4}
$$

Moreover, plugging $y = 1$ yields

$$
D_{\bar{A},\bar{\emptyset}}(1,z) = \frac{1}{1 + \chi(1 \in A)z - zD_{\overline{A-1},\bar{\emptyset}}(1,z)},
$$

which agrees with Equation 2 [2]. When A is finite with $m = \max A$, Equation (4) implies a continued fraction with $(m+1)$ layers:

$$D_{\bar{A},\bar{\emptyset}}(y,z) = \cfrac{1}{1+z-\chi(1\notin A)yz-\cfrac{z}{\ddots\, \cfrac{}{1+z-\chi(m-1\notin A)yz-\cfrac{z}{1+z-z\,D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}}}}, \tag{5}$$

which generalizes the result ($y = 1$) in [2, p. 458].

3. Miscellaneous cases

From Theorem 1(b), a single-fraction closed form depending on $D_{\bar{\emptyset},\bar{\emptyset}}(y,z)$ is guaranteed. In this section, we focus on this single-fraction closed form for several interesting cases. First, let us define 2×2 matrices

$$T_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \quad \text{and} \quad S_x \begin{pmatrix} 0 & 1 \\ -z & x \end{pmatrix}$$

and also $\alpha = 1 + z - zy$ and $\beta = 1 + z$. We will use $T_\alpha, T_\beta, S_\alpha, S_\beta$, and S_0 frequently. To obtain the closed form shown in Theorem 1(b) precisely, we depict the recurrence relations between $D_{\bar{A},\bar{B}}(y,z)$ and $D_{\overline{A-1},\overline{B-1}}(y,z)$ by a 2×2 matrix, and then, using the corresponding matrices, we may computer the single-fraction closed form depending on $D_{\bar{\emptyset},\bar{\emptyset}}(y,z)$. This new approach is different from the methods in [2].

3.1. Case (A, B) = ([s, m + s − 1], ∅)

The subcase $(A, B) = ([2, m+1], \emptyset)$ with $y = 1$ was studied by Eu et al. [2, theorem 3]. First, let us consider the subcase with $s = 1$. By Theorem 1(a) (also referring (5) or even iterating Equation (2)), we obtain the following continued fraction with $m + 1$ layers:

$$D_{\overline{[1,m]},\bar{\emptyset}}(y,z) = \cfrac{1}{\beta-\cfrac{z}{\beta-\cfrac{z}{\ddots\,\cfrac{}{\beta-\cfrac{z}{\beta-z\,D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}}}}}. \tag{6}$$

Instead of simplifying this continued fraction, let us suppose

$$D_{\overline{[1,m]},\bar{\emptyset}}(y,z) = \frac{a_m - b_m z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_m - d_m z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}$$

for any $m \geq 1$, where $a_m, b_m, c_m, d_m \in \mathbb{Z}[y, z]$. Then

$$D_{\overline{[1,m]},\bar{\emptyset}}(y, z) = \frac{c_{m-1} - d_{m-1} z D_{\bar{\emptyset},\bar{\emptyset}}(y, z)}{(1 + z)c_{m-1} - za_{m-1} - [(1 + z)\, d_{m-1} - zb_{m-1}]z D_{\bar{\emptyset},\bar{\emptyset}}(y, z)}.$$

which is due to Equation (2). Hence, $a_m = c_{m-1}, b_m = d_{m-1}, c_m = \beta c_{m-1} - za_{m-1}$, and $d_m = \beta d_{m-1} - zb_{m-1}$. Equivalently, we derive that

$$\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} = S_\beta \begin{pmatrix} a_{m-1} & b_{m-1} \\ c_{m-1} & d_{m-1} \end{pmatrix}.$$

Notice that $D_{\overline{\{1\}},\bar{\emptyset}}(y, z) = \frac{1}{1+z-zD_{\bar{\emptyset},\bar{\emptyset}}(y,z)}$, and then the following corollary is ready.

THEOREM 2. *For any $m \geq 1$, let*

$$\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} = S_\beta^{m-1} T_\beta.$$

Then

$$D_{\overline{[1,m]},\bar{\emptyset}} = \frac{a_m - b_m z D_{\bar{\emptyset},\bar{\emptyset}}(y, z)}{c_m - d_m z D_{\bar{\emptyset},\bar{\emptyset}}(y, z)}$$

and, reversely,

$$D_{\bar{\emptyset},\bar{\emptyset}}(y, z) = \frac{a_m - c_m D_{\overline{[1,m]},\bar{\emptyset}}(y, z)}{b_m - d_m D_{\overline{[1,m]},\bar{\emptyset}}(y, z)} z^{-1}.$$

EXAMPLE 1. Take $m = 2$. Then

$$S_\beta T_\beta = \begin{pmatrix} 1 + z & 1 \\ 1 + z + z^2 & 1 + z \end{pmatrix}.$$

Hence,

$$D_{\overline{[1,2]},\bar{\emptyset}}(y, z) = \frac{1 + z - zD_{\bar{\emptyset},\bar{\emptyset}}(y, z)}{1 + z + z^2 - (1 + z)zD_{\bar{\emptyset},\bar{\emptyset}}(y, z)}, \quad \text{and}$$

$$D_{\bar{\emptyset},\bar{\emptyset}}(y, z) = \frac{1 + z - (1 + z + z^2)D_{\overline{[1,2]},\bar{\emptyset}}(y, z)}{z - (z + z^2)D_{\overline{[1,2]},\bar{\emptyset}}(y, z)}.$$

Now, we consider the subcase with $s > 1$. By Theorem 1(a), we obtain the following formula in the form of a continued fraction with s layers.

$$D_{\overline{[s,m+s-1]},\bar{\emptyset}}(y,z) = \cfrac{1}{\alpha - \cfrac{z}{\alpha - \cfrac{z}{\ddots \, \cfrac{}{\alpha - \cfrac{z}{\alpha - z\, D_{\overline{[1,m]},\bar{\emptyset}}(y,z)}}}}}. \tag{7}$$

THEOREM 3. *For any $s \geq 2$ and $m \geq 1$, let*

$$\begin{pmatrix} a_{s,m} & b_{s,m} \\ c_{s,m} & d_{s,m} \end{pmatrix} = S_\alpha^{s-2} T_\alpha S_0 S_\beta^{m-1} T_\beta.$$

Then

$$D_{\overline{[s,m+s-1]},\bar{\emptyset}}(y,z) = \frac{a_{s,m} - b_{s,m} z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_{s,m} - d_{s,m} z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)},$$

and, reversely,

$$D_{\bar{\emptyset},\bar{\emptyset}}(y,z) = \frac{a_{s,m} - c_{s,m} D_{\overline{[s,m+s-1]},\bar{\emptyset}}(y,z)}{b_{s,m} - d_{s,m} D_{\overline{[s,m+s-1]},\bar{\emptyset}}(y,z)} z^{-1}.$$

Proof: Similar to the subcase with $s = 1$, let

$$\begin{pmatrix} \hat{a}_s & \hat{b}_s \\ \hat{c}_s & \hat{d}_s \end{pmatrix} = S_\alpha^{s-2} T_\alpha,$$

then

$$D_{\overline{[s,m+s-1]},\bar{\emptyset}}(y,z) = \frac{\hat{a}_s - \hat{b}_s z D_{\overline{[1,m]},\bar{\emptyset}}(y,z)}{\hat{c}_s - \hat{d}_s z D_{\overline{[1,m]},\bar{\emptyset}}(y,z)}.$$

By Theorem 2, we have

$$D_{\overline{[s,m+s-1]},\bar{\emptyset}}(y,z) = \frac{\hat{a}_s c_m - z\hat{b}_s a_m - [\hat{a}_s d_m - z\hat{b}_s d_m] z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{\hat{c}_s c_m - z\hat{d}_s a_m - [\hat{c}_s d_m - z\hat{d}_s d_m] z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

Clearly, we obtain

$$\begin{pmatrix} \hat{a}_s c_m - z\hat{b}_s a_m & \hat{a}_s d_m - z\hat{b}_s d_m \\ \hat{c}_s c_m - z\hat{d}_s a_m & \hat{c}_s d_m - z\hat{d}_s d_m \end{pmatrix} = S_\alpha^{s-2} T_\alpha S_0 S_\beta^{m-1} T_\beta,$$

as we desired. The reverse formula is left to the reader to check. ■

Example 2. Take $s = m = 2$. Then

$$T_\alpha S_0 S_\beta T_\beta = \begin{pmatrix} 1+z+z^2 & 1+z \\ 1+(1-y)(z+z^2+z^3) & 1+(1-y)(z+z^2) \end{pmatrix}.$$

Hence,

$$D_{\overline{[2,3]},\bar{\emptyset}}(y,z) = \frac{1+z+z^2-(1+z)zD_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{1+(1-y)(z+z^2+z^3)-[1+(1-y)(z+z^2)]zD_{\bar{\emptyset},\bar{\emptyset}}(y,z)} \quad \text{and}$$

$$D_{\bar{\emptyset},\bar{\emptyset}}(y,z) = \frac{1+z+z^2-[1+(1-y)(z+z^2+z^3)]D_{\overline{[2,3]},\bar{\emptyset}}(y,z)}{(1+z)z-[1+(1-y)(z+z^2)]zD_{\overline{[2,3]},\bar{\emptyset}}(y,z)}.$$

3.2. Case (A, B) = (∅, [s, s + m])

For $s = 0$, we directly apply Theorem 1(a) and obtain:

Corollary 1. *Let m be an nonnegative integer. Then*

$$D_{\bar{\emptyset},\overline{[0,m]}}(y,z) = 1-z^{m+1}+y(z+z^2+\cdots+z^{m+1})+z^{m+1}D_{\bar{\emptyset},\bar{\emptyset}}(y,z).$$

As for $s \geq 1$, we obtain a continued fraction with $s+1$ layers as follows:

$$D_{\bar{\emptyset},\overline{[s,s+m]}}(y,z) = \cfrac{1}{\alpha - \cfrac{z}{\alpha - \cfrac{z}{\ddots \over \alpha - \cfrac{z}{\alpha - z\,D_{\bar{\emptyset},\overline{[0,m]}}(y,z)}}}}.$$

As a similar pattern as Equation 7, the above continued fraction can follow the same argument as in the proof of Theorem 3.

Theorem 4. *For any $s \geq 1$ and $m \geq 0$, let*

$$\begin{pmatrix} a_{s,m} & b_{s,m} \\ c_{s,m} & d_{s,m} \end{pmatrix} = S_\alpha^{s-1} T_\alpha S_0 \begin{pmatrix} 1-z^{m+1}+y(z+z^2+\cdots+z^{m+1}) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= S_\alpha^{s-1} T_\alpha T_{1-z^{m+1}+y(z+z^2+\cdots+z^{m+1})}.$$

Then

$$D_{\bar{\emptyset},\overline{[s,s+m]}}(y,z) = \frac{a_{s,m} - b_{s,m} z^{m+1} D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_{s,m} - d_{s,m} z^{m+1} D_{\bar{\emptyset},\bar{\emptyset}}(y,z)},$$

and, reversely,

$$D_{\bar{\emptyset},\bar{\emptyset}}(y,z) = \frac{a_{s,m} - c_{s,m} D_{\bar{\emptyset},\overline{[s,s+m]}}(y,z)}{b_{s,m} - d_{s,m} D_{\bar{\emptyset},\overline{[s,s+m]}}(y,z)} z^{-(m+1)}.$$

From the above results and examples, once we claim

$$D_{\bar{A},\bar{B}}(y,z) = \frac{a - bz^k D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c - dz^k D_{\bar{\emptyset},\bar{\emptyset}}(y,z)},$$

then we have an equivalent formula

$$D_{\bar{\emptyset},\bar{\emptyset}}(y,z) = \frac{a - cD_{\bar{A},\bar{B}}(y,z)}{b - dD_{\bar{A},\bar{B}}(y,z)(y,z)} z^{-k}.$$

We will not mention these reverse formulas again in the following sections.

3.3. Cases (A, B) = (∅, {m}) and (A, B) = ({s}, ∅)

For $m = 0$, the result can be found in Corollary 1; so, here we assume $m \geq 1$. By Theorem 1(a), we obtain the following formula in the form of a continued fraction with $m + 1$ layers.

$$D_{\bar{\emptyset},\overline{\{m\}}}(y,z) = \cfrac{1}{\alpha - \cfrac{z}{\alpha - \cfrac{z}{\ddots \over \alpha - \cfrac{z}{1 + z^2 - y(z+z^2) - z^2 D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}}}}.$$

Similar to Equation 7 and the argument in the proof of Theorem 3, we obtain the following theorem.

THEOREM 5. *For any $m \geq 1$, let*

$$\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} = S_\alpha^{m-1} T_{1+z^2-y(z+z^2)}.$$

Then

$$D_{\bar{\emptyset},\overline{\{m\}}}(y,z) = \frac{a_m - b_m z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_m - d_m z D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

EXAMPLE 3. Take $m = 2$. Then

$$S_\alpha T_{1+z^2-y(z+z^2)} = \begin{pmatrix} 1 - zy + (1-y)z^2 & 1 \\ 1 - 2zy + (1-y)^2(z^2+z^3) & 1 + z - zy \end{pmatrix}.$$

Hence,

$$D_{\bar{\emptyset},\overline{\{2\}}}(y,z)=\frac{1-zy+(1-y)z^2-z^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{1-2zy+(1-y)^2(z^2+z^3)-(1+z-zy)z^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

Now for the second case with $(A, B) = (\{s\}, \emptyset)$. Using the same technique, the following result is easy to derive.

THEOREM 6. *For any $s \geq 1$, let*

$$\begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} = S_\alpha^{s-1}T_\beta.$$

Then

$$D_{\overline{\{s\}},\bar{\emptyset}}(y,z)=\frac{a_s-b_sz^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_s-d_sz^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

3.4. *Case $(A, B) = (\{s\}, \{s + m - 1\})$*

In this section, we suppose $s \geq 1$ and $m \geq 0$. First, let us deal with the special case $m = 0$. By Theorem 1(a), we obtain an s-layer continued fraction:

$$D_{\overline{\{s\}},\overline{\{s-1\}}}(y,z)=\cfrac{1}{\alpha-\cfrac{z}{\alpha-\cfrac{z}{\ddots\,\overline{\alpha-\cfrac{z}{1+z^2-yz-z^2\,D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}}}}},$$

and the following result:

THEOREM 7. *For any $s \geq 1$, let*

$$\begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} = S_\alpha^{s-2}T_{1+z^2-yz}.$$

Then

$$D_{\overline{\{s\}},\overline{\{s-1\}}}(y,z)=\frac{a_s-b_sz^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_s-d_sz^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

Second, let us deal with another special case with $m = 1$. We are facing an $(s + 1)$-layer continued fraction:

$$D_{\overline{\{s\}},\overline{\{s\}}}(y,z) = \cfrac{1}{\alpha - \cfrac{z}{\alpha - \cfrac{z}{\ddots \cfrac{}{\alpha - \cfrac{z}{1+z^2-yz^2-z^2 D_{\bar{\emptyset},\bar{\emptyset}(y,z)}}}}}}.$$

THEOREM 8. *For any $s \geq 1$, let*

$$\begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} = S_\alpha^{s-1} T_{1+z^2-yz^2}.$$

Then

$$D_{\overline{\{s\}},\overline{\{s\}}}(y,z) = \frac{a_s - b_s z^2 D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_s - d_s z^2 D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

Let us see the final situation with $s \geq 1$ and $m \geq 2$. By Theorem 1(a), we obtain an $(s+1)$-layer continued fraction:

$$D_{\overline{\{s\}},\overline{\{s+m-1\}}}(y,z) = \cfrac{1}{\alpha - \cfrac{z}{\alpha - \cfrac{z}{\ddots \cfrac{}{\alpha - \cfrac{z}{\beta - z\, D_{\bar{\emptyset},\overline{\{m-1\}}}(y,z)}}}}},$$

and then, by Theorem 5, we derive the next property.

THEOREM 9. *For any $s \geq 1$ and $m \geq 2$, let*

$$\begin{pmatrix} a_{s,m} & b_{s,m} \\ c_{s,m} & d_{s,m} \end{pmatrix} = S_\alpha^{s-1} T_\beta S_0 S_\alpha^{m-2} T_{1+z^2-y(z+z^2)}.$$

Then

$$D_{\overline{\{s\}},\overline{\{s+m-1\}}}(y,z) = \frac{a_{s,m} - b_{s,m} z^2 D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_{s,m} - d_{s,m} z^2 D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

3.5. Case $(A, B) = (\{s+m\}, \{s\})$

Suppose $s \geq 0$ and $m \geq 2$. We face the following continued fraction with $s+1$ layers.

$$D_{\overline{\{s+m\}},\overline{\{s\}}}(y,z)=\cfrac{1}{\alpha-\cfrac{z}{\alpha-\cfrac{z}{\ddots\,\cfrac{}{\alpha-\cfrac{z}{1+z^2-y(z+z^2)-z^2\,D_{\overline{\{m\}-1},\bar{\emptyset}}(y,z)}}}}}.$$

When $s=0$, there is no fraction and then a corollary of Theorem 6 is derived:

Corollary 2. *For any $m\geq 2$, let*

$$\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} = S_\alpha^{m-2}T_\beta.$$

Then

$$D_{\overline{\{m\}},\bar{\emptyset}}(y,z)=1+z^2-y(z+z^2)-z^2\,\frac{a_m-b_mz^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_m-d_mz^2D_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

As for $s\geq 1$, by Theorem 6 again, we obtain the following result.

Theorem 10. *For any $s\geq 1$ and $m\geq 2$, let*

$$\begin{pmatrix} a_{s,m} & b_{s,m} \\ c_{s,m} & d_{s,m} \end{pmatrix} = S_\alpha^{s-1}T_{1+z^2-y(z+z^2)}\begin{pmatrix} 0 & 1 \\ -z^2 & 0 \end{pmatrix}S_\alpha^{m-2}T_\beta.$$

Then

$$D_{\overline{\{s+m\}},\overline{\{s\}}}(y,z)=\frac{a_{s,m}-b_{s,m}zD_{\bar{\emptyset},\bar{\emptyset}}(y,z)}{c_{s,m}-d_{s,m}zD_{\bar{\emptyset},\bar{\emptyset}}(y,z)}.$$

3.6. Case (A, B) with $A, B\in\{\emptyset,\mathbb{E},\mathbb{O}\}$

Let $\mathbb{O}$ and $\mathbb{E}$ be the sets of odd and even non-negative integers, respectively. We discuss the cases with nontrivial combinations of (A,B) for $A,B\in\{\emptyset,\mathbb{O},\mathbb{E}\}$. First, for the cases with one of A and B being empty, we obtain the following theorem.

Theorem 11. *We have*

$$D_{\overline{\mathbb{E}},\bar{\emptyset}}(y,z)$$
$$=\frac{1+z+(1-y)z(1+z)-\sqrt{(1+z+(1-y)(z+z^2))^2-4(z+z^2)(1+z(1-y))}}{2z(1+(1-y)z)},$$

$$D_{\bar{\mathbb{O}},\bar{\emptyset}}(y,z)$$
$$=\frac{1+z+(1-y)(z+z^2)-\sqrt{(1+z+(1-y)(z+z^2))^2-4(z+z^2)(1+z(1-y))}}{2z(1+z)},$$

$$D_{\bar{\emptyset},\bar{\mathbb{E}}}(y,z) = \frac{1+z+(1-y)(z-z^2)-\sqrt{(1-zy+(1-y)z^2)^2-4z^2}}{2z}, \quad \textit{and}$$

$$D_{\bar{\emptyset},\bar{\mathbb{O}}}(y,z) = \frac{1-zy+(1-y)z^2-\sqrt{(1-zy+(1-y)z^2)^2-4z^2}}{2z^2}.$$

Proof: Knowing that $\mathbb{E}-1=\mathbb{O}$, $\mathbb{O}-1=\mathbb{E}$ and by Equation (1), we derive two pairs of reduction formulas:

$$D_{\bar{\mathbb{E}},\bar{\emptyset}}(y,z) = 1+z(D_{\bar{\mathbb{O}},\bar{\emptyset}}(y,z)+y-1)D_{\bar{\mathbb{E}},\bar{\emptyset}}(y,z)$$
$$D_{\bar{\mathbb{O}},\bar{\emptyset}}(y,z) = 1+z(D_{\bar{\mathbb{E}},\bar{\emptyset}}(y,z)-1)D_{\bar{\mathbb{O}},\bar{\emptyset}}(y,z).$$

and

$$D_{\bar{\emptyset},\bar{\mathbb{E}}}(y,z) = 1+z(D_{\bar{\emptyset},\bar{\mathbb{O}}}(y,z)+y-1),$$
$$D_{\bar{\emptyset},\bar{\mathbb{O}}}(y,z) = 1+z(D_{\bar{\emptyset},\bar{\mathbb{E}}}(y,z)+y-1)D_{\bar{\emptyset},\bar{\mathbb{O}}}(y,z).$$

The four equations are the exact solutions of these two pairs. ■

An *n-Motzkin path* is a lattice path starting at (0, 0) and ending at (*n*, 0) in the first quadrant. Similar in structure to the Dyck paths, but Motzkin paths consist of one more type of step—*level step:* $L=(1,0)$. Let $\mathcal{M}_n$ denote the set of all *n*-Motzkin paths, $\mathcal{M}=\bigcup_{n\geq 0}\mathcal{M}_n$, and $m_n=|\mathcal{M}_n|$, the so-called Motzkin number. It is well known that the generating function $M(z):=\sum_{n\geq 0}m_n z^n$ satisfies $M(z)=1+zM(z)+zM(z)^2$ and explicitly $M(z)=\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$.

Theorem 11 implies that $D_{\bar{\emptyset},\bar{\mathbb{E}}}(1,z)=D_{\bar{\mathbb{E}},\bar{\emptyset}}(1,z)=1+zM(z)$ and $D_{\bar{\emptyset},\bar{\mathbb{O}}}(1,z)=M(z)$. Eu et al. [2] first discovered $D_{\bar{\mathbb{E}},\bar{\emptyset}}(1,z)=1+zM(z)$ by claiming that $d_{n,\bar{\mathbb{E}},\bar{\emptyset}}$ is equal to the shifted Motzkin number m_{n-1}. They also found that $d_{n,\bar{\mathbb{O}},\bar{\emptyset}}$ is the *n*th Riordan number. We will mimic their proof to demonstrate bijections for $D_{\bar{\emptyset},\bar{\mathbb{E}}}(1,z)=1+zM(z)$ and $D_{\bar{\emptyset},\bar{\mathbb{O}}}(1,z)=M(z)$.

Bijection proof of these two equations. Let us define a map $\phi:\{U,L,D\}\to\{UU,DU,DD\}$ by $\phi(U)=UU, \phi(L)=UD$, and $\phi(D)=DD$. To prove that $D_{\bar{\emptyset},\bar{\mathbb{E}}}(1,z)=1+zM(z)$, it is sufficient to devise a bijection between $\mathcal{M}_{n-1}$ and $\mathcal{D}_{n,\bar{\emptyset},\bar{\mathbb{E}}}$. Given any $(n-1)$-Motzkin path $M=S_1S_2\ldots S_{n-1}$, we define $\phi(M)=\phi(S_1)\phi(S_2)\ldots\phi(S_{n-1})$. The corresponding *n*-Dyck path of M is $P=U\phi(M)D$. For example, $M=LULUDD$ corresponds to $P=UUDUUUDUUDDDDD$. It is clear that P contains no valleys of even heights, and that such map $\mathcal{M}_{n-1}\to\mathcal{D}_{n,\bar{\emptyset},\bar{\mathbb{E}}}$ is bijective.

To obtain $D_{\bar{\emptyset},\bar{\mathbb{O}}}(1,z)=M(z)$, given any *n*-Motzkin path $M=S_1S_2\ldots S_n$, the corresponding *n*-Dyck path of M is $P=\phi(M)$. It is clear that P

contains no valleys of odd heights, and that such map $\mathcal{M}_n \to \mathcal{D}_{n,\bar{\emptyset},\bar{\mathbb{O}}}$ is bijective.

Now, we consider the cases $(A, B) \in \{(\mathbb{E}, \mathbb{E}), (\mathbb{O}, \mathbb{O}), (\mathbb{E}, \mathbb{O}), (\mathbb{O}, \mathbb{E})\}$.

THEOREM 12.

$$\begin{aligned} D_{\bar{\mathbb{E}},\bar{\mathbb{E}}}(y,z) &= \frac{1+2z+z^2(y-1)-\sqrt{(1+z^2(1-y))^2-4z^2}}{2z}, \\ D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(y,z) &= \frac{1+z^2(1-y)-\sqrt{(1+z^2(1-y))^2-4z^2}}{2z^2}, \\ D_{\bar{\mathbb{E}},\bar{\mathbb{O}}}(y,z) &= \frac{1-zy+z^2-\sqrt{(1-zy+z^2)^2-4z^2}}{2z^2}, \\ D_{\bar{\mathbb{O}},\bar{\mathbb{E}}}(y,z) &= \frac{1+z(2-y)-z^2-\sqrt{(1-zy+z^2)^2-4z^2}}{2z}. \end{aligned}$$

Proof: Again, we use Equation (1) to derive two pairs of reduction formulas:

$$\begin{aligned} D_{\bar{\mathbb{E}},\bar{\mathbb{E}}}(y,z) &= 1+z(D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(y,z)+y-1), \\ D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(y,z) &= 1+z(D_{\bar{\mathbb{E}},\bar{\mathbb{E}}}-1)D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(y,z). \end{aligned}$$

and

$$\begin{aligned} D_{\bar{\mathbb{E}},\bar{\mathbb{O}}}(y,z) &= 1+z(D_{\bar{\mathbb{O}},\bar{\mathbb{E}}}(y,z)+y-1)D_{\bar{\mathbb{E}},\bar{\mathbb{O}}}(y,z), \\ D_{\bar{\mathbb{O}},\bar{\mathbb{E}}}(y,z) &= 1+z(D_{\bar{\mathbb{E}},\bar{\mathbb{O}}}(y,z)-1). \end{aligned}$$

The four equations are the exact solutions of these two pairs. ∎

3.7. A and B being congruence classes

Give any integer $k \geq 2$ and $J \subset [0, k-1]$, define the congruence class $J_k := \{n \in \mathbb{Z} \mid n \equiv j \pmod{k} \text{ for some } j \in J\}$. For example, $\{m\}_k := \{n \in \mathbb{Z} \mid n \equiv m \pmod{k}\}$ for some $m \in [0, k-1]$. Here we generalize the cases of odd-even in section 3.6 and make A, B to be these congruence classes. Let $\gamma_A^i := 1 + z - \chi(i \notin A)yz$, for short. By Equation (2), we derive two theorems as follows:

THEOREM 13. *Give any $J \subset [0, k-1]$, the generating function $D_{\overline{J_k},\overline{\{0\}_k}}(y,z)$ satisfies the following functional equation*

$$D_{\overline{J_k},\overline{\{0\}_k}}(y,z) = 2 - \gamma_J^1 + \cfrac{z}{\gamma_J^2 - \cfrac{z}{\ddots \gamma_J^{k-1} - \cfrac{z}{\gamma_J^k - zD_{\overline{J_k},\overline{\{0\}_k}}(y,z)}}}$$

and then $D_{\overline{J_k},\overline{\{0\}_k}}(y,z)$ equals $\frac{u+\sqrt{v}}{w}$ as the solution, where $u, v, w \in \mathbb{Z}[y,z]$.

Proof: Iterating Equation (2), the functional equation in above can be easily obtained. This functional equation suggest the matrix as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\prod_{i=1}^{k-1} S_{\gamma_J^i} \right) T_{\gamma_J^k},$$

and then

$$D_{\overline{J_k}, \overline{\{0\}_k}}(y,z) = 2 - \frac{c - dzD_{\overline{J_k}, \overline{\{0\}_k}}(y,z)}{a - bzD_{\overline{J_k}, \overline{\{0\}_k}}(y,z)}$$

This equation is equivalent to a quadratic polynomial with coefficients in $\mathbb{Z}[y,z]$. Solving the quadratic, we can represent $D_{\overline{J_k}, \overline{\{0\}_k}}(y,z)$ by the formula shown in the theorem. ■

THEOREM 14. *Let $m \in [1, k-1]$. The generating function $D_{\overline{J_k}, \overline{\{m\}_k}}(y,z)$ satisfies the following functional equation*

$$D_{\overline{J_k}, \overline{\{m\}_k}}(y,z) = \cfrac{1}{\gamma_J^1 - \cfrac{z}{\gamma_J^{m-1} - \cfrac{\ddots z}{\gamma_J^m - 2z + zG(y,z)}}},$$

where

$$G(y,z) = \gamma_J^{m+1} - \cfrac{z}{\gamma_J^{m+2} - \cfrac{z}{\gamma_J^{k-1} - \cfrac{\ddots z}{\gamma_J^k - zD_{\overline{J_k}, \overline{\{m\}_k}}(y,z)}}},$$

and then $D_{\overline{J_k}, \overline{\{m\}_k}}(y,z)$ equals $\frac{u+\sqrt{v}}{w}$ as the solution, where $u, v, w \in \mathbb{Z}[y,z]$.

Proof: Both continued fractions are obtained by iterating Equation (2). Find the corresponding products of matrices in single-fraction form for these two equations and then combine them. We obtain that

$$D_{\overline{J_k}, \overline{\{m\}_k}}(y,z) = \frac{a - bzD_{\overline{J_k}, \overline{\{m\}_k}}(y,z)}{c - dzD_{\overline{J_k}, \overline{\{m\}_k}}(y,z)},$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = U \begin{pmatrix} 1 & 0 \\ \gamma_J^m(y,z) - 2z & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix} V T_{\gamma_J^k(y,z)},$$

where $U = \prod_{i=1}^{m-1} S_{\gamma_J^i}$ and $V = \prod_{i=m+1}^{k-1} S_{\gamma_J^i}$. The rest argument is as same as the previous proof. ■

THEOREM 15. *Let $m \in [1, k-1]$. The generating function $D_{\overline{J_k}, \overline{\{m\}_k}}(y,z)$ satisfies the following equation*

$$D_{\overline{J_k},\overline{\{m\}_k}}(y,z)=\cfrac{1}{\gamma_J^1-\cfrac{z}{\gamma_J^{m-1}-\cfrac{\ddots z}{\gamma_J^m-2z+zG(y,z)}}},$$

where

$$G(y,z)=\gamma_J^{m+1}-\cfrac{z}{\gamma_J^{m+2}-\cfrac{z}{\gamma_J^{k-1}-\cfrac{\ddots z}{\gamma_J^k-zD_{\overline{J_k},\overline{\{m\}_k}}(y,z)}}}.$$

and then $D_{\overline{J_k},\overline{\{m\}_k}}(y,z)$ *equals* $\frac{u+\sqrt{v}}{w}$ *as the solution, where* $u,v,w\in\mathbb{Z}[y,z]$.

EXAMPLE 4. Let us consider the case with $k=3, m\in\{0,1,2\}$ and $J\in\{\{j\}\mid j=1,2,3\}$. We find

$$D_{\overline{\{1\}_3},\overline{\{0\}_3}}(y,z)=1-z+\cfrac{z}{1+(1-y)z-\cfrac{z}{1+(1-y)z-z\,D_{\overline{\{1\}_3},\overline{\{0\}_3}}(y,z)}},$$

$$D_{\overline{\{2\}_3},\overline{\{0\}_3}}(y,z)=1-(1-y)z+\cfrac{z}{1+z-\cfrac{z}{1+(1-y)z-z\,D_{\overline{\{2\}_3},\overline{\{0\}_3}}(y,z)}},$$

$$D_{\overline{\{0\}_3},\overline{\{0\}_3}}(y,z)=1-(1-y)z+\cfrac{z}{1+(1-y)z-\cfrac{z}{1+z-z\,D_{\overline{\{0\}_3},\overline{\{0\}_3}}(y,z)}},$$

$$D_{\overline{\{1\}_3},\overline{\{1\}_3}}(y,z)=\cfrac{1}{1+(1-y)z^2-\cfrac{z^2}{1+(1-y)z-z\,D_{\overline{\{1\}_3},\overline{\{1\}_3}}(y,z)}},$$

$$D_{\overline{\{2\}_3},\overline{\{1\}_3}}(y,z)=\cfrac{1}{1-zy+z^2-\cfrac{z^2}{1+(1-y)z-z\,D_{\overline{\{2\}_3},\overline{\{1\}_3}}(y,z)}},$$

$$D_{\overline{\{0\}_3},\overline{\{1\}_3}}(y,z)=\cfrac{1}{1-zy+(1-y)z^2-\cfrac{z^2}{1+z-z\,D_{\overline{\{0\}_3},\overline{\{1\}_3}}(y,z)}},$$

$$D_{\overline{\{1\}_3},\overline{\{2\}_3}}(y,z)=\cfrac{1}{1+z-\cfrac{z}{1-zy+(1-y)z^2-z^2\,D_{\overline{\{1\}_3},\overline{\{2\}_3}}(z)}},$$

$$D_{\overline{\{2\}_3},\overline{\{2\}_3}}(y,z) = \cfrac{1}{1+(1-y)z - \cfrac{z}{1+(1-y)z^2 - z^2\, D_{\overline{\{2\}_3},\overline{\{2\}_3}}(y,z)}},$$

$$D_{\overline{\{0\}_3},\overline{\{2\}_3}}(y,z) = \cfrac{1}{1+(1-y)z - \cfrac{z}{1-zy+z^2 - z^2\, D_{\overline{\{0\}_3},\overline{\{2\}_3}}(y,z)}}.$$

Solving these equations we get the following generating functions:

$$D_{\overline{\{1\}_3},\overline{\{0\}_3}}(y,z) = \frac{(1-\alpha)(z^2-z)+\alpha^2 - \sqrt{[(1-\alpha)(z^2-z)+\alpha^2]^2 - 4\alpha z[z^2-(\alpha^2-\alpha+1)z+\alpha^2]}}{2\alpha z},$$

$$D_{\overline{\{2\}_3},\overline{\{0\}_3}}(y,z) = \frac{(3-\alpha)z^2+2z+\alpha - \sqrt{[(3-\alpha)z^2+2z+\alpha]^2 - 4(1+z)z[(-\alpha^2+4\alpha-2)z+(2-\alpha)\alpha]}}{2(1+z)z},$$

$$D_{\overline{\{0\}_3},\overline{\{0\}_3}}(y,z) = \frac{\alpha-(\alpha^2-3\alpha+1)z+z^2 - \sqrt{[\alpha-(\alpha^2-3\alpha+1)z+z^2]^2 - 4\alpha z[z^2-(\alpha^2-3\alpha+1)z+\alpha(2-\alpha)]}}{2\alpha z},$$

$$D_{\overline{\{1\}_3},\overline{\{1\}_3}}(y,z) = \frac{\alpha+(1-\alpha+\alpha^2)z+z^2 - \sqrt{[\alpha+(1-\alpha+\alpha^2)z+z^2]^2 - 4(1-z+z\alpha)z\alpha}}{2(1-z+z\alpha)z},$$

$$D_{\overline{\{2\}_3},\overline{\{1\}_3}}(y,z) = \frac{(1-\alpha)(z-z^2)+\alpha^2 - \sqrt{[(1-\alpha)(z-z^2)+\alpha^2]^2 - 4(\alpha-z+z^2)z\alpha}}{2(\alpha-z+z^2)z},$$

$$D_{\overline{\{0\}_3},\overline{\{1\}_3}}(y,z) = \frac{\alpha+(2\alpha-1)z+(\alpha-3)z^2 - \sqrt{[\alpha+(2\alpha-1)z+(\alpha-3)z^2]^2 - 4z(\alpha+(\alpha-2)z)(1+z)}}{2(\alpha+(\alpha-2)z)z},$$

$$D_{\overline{\{1\}_3},\overline{\{2\}_3}}(y,z) = \frac{\alpha+(2\alpha-3)z+(1-\alpha)z^2 - \sqrt{[\alpha+(2\alpha-3)z+(1-\alpha)z^2]^2 - 4(1+z)z(\alpha+(\alpha-2)z)}}{2(1+z)z^2},$$

$$D_{\overline{\{2\}_3},\overline{\{2\}_3}}(y,z) = \frac{\alpha+(\alpha^2-\alpha-1)z+z^2 - \sqrt{[\alpha+(\alpha^2-\alpha-1)z+z^2]^2 - 4\alpha z^2(1-z+z\alpha)}}{2\alpha z^2},$$

$$D_{\overline{\{0\}_3},\overline{\{2\}_3}}(y,z) = \frac{(1+\alpha)(z^2-z)+\alpha^2 - \sqrt{[(1+\alpha)(z^2-z)+\alpha^2]^2 - 4\alpha z^2(\alpha-z+z^2)}}{2\alpha z^2}.$$

Taking $y = 1$, we list the first few terms of them as follows:

$$
\begin{aligned}
D_{\overline{\{1\}_3},\overline{\{0\}_3}}(1,z) &= \frac{1-\sqrt{1-4z+4z^2-4z^3}}{2z}, \\
&= 1+z^2+2z^3+4z^4+9z^5+22z^6+56z^7+146z^8+388z^9 \\
&\quad + \cdots, \ \textit{(A025265 in [4])} \\
D_{\overline{\{2\}_3},\overline{\{0\}_3}}(1,z) &= \frac{1+z+2z^2-\sqrt{1-2z-3z^2+4z^4}}{2(1+z)z}, \\
&= 1+z+z^3+2z^4+4z^5+10z^6+23z^7+56z^8+138z^9 \\
&\quad + \cdots, \ \textit{(A127389 in [4])} \\
D_{\overline{\{0\}_3},\overline{\{0\}_3}}(1,z) &= \frac{1+z+z^2-\sqrt{1-2z-z^2-2z^3+z^4}}{2z}, \\
&= 1+z+z^2+z^3+2z^4+4z^5+8z^6+17z^7+37z^8+82z^9 \\
&\quad + \cdots, \ \textit{(A025241 in [4])} \\
D_{\overline{\{1\}_3},\overline{\{1\}_3}}(1,z) &= \frac{1+z-z^2-\sqrt{1-2z-z^2-2z^3+z^4}}{2z} = D_{\overline{\{0\}_3},\overline{\{0\}_3}}(1,z)-z, \\
D_{\overline{\{2\}_3},\overline{\{1\}_3}}(1,z) &= \frac{1-\sqrt{1-4z+4z^2-4z^3}}{2(1-z+z^2)z}, \\
&= 1+z+z^2+2z^3+5z^4+12z^5+29z^6+73z^7+190z^8+\cdots, \\
D_{\overline{\{0\}_3},\overline{\{1\}_3}}(1,z) &= \frac{1+z-2z^2-\sqrt{1-2z-3z^2+4z^4}}{2(1-z)z}, \\
&= 1+z+2z^2+3z^3+6z^4+12z^5+26z^6+59z^7+138z^8+\cdots, \\
D_{\overline{\{1\}_3},\overline{\{2\}_3}}(1,z) &= \frac{1-z-\sqrt{1-2z-3z^2+4z^4}}{2(1+z)z^2} = [D_{\overline{\{2\}_3},\overline{\{0\}_3}}(1,z)-1]z^{-1}, \\
D_{\overline{\{2\}_3},\overline{\{2\}_3}}(1,z) &= \frac{1-z+z^2-\sqrt{1-2z-z^2-2z^3+z^4}}{2z^2} \\
&= [D_{\overline{\{0\}_3},\overline{\{0\}_3}}(1,z)-1]z^{-1}, \\
D_{\overline{\{0\}_3},\overline{\{2\}_3}}(1,z) &= \frac{1-2z+2z^2-\sqrt{1-4z+4z^2-4z^3}}{2z^2} \\
&= [D_{\overline{\{1\}_3},\overline{\{0\}_3}}(1,z)+z-1]z^{-1}.
\end{aligned}
$$

Remark 1. The sequences formed by the coefficients of $D_{\overline{\{2\}_3},\overline{\{1\}_3}}(1,z)$ and $D_{\overline{\{0\}_3},\overline{\{1\}_3}}(1,z)$ are not in Sloane. The coefficients of $D_{\overline{\{0\}_3},\overline{\{0\}_3}}(1,z)$, excluding the constant term, form a sequence called the generalized Catalan number, which is the sequence A025241 in [4]. Note that $d_{n,\overline{\{0\}_3},\overline{\{0\}_3}} = d_{n,\bar{\mathbb{O}},\bar{\mathbb{E}}}$ for $n \geq 2$. It would be interesting to have a bijective proof of this equation.

We have done the case for $B = \{m\}_k$ for any $k \in [0, k-1]$. Actually, we can derive the same result for any $(A, B) = (J_k, J_k')$ by using more continued fractions as in Theorem 15.

THEOREM 16.

(a) *Let* $J, J' \subset [0, k-1]$. *Then* $D_{\overline{J_k},\overline{J_k'}}(y,z) = \frac{u+\sqrt{v}}{w}$ *for some* $u, v, w \in \mathbb{Z}[y,z]$.
(b) *Let* $J \subset [0, k-1]$. *Then* $D_{\overline{J_k},\emptyset}(y,z)$ (*as well as* $D_{\bar{\emptyset},\overline{J_k}}(y,z)$) *is equal to* $\frac{u+\sqrt{v}}{w}$ *for some* $u, v, w \in \mathbb{Z}[y,z]$.
(c) *Let* $J \subset [0, k-1]$ *and* I *be a finite set of integers. Then* $D_{\overline{J_k},\overline{I}}(y,z)$ *is equal to* $\frac{a-bz^t D_{\overline{J_k},\bar{\emptyset}}(y,z)}{c-dz^t D_{\overline{J_k},\bar{\emptyset}}(y,z)}$ *for some* $a, b, c, d \in \mathbb{Z}[y,z]$ *and* $t \in \mathbb{P}$. *It is similar for* $D_{\overline{I},\overline{J_k}}(y,z)$.

4. Means and variances

First, we study the mean and the variance of peaks of all n-Dyck paths with $(A, B) = (\emptyset, [0, m])$.

THEOREM 17. *Let m be a nonnegative integer, then*

$$md_{n,*,\bar{\emptyset},\overline{[0,m]}} = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1, 2, \ldots, m+1 \\ \dfrac{1}{2}(n-m) & \text{if} \quad n \geq m+2 \end{cases}$$

and

$$vd_{n,*,\bar{\emptyset},\overline{[0,m]}} = \begin{cases} 0 & \text{if} \quad 0 \leq n \leq m+2 \\ \dfrac{1}{4}\dfrac{(n-m)(n-m-2)}{2(n-m)-3} & \text{if} \quad n \geq m+3 \end{cases}.$$

Proof: By Corollary 1, simple computations implies that

$$\left.\frac{\partial D_{\bar{\emptyset},\overline{[0,m]}}(y,z)}{\partial y}\right|_{y=1} = z + z^2 + \cdots + z^{m+1} + \sum_{k\geq 2} \frac{(2k-3)!!}{(k-1)!} 2^{k-2} z^{m+k},$$

$$\left.\frac{\partial^2 D_{\bar{\emptyset},\overline{[0,m]}}(y,z)}{\partial y^2}\right|_{y=1} = \sum_{k\geq 0} \frac{(2k+1)!!}{k!} 2^{k+1} z^{m+k+3},$$

and

$$D_{\bar{\emptyset},\overline{[0,m]}}(1,z) = 1 + z + z^2 + \cdots + z^{m+1} + \sum_{k\geq 2} \frac{(2k-3)!!}{k!} 2^{k-1} z^{m+k}.$$

Hence,

$$md_{n,*,\bar{\emptyset},\overline{[0,m]}} = \frac{[z^n] \left.\dfrac{\partial D_{\bar{A},\bar{B}}(y,z)}{\partial y}\right|_{y=1}}{[z^n] D_{\bar{A},\bar{B}}(1,z)} = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1, 2, \ldots, m+1 \\ \dfrac{1}{2}(n-m) & \text{if} \quad n \geq m+2 \end{cases}$$

and

$$vd_{n,*,\bar{\emptyset},\overline{[0,m]}} = \frac{[z^n] \left.\dfrac{\partial^2 D_{\bar{A},\bar{B}}(y,z)}{\partial y^2}\right|_{y=1}}{[z^n] D_{\bar{A},\bar{B}}(1,z)} + md_{n,*,*,\bar{A},\bar{B}} - (md_{n,*,*,\bar{A},\bar{B}})^2$$
$$= \begin{cases} 0 & \text{if} \quad 0 \leq n \leq m+2 \\ \dfrac{1}{4} \dfrac{(n-m)(n-m-2)}{2(n-m)-3} & \text{if} \quad n \geq m+3. \end{cases} \qquad \blacksquare$$

For any $(A, B) \in \{(\mathbb{O}, \mathbb{O}), (\mathbb{E}, \mathbb{E})\}$, we may obtain the mean and the variance of peaks of n-Dyck paths with heights of peaks avoid set A and valleys avoid set B.

Obviously, $[z^n] D_{n,\bar{\mathbb{E}},\bar{\mathbb{E}}}(z) = 0$ when $n \equiv 0 \pmod 2$; $[z^n] D_{n,\bar{\mathbb{O}},\bar{\mathbb{O}}}(z) = 0$ when $n \equiv 1 \pmod 2$.

THEOREM 18. *Let $n \equiv 1 \pmod 2$, then*

$$md_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}} = \begin{cases} 1 & \text{if} \quad n = 1 \\ \dfrac{1}{4}(n+1) & \text{if} \quad n \geq 3 \end{cases}$$

and

$$vd_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}} = \begin{cases} 0 & \text{if} \quad n = 1, 3 \\ \dfrac{1}{16} \dfrac{(n+1)(n-3)}{n-2} & \text{if} \quad n \geq 5. \end{cases}$$

Let $n \equiv 0 \pmod 2$, then

$$md_{n,*,\bar{\mathbb{O}},\bar{\mathbb{O}}} = \begin{cases} 0 & \text{if} \quad n = 0 \\ \dfrac{1}{4}(n+2) & \text{if} \quad n \geq 2, \end{cases}$$

and

$$vd_{n,*,\bar{\mathbb{O}},\bar{\mathbb{O}}} = \begin{cases} 0 & \text{if} \quad n = 0, 2 \\ \dfrac{1}{16}\dfrac{(n+2)(n-2)}{n-1} & \text{if} \quad n \geq 4. \end{cases}$$

Proof: It is easy to obtain that

$$\left.\frac{\partial D_{\bar{\mathbb{E}},\bar{\mathbb{E}}}(y,z)}{\partial y}\right|_{y=1} = z + \sum_{k\geq 2} \frac{(2k-3)!!}{(k-1)!} 2^{k-2} z^{2k-1}, \quad \left.\frac{\partial^2 D_{\bar{\mathbb{E}},\bar{\mathbb{E}}}(y,z)}{\partial y^2}\right|_{y=1} = \sum_{k\geq 0} \frac{(2k+1)!!}{k!} 2^{k+1} z^{2k+5}$$

and

$$D_{\bar{\mathbb{E}},\bar{\mathbb{E}}}(1,z) = 1 + z + \sum_{k\geq 2} \frac{(2k-3)!!}{k!} 2^{k-1} z^{2k-1}.$$

Hence,

$$md_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}} = \frac{[z^n] \left.\dfrac{\partial D_{\bar{A},\bar{B}}(y,z)}{\partial y}\right|_{y=1}}{[z^n] D_{\bar{A},\bar{B}}(1,z)} = \begin{cases} 1 & \text{if} \quad n = 1 \\ \dfrac{1}{4}(n+1) & \text{if} \quad n \geq 3 \text{ and } n \equiv 1 \pmod 2 \end{cases}$$

and

$$vd_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}} = \frac{[z^n] \left.\dfrac{\partial^2 D_{\bar{A},\bar{B}}(y,z)}{\partial y^2}\right|_{y=1}}{[z^n] D_{\bar{A},\bar{B}}(1,z)} + md_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}} - (md_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}})^2 = \begin{cases} 0 & \text{if} \quad n = 1, 3 \\ \dfrac{1}{16}\dfrac{(n+1)(n-3)}{n-2} & \text{if} \quad n \geq 5 \text{ and } n \equiv 1 \pmod 2. \end{cases}$$

Similarly, we may obtain

$$\left.\frac{\partial D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(y,z)}{\partial y}\right|_{y=1} = \sum_{k\geq 2} \frac{(2k-3)!!}{(k-1)!} 2^{k-2} z^{2k-2}, \quad \left.\frac{\partial^2 D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(y,z)}{\partial y^2}\right|_{y=1} = \sum_{k\geq 0} \frac{(2k+1)!!}{k!} 2^{k+1} z^{2k+4}$$

and

$$D_{\bar{\mathbb{O}},\bar{\mathbb{O}}}(1,z) = 1 + \sum_{k\geq 2} \frac{(2k-3)!!}{k!} 2^{k-1} z^{2k-2}.$$

Thus,

$$md_{n,*,\bar{\mathbb{O}},\bar{\mathbb{O}}} = \frac{[z^n] \left.\dfrac{\partial D_{\bar{A},\bar{B}}(y,z)}{\partial y}\right|_{y=1}}{[z^n] D_{\bar{A},\bar{B}}(1,z)} = \begin{cases} 0 & \text{if} \quad n = 0 \\ \dfrac{1}{4}(n+2) & \text{if} \quad n \geq 2 \text{ and } n \equiv 0 \pmod 2 \end{cases}$$

and

$$vd_{n,*,\bar{\mathbb{O}},\bar{\mathbb{O}}} = \frac{[z^n] \left.\dfrac{\partial^2 D_{\bar{A},\bar{B}}(y,z)}{\partial y^2}\right|_{y=1}}{[z^n] D_{\bar{A},\bar{B}}(1,z)} + md_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}} - (md_{n,*,\bar{\mathbb{E}},\bar{\mathbb{E}}})^2 = \begin{cases} 0 & \text{if} \quad n = 0, 2 \\ \dfrac{1}{16} \dfrac{(n+2)(n-2)}{n-1} & \text{if} \quad n \geq 4 \text{ and } n \equiv 0 \pmod 2). \end{cases} \blacksquare$$

5. Equivalence classes

Following [2], we say that two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are shift equivalent, denoted by $\langle a_n \rangle \equiv_s \langle b_n \rangle$, if there exist nonnegative integers p and q such that $a_{p+n} = b_{q+n}$ for all $n \in \mathbb{N}$. Ignoring the number of peaks, we study shift equivalent classes for the sequence $\langle d_{n,\bar{A},\bar{B}} \rangle$, in this section. One of results is trivial as follows.

THEOREM 19. *Let $A \subset \mathbb{P}$ and $B \subset \mathbb{N}$ with $0 \in B$. Then $d_{n+1,\bar{A},\bar{B}} = d_{n,\overline{A-1},\overline{B-1}}$ for any $n \in \mathbb{N}$. Reversely, given $A' \subset \mathbb{P}$ and $B' \subset \mathbb{N}$, we have $d_{n,\overline{A'},\overline{B'}} = d_{n+1,\overline{A'+1},\overline{(B'+1)\cup\{0\}}}$ for any $n \in \mathbb{N}$.*

In this theorem, the condition $0 \in B$ almost derives $1 \in A$ with the only one exception, namely the path $P = UD$. If $[0, m-1] \subset B$ and $m \notin B$, we can repeat this theorem m times and obtain an avoiding pair $(A - m, B - m)$ with $0 \notin B - m$. Thus, we study further on the case with $0 \notin B$. Eu et al. [2, p. 462] studied the case with $B = \emptyset$ (thus, $0 \notin B$) and obtained a shift equivalence property as follows.

THEOREM 20. *Let $A \subset \mathbb{P}$ with $2 \in A$ but $1 \notin A$. Then $d_{n+1,\bar{A},\bar{\emptyset}} = d_{n,\overline{(A-\{2\})-1},\bar{\emptyset}}$ for any $n \in \mathbb{N}$.*

This theorem can be applied iteratively when $B = \emptyset$ and $[2, m] \subset A$ but $1 \notin A$. Now we generalize their result and obtain the following property.

THEOREM 21. *Let $A \subset \mathbb{P}$ and $B \subset \mathbb{N}$ with $2 \in A$ but $1 \notin A$ and $0, 1 \notin B$, then $d_{n+1,\bar{A},\bar{B}} = d_{n,\overline{(A\setminus\{2\})-1},\overline{B-1}}$ for all $n \in \mathbb{N}$. Reversely, given any $A' \subset \mathbb{P}$ and $B' \subset \mathbb{N}$ with $1 \notin A'$ and $0 \notin B'$. Then $d_{n,\overline{A'},\overline{B'}} = d_{n+1,\overline{(A'+1)\cup\{2\}},\overline{B'+1}}$ for all $n \in \mathbb{N}$.*

Proof: The first statement is equivalent to the second one; so we only prove the second statement. We follow [2] to offer a bijection between $\mathcal{D}_{n,\overline{A'},\overline{B'}}$ and $\mathcal{D}_{n+1,\overline{(A'+1)\cup\{2\}},\overline{B'+1}}$. Given any $P \in \mathcal{D}_{n,\overline{A'},\overline{B'}}$, let us construct an $(n+1)$-Dyck path P^* by first lifting up P (i.e., UPD), followed by replacing every peak UD of height two in UPD with a valley DU. The lifted path UPD avoids peaks of heights in $A' + 1$ and valleys of heights in $B' + 1$; so does P^*. Moreover, P^* avoids peaks of height two because all such peaks in UPD turn to be valleys of height zero in P^*. In fact, all valleys of height zero arise in this way.

The inverse of this bijection is described as follows. Given any $P^* \in \mathcal{D}_{n+1,\overline{(A'+1)\cup\{2\}},\overline{B'+1}}$, to obtain P we need only replace every valley DU of height zero in P^* with a peak UD and then remove the first U and the last D. Since some P^* might have a valley of height zero, the corresponding P would have a peak of height one and a valley of height zero. Therefore, it is necessary to require $1 \notin A'$ and $0 \notin B'$. It is easy to check these two maps are the inverse of each other. ■

Using the technique of the previous proof, we derive another general property:

THEOREM 22. *Let $n \in \mathbb{N}$. For any $A \subset \mathbb{P}$ and $B \subset \mathbb{N}$ with some positive integer $m \notin A \cup B, m+1 \in A$ and $m-1 \notin B$, we have $d_{n,\bar{A},\bar{B}} = d_{n,\overline{A-\{m+1\}},\overline{B\cup\{m-1\}}}$. Reversely, given any $A' \subset \mathbb{P}$ and $B' \subset \mathbb{N}$ with some positive integer $m \notin A \cup B, m+1 \notin A$ and $m-1 \in B$. Then we have $d_{n,\overline{A'},\overline{B'}} = d_{n,\overline{A'\cup\{m+1\}},\overline{B-\{m-1\}}}$.*

Proof: We only prove the first statement. The proof relies on a bijection between $\mathcal{D}_{n,\bar{A},\bar{B}}$ and $\mathcal{D}_{n,\overline{A-\{m+1\}},\overline{B\cup\{m-1\}}}$. The forward map simply replace those valleys DU of height $m-1$ with UD, which are then peaks of height $m+1$. Since no valleys of height $m-1$ remain and some valleys of height m might be created, so the new Dyck path satisfies avoiding pair $(\overline{A-\{m+1\}}, \overline{B\cup\{m-1\}})$. It seems that requiring $m \in B$ is also good enough, but actually $m \in B$ achieves too much by making every valley of height m in the new Dyck path to be the form DUD or UDU.

On the other hand, the reverse map replaces those peaks UD of height $m+1$ with DU, which are then valleys of height $m-1$. Since peaks of height

$m-1$ remain and some valleys of height m might be created, so the new avoiding pair is $(\overline{A-\{m+1\}}, \overline{B\cup\{m-1\}})$. It is easy to check these two maps are inverse of each other. Hence, the proof follows. ■

Theorem 21 can also be derived from Theorems 22 and 19. The following more useful properties are obtained by repeating Theorem 22.

COROLLARY 3.

(a) Let $n \in \mathbb{N}$. For any $A, A' \subset \mathbb{P}$ nd $B \subset \mathbb{N}$ with $1 \notin A'$ and $A \cap A' = (A \cup B) \cap (A'-1) = B \cap (A'-2) = \emptyset$, we have $d_{n,\overline{A\cup A'},\bar{B}} = d_{n,\bar{A},\overline{B\cup(A'-2)}}$. Reversely, for any $A \subset \mathbb{P}$ and $B, B' \subset \mathbb{N}$ with $B \cap B' = (A \cup B) \cap (B'+1) = A \cap (B'+2) = \emptyset$, we have $d_{n,\bar{A},\overline{B\cup B'}} = d_{n,\overline{A\cup(B'+2)},\bar{B}}$.
(b) Let $n \in \mathbb{N}$. For any $A \subset \mathbb{P}$ with $1 \notin A$, we have $d_{n,\bar{A},\bar{\emptyset}} = d_{n,\bar{\emptyset},\overline{A-2}}$. Reversely, for any $B \subset \mathbb{N}$, we have $d_{n,\bar{\emptyset},\bar{B}} = d_{n,\overline{B+2},\bar{\emptyset}}$.

We say avoiding pairs (A, B) and (A', B') are equivalent if the corresponding sequences $\langle d_{n,\bar{A},\bar{B}}\rangle$ and $\langle d_{n,\overline{A'},\overline{B'}}\rangle$ are shift equivalent. The above properties about shift equivalence can be used to claim equivalent classes on avoiding pairs. Does these properties draw the whole picture of equivalence? The problem is still open. At least we use these properties to make several examples about equivalent classes as follows.

Start with any avoiding pair (A, B). First of all, by Theorem 22 and Corollary 3, we transfer as many elements as possible from A to B. Now suppose $[0, m-1] \subset B$ and $m \notin B$, by Theorem 19 we obtain $(A-m, B-m)$ with $0 \notin B-m$ as an equivalent avoiding pair. Let us rename the current avoiding pair (A, B). It could not satisfies the hypothesis in Theorem 22, because we have already transferred as many elements as possible from A to B. This fact suggests that (A, B) satisfies one of the following three types:

1. $0, 1 \notin B$ and $1, 2 \notin A$,
2. $0, 1 \notin B$ and $1 \in A$, and
3. $0 \notin B$ but $1 \in B$.

Finally, we list several interesting equivalence classes as follows, in which every leading sequence is classified according to these three types.

Type 1:

$$\begin{aligned}\langle d_{n,\bar{\emptyset},\bar{\emptyset}}\rangle &\equiv_s \langle d_{n,\bar{\emptyset},\overline{\{0\}}}\rangle \equiv_s \langle d_{n,\bar{\emptyset},\overline{\{0,1\}}}\rangle \equiv_s \cdots \\ &\equiv_s \langle d_{n,\overline{\{2\}},\bar{\emptyset}}\rangle \equiv_s \langle d_{n,\overline{\{2,3\}},\bar{\emptyset}}\rangle \equiv_s \cdots \equiv_s \langle \text{Catalan number}\rangle, \\ \langle d_{n,\overline{\{0\}_3},\overline{\{2\}_3}}\rangle &\equiv_s \langle d_{n,\overline{\{1\}_3},\overline{\{0\}_3}}\rangle \equiv_s \cdots \equiv_s \langle \text{A025265 in [4]}\rangle\end{aligned}$$

Type 2:

$$\langle d_{n,\overline{\{1\}_3},\overline{\{2\}_3}}\rangle \equiv_s \langle d_{n,\overline{\{2\}_3},\overline{\{0\}_3}}\rangle \equiv_s \cdots \equiv_s \langle \text{A127389 in[4]}\rangle$$

Type 3:

$$
\begin{aligned}
\langle d_{n,\bar{\emptyset},\bar{\mathbb{O}}}\rangle &\equiv_s \langle d_{n,\overline{\{2\}},\overline{\mathbb{O}+1}}\rangle \equiv_s \langle d_{n,\overline{\{2,3\}},\overline{\mathbb{O}+2}}\rangle \equiv_s \cdots \\
&\equiv_s \langle d_{n,\bar{\emptyset},\bar{\mathbb{E}}}\rangle \equiv_s \langle d_{n,\overline{\{2\}},\overline{\mathbb{E}+1}}\rangle \equiv_s \cdots \\
&\equiv_s \langle d_{n,\overline{\mathbb{O}\setminus\{1\}},\bar{\emptyset}}\rangle \equiv_s \langle d_{n,\bar{\mathbb{E}},\bar{\emptyset}}\rangle \equiv_s \cdots \equiv_s \langle \text{Motakin numbers}\rangle, \\
\langle d_{n,\bar{\mathbb{E}},\bar{\mathbb{O}}}\rangle &\equiv_s \langle d_{n,\bar{\mathbb{O}},\bar{\mathbb{E}}}\rangle \equiv_s \cdots \equiv_s \langle \text{generalized Catalan number}\rangle, \\
\langle d_{n,\bar{\emptyset},\overline{\{1,2\}_3}}\rangle &\equiv_s \langle d_{n,\bar{\emptyset},\overline{\{0,1\}_3}}\rangle \equiv_s \langle d_{n,\bar{\emptyset},\overline{\{0,2\}_3}}\rangle \equiv_s \cdots \\
&\equiv_s \langle d_{n,\overline{\{1\}_3\setminus\{1\}},\overline{\{1\}_3}}\rangle \equiv_s \langle d_{n,\overline{\{2\}_3},\overline{\{2\}_3}}\rangle \equiv_s \langle d_{n,\overline{\{0\}_3},\overline{\{0\}_3}}\rangle \equiv_s \cdots \\
&\equiv_s \langle d_{n,\overline{\{1\}_3},\overline{\{1\}_3}}\rangle \equiv_s \cdots \equiv_s \langle \text{generalized Catalan number}\rangle, \\
\langle d_{n,\bar{\mathbb{O}},\bar{\mathbb{O}}}\rangle &\equiv_s \langle d_{n,\bar{\mathbb{E}},\bar{\mathbb{E}}}\rangle \equiv_s \langle d_{n,\overline{(\mathbb{E}+1)\cup\{2\}},\overline{\mathbb{E}+1}}\rangle \equiv_s \cdots \equiv_s \langle \text{Sloane A126120}\rangle.
\end{aligned}
$$

Acknowledgments

Shu-Chung Liu was partially supported by NSC 96-2115-M-134-003-MY2. Yeong-Nan Yeh was partially supported by NSC 96-2115-M-001-005.

References

1. E. Deutsch, Dyck path enumeration, *Discrete Math.* 204:167–202 (1999).
2. S.-P. Eu, S.-C. Liu, and Y.-N. Yeh, Dyck paths with Peaks Avoiding or Restricted to a Given Set, *Stud. Appl. Math.* 111:453–465 (2003).
3. P. Peart and W.-J. Woan, Dyck paths with no peaks at height k, *J. Integer Seq.* 4, Article 01.1.3 (2001).
4. N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at http://www.research.att.com/∼njas/sequences/Seis.html.

National Hsinchu University of Education, Taiwan
Academia Sinica, Taiwan

(Received February 25, 2008)