

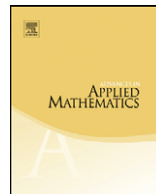


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Tutte polynomials and G -parking functions

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ABSTRACT

In this paper, we give a new expression for the Tutte polynomial of a general connected graph G in terms of statistics of G -parking functions. In particular, given a G -parking function f , let $cb_G(f)$ be the number of critical-bridge vertices of f and denote $w_G(f) = |E(G)| - |V(G)| - \sum_{v \in V(G)} f(v)$. We prove that $T_G(x, y) = \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)}$, where \mathcal{P}_G is the set of G -parking functions. Our proof avoids any use of spanning trees and is independent of bijections between the set of G -parking functions and the set of spanning trees.

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1. Introduction

The classical parking functions are defined as follows. There are n parking spaces which are arranged in a line, numbered 0 to $n - 1$ left to right and n drivers labeled $1, \dots, n$. Each driver i has an initial parking preference a_i . Drivers enter the parking area in the order in which they are labeled. Each driver proceeds to his preferred space and parks here if it is free, or parks at the next unoccupied space to the right. If all the drivers park successfully by this rule, then the sequence (a_1, \dots, a_n) is called a parking function.

Konheim and Weiss [5] introduced the concept of parking functions of length n in the study of the linear probes of random hashing function. Riordan [9] studied a relation of parking problems to ballot problems. The most notable result about parking functions is bijections from the set of classical parking functions to the set of labeled trees on $n + 1$ vertices.

Recently, Postnikov and Shapiro [8] based on the work of Cori, Rossin and Salvay [3] and defined the G -parking functions of a graph. Let G be a connected graph with vertex set $V(G) = \{0, 1, 2, \dots, n\}$

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and edge set $E(G)$. We allow G to have multiple edges and loops. For any $I \subseteq V(G) \setminus \{0\}$ and $v \in I$, define $\text{outdeg}_{I,G}(v)$ to be the number of edges from the vertex v to a vertex outside of the subset I in G . We give the definition of G -parking functions as follows.

Definition 1.1. A G -parking function is a function $f : V(G) \setminus \{0\} \rightarrow \mathbb{N}$, such that for every $I \subseteq V(G) \setminus \{0\}$ there exists a vertex $v \in I$ such that $0 \leq f(v) < \text{outdeg}_{I,G}(v)$.

Note that $f(0)$ is undefined. For convenience, we always let $f(0) = -1$.

Plautz and Calderer [7] related G -parking functions and the Tutte polynomial $T_G(x, y)$ of G . For each G -parking function f , let $w_G(f) = |E(G)| - |V(G)| - \sum_{v \in V(G)} f(v)$. Using deletion and contraction, they proved that $T_G(1, y) = \sum_{f \in \mathcal{P}_G} y^{w_G(f)}$, where \mathcal{P}_G is the set of G -parking functions. In the bi-variate expression of Tutte polynomial $T_G(x, y)$ defined by Tutte [10], the exponents of x and y are the number of internally active edges and the number of externally active edges in a spanning tree of G , respectively. Thus, the statistic $w_G(f)$ on G -parking functions is equally distributed as the number of externally active edges in a spanning tree of G . Chebikin and Pylyavskyy [2] established a family of bijections from the set of G -parking functions to the set of spanning trees of G . Hence, one expects to get a new expression for the Tutte polynomial of a general connected graph G in terms of statistics of G -parking functions. Other authors have attempted to formulate it. Kostić and Yan [6] proposed the notion of a G -multiparking function, a natural extension of the notion of a G -parking function and gave a representation to the polynomial $T_G(1 + x, y)$ by the reversed sum of G -multiparking functions. Given a classical parking function $\alpha = (a_1, \dots, a_n)$, let $cm(\alpha)$ be the number of critical left-to-right maximum subscripts in α . In particular, they gave an expression to the Tutte polynomial $T_{K_{n+1}}(x, y)$ of the complete graph K_{n+1} as follows:

$$T_{K_{n+1}}(x, y) = \sum_{\alpha \in \mathcal{P}_n} x^{cm(\alpha)} y^{\binom{n}{2} - \sum_{i=1}^n a_i},$$

where \mathcal{P}_n is the set of classical parking functions of length n . Recently, Eu, Fu and Lai [4] considered a class of multigraphs in connection with \mathbf{x} -parking functions, where $\mathbf{x} = (a, b, \dots, b)$. They gave the Tutte polynomial of the multigraphs in terms of \mathbf{x} -parking functions.

In this paper, we give a bivariate expression for the Tutte polynomial in terms of statistics of G -parking functions. First, note that the Tutte polynomials obey the deletion-contraction recurrence. In the bi-variate expression of the Tutte polynomial $T_G(x, y)$, the exponent of x has close relations with the bridges picked for deletion and contraction. Hence, given a G -parking function f , we give the definition of a bridge vertex of f . We find that a bridge vertex of a G -parking function which characterizes a global property of this function. Fixing a G -parking function f and a vertex v in G , we define $W_{G,f,v}$ (resp. $S_{G,f,v}$) as the set of G -parking functions which are weak (resp. strong) v -identical to f . Then v is a bridge vertex of a G -parking function f if and only if $W_{G,f,v} = S_{G,f,v}$. Finally, we give the definition of critical-bridge vertices of a G -parking function. Let $cb_G(f)$ denote the number of critical-bridge vertices of a G -parking function f . Define $P_G(x, y)$ to be a bivariate polynomial such that $P_G(x, y) = \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)}$. We show that $P_G(x, y)$ also obeys the deletion-contraction recurrence and has the same initial conditions as $T_G(x, y)$. Hence, $T_G(x, y) = P_G(x, y)$. This implies that the statistic $cb_G(f)$ on G -parking functions is equally distributed as the number of internally active edges in spanning trees of G . Our proof avoids any use of spanning trees and is independent of bijections between the set of G -parking functions and the set of spanning trees. Let K_{n+1} denote the complete graph with vertex set $V(K_{n+1}) = \{0, 1, \dots, n\}$. For any a function f from $V(K_{n+1}) \setminus \{0\}$ to \mathbb{N} , let $\alpha_f = (a_1, \dots, a_n)$ be a sequence of integers such that $a_i = f(n + 1 - i)$. Then f is a K_{n+1} -parking function if and only if α_f is a classical parking function. Furthermore, given a K_{n+1} -parking function f , we find that $v \in \{1, \dots, n\}$ is a critical-bridge vertex of f if and only if $n + 1 - v$ is a critical left-to-right maximum subscript in α_f . As an application of the main theorem in this paper, we rederive the result about the Tutte polynomial of the complete graph K_{n+1} in [6].

This paper is organized as follows. In Section 2, we give the definition of critical-bridge vertices of G -parking functions. In Section 3, we express the Tutte polynomial $T_G(x, y)$ of G in terms of statistics of G -parking functions. In Section 4, we give some research directions.

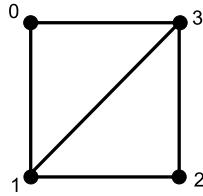


Fig. 1. The graph G.

Table 1
The G -parking functions f and $\pi_{G,f}$.

| G -parking functions f | $\pi_{G,f}$ | G -parking functions f | $\pi_{G,f}$ |
|----------------------------|----------------|----------------------------|----------------|
| $f_1 = (-1, 0, 0, 0)$ | $(0, 1, 2, 3)$ | $f_5 = (-1, 0, 1, 1)$ | $(0, 1, 3, 2)$ |
| $f_2 = (-1, 0, 0, 1)$ | $(0, 1, 2, 3)$ | $f_6 = (-1, 1, 0, 0)$ | $(0, 3, 1, 2)$ |
| $f_3 = (-1, 0, 0, 2)$ | $(0, 1, 2, 3)$ | $f_7 = (-1, 1, 1, 0)$ | $(0, 3, 1, 2)$ |
| $f_4 = (-1, 0, 1, 0)$ | $(0, 1, 3, 2)$ | $f_8 = (-1, 2, 0, 0)$ | $(0, 3, 2, 1)$ |

2. Critical-bridge vertices of G -parking functions

Throughout the paper, we let G be a connected graph with vertex set $V(G) = \{0, 1, 2, \dots, n\}$ and edge set $E(G)$ and allow G to have multiple edges and loops. Define \mathcal{P}_G as the set of G -parking functions.

Given a G -parking function f , we associate f with a sequence

$$\pi_{G,f} = (\pi_{G,f}(0), \pi_{G,f}(1), \pi_{G,f}(2), \dots, \pi_{G,f}(|V(G)| - 1))$$

of length $|V(G)|$ on vertices of G by the following algorithm, where $|V(G)|$ denotes the number of the vertices in G :

Algorithm A.

Step 1. Let $\pi_{G,f}(0) = 0$.

Step 2. Assume that $\pi_{G,f}(0), \pi_{G,f}(1), \pi_f(2), \dots, \pi_{G,f}(i)$ are determined. Let

$$V_i = V(G) \setminus \{\pi_{G,f}(0), \pi_{G,f}(1), \pi_{G,f}(2), \dots, \pi_{G,f}(i)\}.$$

Let v be a vertex of G such that $v = \min\{w \in V_i \mid 0 \leq f(w) < \text{outdeg}_{V_i,G}(w)\}$. Thus, let $\pi_{G,f}(i + 1) = v$.

The sequence $\pi_{G,f}$ obtained by Algorithm A is the same as the process order on vertices of G defined by Kostic and Yan [11]. Note that $\pi_{G,f}$ can be viewed as a bijection from $\{0, 1, \dots, |V(G)| - 1\}$ to $V(G)$.

Example 2.1. Consider the following graph G .

We list all the G -parking functions f and $\pi_{G,f}$ in Table 1.

Now, we fix a G -parking function f and $v \in V(G) \setminus \{0\}$. Let $\pi_{G,f}$ be the sequence obtained by Algorithm A. Suppose $i = \pi_{G,f}^{-1}(v)$. We construct a graph $G(f, v)$ from the graph G by the following three steps:

1. Identify all the vertices of G in the set $\{\pi_{G,f}(k) \mid 0 \leq k \leq i - 1\}$ as a new vertex 0 and delete all resulting loops.

Table 2
The graph $G(f_i, v)$ for any $v \in \{1, 2, 3\}$ and $i \in \{1, 2, \dots, 8\}$.

| $\pi_{G,f}$ | $G(f, v)$ | $\pi_{G,f}$ | $G(f, v)$ |
|------------------------------|-----------|------------------------------|-----------|
| $\pi_{G,f_1} = (0, 1, 2, 3)$ | | $\pi_{G,f_5} = (0, 1, 3, 2)$ | |
| $\pi_{G,f_2} = (0, 1, 2, 3)$ | | $\pi_{G,f_6} = (0, 3, 1, 2)$ | |
| $\pi_{G,f_3} = (0, 1, 2, 3)$ | | $\pi_{G,f_7} = (0, 3, 1, 2)$ | |
| $\pi_{G,f_4} = (0, 1, 3, 2)$ | | $\pi_{G,f_8} = (0, 3, 2, 1)$ | |

- Let U be the set of vertices w satisfying $\pi_{G,f}^{-1}(w) > i$ and there is a vertex $\bar{v} > w$ with $\pi_{G,f}^{-1}(\bar{v}) \leq i$ such that $\{u, w\} \notin E(G)$ for all $\pi_{G,f}^{-1}(\bar{v}) \leq \pi_{G,f}^{-1}(u) \leq i$. For every $w \in U$, delete all the edges connecting the new vertex 0 to w .
- Let \tilde{U} be the set of vertices of G such that $\tilde{U} = \{\pi_{G,f}(k) \mid k \geq i + 1 \text{ and } \pi_{G,f}(k) \geq v\}$. For every $w \in \tilde{U}$, delete duplicate edges connecting the new vertex 0 to w .

Example 2.2. Let G be the graph in Fig. 1. For any $v \in \{1, 2, 3\}$ and $i \in \{1, 2, \dots, 8\}$, we draw a graph $G(f_i, v)$ in Table 2.

Definition 2.3. Given a G -parking function f and a vertex $v \in V(G) \setminus \{0\}$, let $G(f, v)$ be a graph constructed as above. If the edge connecting the vertex 0 to v in $G(f, v)$ is a bridge, then we say that v is a bridge vertex of the G -parking function f . Let $B_G(f)$ be the set of bridge vertices of a G -parking function f .

It is easy to see that $B_G(f) = B_G(g)$ if f and g are two G -parking functions with $\pi_{G,f} = \pi_{G,g}$.

Table 3
The G -parking functions f , $\pi_{G,f}$ and $R_{G,f}$.

| G -parking functions f | $\pi_{G,f}$ | $R_{G,f}$ |
|----------------------------|----------------|-----------------|
| $f_1 = (-1, 0, 0, 0)$ | $(0, 1, 2, 3)$ | $(-1, 0, 0, 0)$ |
| $f_2 = (-1, 0, 0, 1)$ | $(0, 1, 2, 3)$ | $(-1, 0, 0, 1)$ |
| $f_3 = (-1, 0, 0, 2)$ | $(0, 1, 2, 3)$ | $(-1, 0, 0, 2)$ |
| $f_4 = (-1, 0, 1, 0)$ | $(0, 1, 3, 2)$ | $(-1, 0, 0, 1)$ |
| $f_5 = (-1, 0, 1, 1)$ | $(0, 1, 3, 2)$ | $(-1, 0, 1, 1)$ |
| $f_6 = (-1, 1, 0, 0)$ | $(0, 3, 1, 2)$ | $(-1, 0, 1, 0)$ |
| $f_7 = (-1, 1, 1, 0)$ | $(0, 3, 1, 2)$ | $(-1, 0, 1, 1)$ |
| $f_8 = (-1, 2, 0, 0)$ | $(0, 3, 2, 1)$ | $(-1, 0, 0, 2)$ |

Example 2.4. Let G be the graph in Fig. 1. See Table 2. We have

$$\begin{aligned}
 B_G(f_1) &= \{3\}, & B_G(f_2) &= \{3\}, & B_G(f_3) &= \{3\}, & B_G(f_4) &= \{2, 3\}, \\
 B_G(f_5) &= \{2, 3\}, & B_G(f_6) &= \{2, 3\}, & B_G(f_7) &= \{2, 3\}, & B_G(f_8) &= \{1, 2, 3\}.
 \end{aligned}$$

Given a G -parking function f , we give another characterization of bridge vertices of f . Let $\pi_{G,f}$ be the sequence obtained by Algorithm A. Define $R_{G,f}$ as a function on the vertices of G such that $R_{G,f}(i) = f(\pi_{G,f}(i))$. It is easy to see that $R_{G,f}$ is a rearrangement of the G -parking function f . Hence, for each $f \in \mathcal{P}_G$, we obtain a pair $(\pi_{G,f}, R_{G,f})$.

Example 2.5. Let G be the graph in Fig. 1. We list all the G -parking functions f as well as the corresponding $\pi_{G,f}$ and $R_{G,f}$ in Table 3.

Definition 2.6. Let f be a G -parking function and $v \in V(G) \setminus \{0\}$. Suppose $i = \pi_{G,f}^{-1}(v)$. A G -parking function g is *weak v -identical to f* if it satisfies the following conditions:

- (1) $R_{G,g}(j) = R_{G,f}(j)$ and $\pi_{G,g}(j) = \pi_{G,f}(j)$ for all $j \in \{0, 1, 2, \dots, i - 1\}$,
- (2) $\pi_{G,g}(i) \geq v$, and
- (3) $g(v) \geq f(v)$.

Furthermore, g is *strong v -identical to f* if g is weak v -identical to f and $\pi_{G,g}(i) = v$.

Given $f \in \mathcal{P}_G$ and $v \in V(G) \setminus \{0\}$, let $W_{G,f,v}$ (resp. $S_{G,f,v}$) be the set of G -parking functions which are weak (resp. strong) v -identical to f . It is easy to see that $S_{G,f,v} \subseteq W_{G,f,v}$.

Example 2.7. Let us consider the graph G in Fig. 1. Let $[n] := \{1, 2, \dots, n\}$. By Table 3, we easily obtain the results in Table 4.

In Table 4, we find that $W_{G,f,v} = S_{G,f,v}$ if v is a bridge vertex of a G -parking function f . In fact, we prove the following lemma.

Lemma 2.8. *Let f be a G -parking function and $v \in V(G) \setminus \{0\}$. Then v is a bridge vertex of f in G if and only if $W_{G,f,v} = S_{G,f,v}$.*

Proof. Let f be a G -parking function and $v \in V(G) \setminus \{0\}$. Let $\pi_{G,f}$ be the sequence obtained by Algorithm A. Suppose $i = \pi_{G,f}^{-1}(v)$. Let $U = \{\pi_{G,f}(k) \mid k \geq i\}$. Write the graph $G(f, v)$ as \bar{G} for short.

Suppose v is a bridge vertex of f . Then the edge $\{0, v\}$ is a bridge in \bar{G} . Delete this edge from \bar{G} and let I denote the set of vertices in the component containing v . It is easy to see that $\text{outdeg}_{I,G}(w) = \text{outdeg}_{U,\bar{G}}(w)$ for all $w \in I$. For any $g \in W_{G,f,v}$, we consider the sequence $\pi_{G,g}$. Clearly, $U = \{\pi_{G,g}(k) \mid k \geq i\}$ since $g \in W_{G,f,v}$. Assume $\pi_{G,g}(i) = \bar{v} > v$. Then $\bar{v} \notin I$ since

Table 4

$W_{G,f_i,v}$ and $S_{G,f_i,v}$ for any $v \in \{1, 2, 3\}$ and $i \in [8]$.

| | | | |
|--|--|--|--|
| $W_{G,f_1,1} = \{f_i \mid i \in [8]\}$ | $S_{G,f_1,1} = \{f_i \mid i \in [5]\}$ | $W_{G,f_5,1} = \{f_i \mid i \in [8]\}$ | $S_{G,f_5,1} = \{f_i \mid i \in [5]\}$ |
| $W_{G,f_1,2} = \{f_i \mid i \in [5]\}$ | $S_{G,f_1,2} = \{f_i \mid i \in [3]\}$ | $W_{G,f_5,2} = \{f_5\}$ | $S_{G,f_5,2} = \{f_5\}$ |
| $W_{G,f_1,3} = \{f_i \mid i \in [3]\}$ | $S_{G,f_1,3} = \{f_i \mid i \in [3]\}$ | $W_{G,f_5,3} = \{f_5\}$ | $S_{G,f_5,3} = \{f_5\}$ |
| $W_{G,f_2,1} = \{f_i \mid i \in [8]\}$ | $S_{G,f_2,1} = \{f_i \mid i \in [5]\}$ | $W_{G,f_6,1} = \{f_6, f_7, f_8\}$ | $S_{G,f_6,1} = \{f_6, f_7\}$ |
| $W_{G,f_2,2} = \{f_i \mid i \in [5]\}$ | $S_{G,f_2,2} = \{f_i \mid i \in [3]\}$ | $W_{G,f_6,2} = \{f_6, f_7\}$ | $S_{G,f_6,2} = \{f_6, f_7\}$ |
| $W_{G,f_2,3} = \{f_2, f_3\}$ | $S_{G,f_2,3} = \{f_2, f_3\}$ | $W_{G,f_6,3} = \{f_6, f_7, f_8\}$ | $S_{G,f_6,3} = \{f_6, f_7, f_8\}$ |
| $W_{G,f_3,1} = \{f_i \mid i \in [8]\}$ | $S_{G,f_3,1} = \{f_i \mid i \in [5]\}$ | $W_{G,f_7,1} = \{f_6, f_7, f_8\}$ | $S_{G,f_7,1} = \{f_6, f_7\}$ |
| $W_{G,f_3,2} = \{f_i \mid i \in [5]\}$ | $S_{G,f_3,2} = \{f_i \mid i \in [3]\}$ | $W_{G,f_7,2} = \{f_7\}$ | $S_{G,f_7,2} = \{f_7\}$ |
| $W_{G,f_3,3} = \{f_3\}$ | $S_{G,f_3,3} = \{f_3\}$ | $W_{G,f_7,3} = \{f_6, f_7, f_8\}$ | $S_{G,f_7,3} = \{f_6, f_7, f_8\}$ |
| $W_{G,f_4,1} = \{f_i \mid i \in [8]\}$ | $S_{G,f_4,1} = \{f_i \mid i \in [5]\}$ | $W_{G,f_8,1} = \{f_8\}$ | $S_{G,f_8,1} = \{f_8\}$ |
| $W_{G,f_4,2} = \{f_4\}$ | $S_{G,f_4,2} = \{f_4\}$ | $W_{G,f_8,2} = \{f_8\}$ | $S_{G,f_8,2} = \{f_8\}$ |
| $W_{G,f_4,3} = \{f_4, f_5\}$ | $S_{G,f_4,3} = \{f_4, f_5\}$ | $W_{G,f_8,3} = \{f_8\}$ | $S_{G,f_8,3} = \{f_8\}$ |

$\text{outdeg}_{U,G}(\bar{v}) > 0$. Note that $g(w) \geq \text{outdeg}_{I,G}(w)$ for all $w \in I$. This implies g is not a G -parking function, a contradiction. Thus, $\bar{v} = v$ and $g \in S_{G,f,v}$. Hence, $W_{G,f,v} = S_{G,f,v}$.

Conversely, we assume that $v \neq 0$ and v is not a bridge vertex of f . The vertex set of \bar{G} is $U \cup \{0\}$. Note that there is a \bar{G} -parking function \bar{f} such that $\bar{f}(v) \geq 1$ by assumption. We define a function g on vertices of G by the following ways:

$$g(u) = \begin{cases} f(u) & \text{if } u \in V(G) \setminus U, \\ \bar{f}(u) + \text{outdeg}_{U,G}(u) - 1 & \text{if } u \in U, u \geq v \text{ and } \text{outdeg}_{U,\bar{G}}(u) \geq 1, \\ \bar{f}(u) + \text{outdeg}_{U,G}(u) & \text{if } u \in U, u \geq v \text{ and } \text{outdeg}_{U,\bar{G}}(u) = 0, \\ \bar{f}(u) + \text{outdeg}_{U,G}(u) & \text{if } u \in U \text{ and } u < v. \end{cases}$$

Let $\bar{\pi}_{\bar{G},\bar{f}}$ be the sequence obtained by Algorithm A. For any $I \subseteq V(G) \setminus \{0\}$, we discuss the following two cases:

Case 1. $I \subseteq U$. There exists a unique index m in the sequence $\bar{\pi}_{\bar{G},\bar{f}}$ such that

$$I \subseteq \{\bar{\pi}_{\bar{G},\bar{f}}(k) \mid m \leq k \leq |U|\} \quad \text{and} \quad \bar{\pi}_{\bar{G},\bar{f}}(m) \in I.$$

We denote the set $\{\bar{\pi}_{\bar{G},\bar{f}}(k) \mid m \leq k \leq |U|\}$ by \bar{U}_m and let $\bar{\pi}_{\bar{G},\bar{f}}(m) = w$. Algorithm A tells us that $\bar{f}(w) < \text{outdeg}_{\bar{U}_m,\bar{G}}(w) \leq \text{outdeg}_{I,\bar{G}}(w)$. When $w < v$, $g(w) = \bar{f}(w) + \text{outdeg}_{U,G}(w) < \text{outdeg}_{I,\bar{G}}(w) + \text{outdeg}_{U,G}(w) = \text{outdeg}_{I,G}(w)$. When $w \geq v$ and $\text{outdeg}_{U,\bar{G}}(w) = 0$, $g(w) = \bar{f}(w) < \text{outdeg}_{I,\bar{G}}(w) = \text{outdeg}_{I,G}(w)$. When $w \geq v$ and $\text{outdeg}_{U,\bar{G}}(w) = 1$, $g(w) = \bar{f}(w) + \text{outdeg}_{U,G}(w) - 1 < \text{outdeg}_{I,\bar{G}}(w) + \text{outdeg}_{U,G}(w) - 1 = \text{outdeg}_{I,G}(w)$.

Case 2. $I \setminus U \neq \emptyset$. There exists a unique index h in the sequence $\pi_{G,f}$ such that

$$I \subseteq \{\pi_{G,f}(k) \mid h \leq k \leq |V(G)| - 1\} \quad \text{and} \quad \pi_{G,f}(h) \in I.$$

We denote the set $\{\pi_{G,f}(k) \mid h \leq k \leq |V(G)| - 1\}$ by \bar{U}_h and let $\pi_{G,f}(h) = s$. Algorithm A tells us that $g(s) = f(s) < \text{outdeg}_{\bar{U}_h,G}(s) \leq \text{outdeg}_{I,G}(s)$.

Hence, g is a G -parking function. For any $u \in U$ and $u \neq v$, if $u > v$ and $\text{outdeg}_{U,\bar{G}}(u) \geq 1$, then $g(u) = \bar{f}(u) + \text{outdeg}_{U,G}(u) - 1 \geq \text{outdeg}_{U,G}(u) - 1$; if $u < v$, then $g(u) \geq \text{outdeg}_{U,G}(u)$. $g(v) = \bar{f}(v) + \text{outdeg}_{U,G}(v) - 1 \geq \text{outdeg}_{U,G}(v) > f(v)$ since $\bar{f}(v) \geq 1$. Note that $g(u) = f(u)$ for all $u \in V(G) \setminus U$. Thus, $g \in W_{G,f,v}$ and $g \notin S_{G,f,v}$, a contradiction. \square

Table 5
The G -parking functions f and $C_G(f)$.

| G -parking functions f | $C_G(f)$ |
|----------------------------|------------------|
| $f_1 = (-1, 0, 0, 0)$ | $\{0, 1, 2\}$ |
| $f_2 = (-1, 0, 0, 1)$ | $\{0, 1, 2\}$ |
| $f_3 = (-1, 0, 0, 2)$ | $\{0, 1, 2, 3\}$ |
| $f_4 = (-1, 0, 1, 0)$ | $\{0, 1, 2\}$ |
| $f_5 = (-1, 0, 1, 1)$ | $\{0, 1, 2, 3\}$ |
| $f_6 = (-1, 1, 0, 0)$ | $\{0, 1, 3\}$ |
| $f_7 = (-1, 1, 1, 0)$ | $\{0, 1, 2, 3\}$ |
| $f_8 = (-1, 2, 0, 0)$ | $\{0, 1, 2, 3\}$ |

Table 6
The G -parking functions f , $CB_G(f)$, $cb_G(f)$ and $w_G(f)$.

| G -parking function f | $CB_G(f)$ | $(cb_G(f), w_G(f))$ |
|---------------------------|---------------|---------------------|
| $f_1 = (-1, 0, 0, 0)$ | \emptyset | $(0, 2)$ |
| $f_2 = (-1, 0, 0, 1)$ | \emptyset | $(0, 1)$ |
| $f_3 = (-1, 0, 0, 2)$ | $\{3\}$ | $(1, 0)$ |
| $f_4 = (-1, 0, 1, 0)$ | $\{2\}$ | $(1, 1)$ |
| $f_5 = (-1, 0, 1, 1)$ | $\{2, 3\}$ | $(2, 0)$ |
| $f_6 = (-1, 1, 0, 0)$ | $\{3\}$ | $(1, 1)$ |
| $f_7 = (-1, 1, 1, 0)$ | $\{2, 3\}$ | $(2, 0)$ |
| $f_8 = (-1, 2, 0, 0)$ | $\{1, 2, 3\}$ | $(3, 0)$ |

Definition 2.9. Let f be a G -parking function and $v \in V(G)$. Suppose $i = \pi_{G,f}^{-1}(v)$. Let $U_{G,f,v} = \{\pi_{G,f}(j) \mid i \leq j \leq |V(G)| - 1\}$. Write $U_{G,f,v}$ as U for short. v is said to be a *critical vertex* of f if $f(v) = \text{outdeg}_{U,G}(v) - 1$. Define $C_G(f)$ as the set of critical vertices of a G -parking function f .

Clearly, $C_G(f) \neq \emptyset$ since $0 \in C_G(f)$.

Example 2.10. Let us consider the graph G in Fig. 1. We list all the G -parking functions f and $C_G(f)$ in Table 5.

Definition 2.11. Let f be a G -parking function. A vertex v in $V(G) \setminus \{0\}$ is a *critical-bridge vertex* of f if v is a bridge vertex of f and $v \in C_G(f)$. Define $CB_G(f)$ as the set of critical-bridge vertices of a G -parking function f . Let $cb_G(f) = |CB_G(f)|$ and $w_G(f) = |E(G)| - |V(G)| - \sum_{v \in V(G)} f(v)$.

Clearly, $CB_G(f) = B_G(f) \cap C_G(f)$.

Example 2.12. Let G be the graph in Fig. 1. We list all the G -parking functions as well as the set $CB_G(f)$, the corresponding parameters $cb_G(f)$ and $w_G(f)$ in Table 6.

We note that the Tutte polynomial $T_G(x, y)$ of G in Fig. 1 satisfies

$$T_G(x, y) = x^3 + 2x^2 + x + 2xy + y + y^2 = \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)}.$$

Lemma 2.13. Suppose w and v are two vertices of G and the edge $e = \{w, v\}$ is a bridge of G . Furthermore, suppose the vertices w and 0 are in the same component after deleting the edge e . Then $v \in CB_G(f)$ for all G -parking functions f .

Proof. Delete the edge $e = \{w, v\}$ and let I denote the set of vertices of the component containing v . Clearly, $\text{outdeg}_{I,G}(u) = 0$ for all $u \in I$ and $u \neq v$. Let f be a G -parking function. Consider the sequence $\pi_{G,f}$. Suppose $i = \pi_{G,f}^{-1}(v)$. Let $U = \{\pi_{G,f}(k) \mid k \geq i\}$. Assume that \bar{u} is the first vertex in the

sequence $\pi_{G,f}$ such that $\bar{u} \in I$ and $\pi_{G,f}^{-1}(\bar{u}) < \pi_{G,f}^{-1}(v)$. Let $j = \pi_{G,f}^{-1}(\bar{u})$ and $\bar{U} = \{\pi_{G,f}(k) \mid k \geq j\}$. Then $\text{outdeg}_{I,G}(\bar{u}) \geq \text{outdeg}_{\bar{U},G}(\bar{u}) > 0$ by Algorithm A, a contradiction. This implies $I \subseteq U$. Algorithm A tells us that $0 \leq f(v) < \text{outdeg}_{U,G}(v)$. So, $w \neq U$. Hence, in the graph $G(f, v)$, the edge $\{0, v\}$ is a bridge and $v \in B_G(f)$. Since $\text{outdeg}_{U,G}(v) = 1$ and f is a G -parking function, we must have $f(v) = 0$. Therefore, $v \in CB_G(f)$. \square

Lemma 2.14. *Let f be a G -parking function such that $f(v) = 0$ for all $v \in V(G) \setminus \{0\}$. Then the number of bridges in G is $cb_G(f)$.*

Proof. Lemma 2.13 implies that the number of bridges in G is less than or equal to $cb_G(f)$. It is sufficient to prove that the number of bridges in G is bigger than or equal to $cb_G(f)$. We suppose $v \in V(G) \setminus \{0\}$ is a critical-bridge vertex of f . Consider the sequence $\pi_{G,f}$ obtained by Algorithm A. Suppose $i = \pi_{G,f}^{-1}(v)$. Let U be the set of vertices of G such that $U = \{\pi_{G,f}(k) \mid k \geq i\}$. Furthermore, delete the edge $\{0, v\}$ from the graph $G(f, v)$ and let I denote the set of vertices in the component containing v . It is easy to see that $\text{outdeg}_{I,G}(w) = \text{outdeg}_{U,G}(w)$ for all $w \in I$. For all $w \in I$ and $w \neq v$, if $w < v$, since $f(w) = 0$, we have $\text{outdeg}_{U,G}(w) = 0$ by Algorithm A; if $w > v$, since v is a bridge vertex of f , we have $\text{outdeg}_{U,G}(w) = 0$. Thus, $\text{outdeg}_{I,G}(w) = 0$ for all $w \in I$ and $w \neq v$. Since v is a critical vertex of f and $f(v) = 0$, we get $\text{outdeg}_{I,G}(v) = \text{outdeg}_{U,G}(v) = 1$. Thus, let u be the vertex in $V(G) \setminus U$ such that there is an edge connecting u to v . Then the edge $\{u, v\}$ is a bridge in G . Hence, the number of bridges in G is $cb_G(f)$. \square

3. A new expression of the Tutte polynomial

In this section, we will prove the main theorem of this paper. Suppose that e is an edge connecting the vertex v to the vertex w in G , where $v < w$. Define a graph $G \setminus e$ as follows. The graph $G \setminus e$ is obtained from G contracting the vertices w and v ; that is, to get $G \setminus e$ we delete the edge e and identify two vertices v and w as a new vertex v . Define $G - e$ as a graph obtained by deleting the edge e from G .

Let $\Gamma_G(0)$ be the set of vertices which are adjacent to the vertex 0 in G . If there is a loop on the vertex 0 , then $0 \in \Gamma_G(0)$. Let u be a vertex in $\Gamma_G(0)$ such that $u = \min\{v \mid v \in \Gamma_G(0) \setminus \{0\}\}$. Let e be the edge of G connecting the vertex 0 to u . Define $\mathcal{P}_{G,0}$ as the set of G -parking functions f such that $f(u) = 0$. Let $\mathcal{P}_{G,1}$ denote the set of G -parking functions f such that $f(u) \geq 1$.

For each $f \in \mathcal{P}_{G,0}$, let $\phi(f)$ be a function from the set $V(G) \setminus \{u\}$ to $\mathbb{N} \cup \{-1\}$ such that $\phi(f)(w) = f(w)$ for any $w \in V(G) \setminus \{u\}$.

Lemma 3.1.

- (1) *The mapping ϕ is a bijection from $\mathcal{P}_{G,0}$ to $\mathcal{P}_{G \setminus e}$ with $w_{G \setminus e}(\phi(f)) = w_G(f)$.*
- (2) *For any $f \in \mathcal{P}_{G,0}$, we have $CB_G(f) \setminus \{u\} = CB_{G \setminus e}(\phi(f))$.*

Proof. (1) For any $I \subset V(G \setminus e)$ with $0 \notin I$ and $w \in I$, we have $\text{outdeg}_{I,G \setminus e}(w) = \text{outdeg}_{I,G}(w)$. This implies $\phi(f)$ is a $(G \setminus e)$ -parking function. Conversely, for any $g \in \mathcal{P}_{G \setminus e}$, we consider the function $f : V(G) \rightarrow \mathbb{N} \cup \{-1\}$ such that $f(w) = g(w)$ for any $w \in V(G \setminus e)$ and $f(u) = 0$. For any $I \subset V(G)$ with $0 \notin I$, if $u \in I$, then $f(u) < \text{outdeg}_{I,G}(u)$ since $f(u) = 0$ and $\text{outdeg}_{I,G}(u) \geq 1$; otherwise, we have $\text{outdeg}_{I,G \setminus e}(w) = \text{outdeg}_{I,G}(w)$ for all $w \in I$, this implies $0 \leq f(w) < \text{outdeg}_{I,G}(w)$ for some $w \in I$ since g is a $(G \setminus e)$ -parking function. Hence, f is a G -parking function and $\phi^{-1}(g) = f$. Clearly, $w_{G \setminus e}(g) = w_G(f)$.

(2) For any $f \in \mathcal{P}_{G,0}$, we have $\pi_{G,f}(1) = u$ since $u < w$ for all $w \in \Gamma_G(0) \setminus \{u\}$ and $f(u) = 0$. For the sequences $\pi_{G,f}$ and $\pi_{G \setminus e, \phi(f)}$, we have $\pi_{G \setminus e, \phi(f)}(0) = \pi_{G,f}(0) = 0$ and $\pi_{G \setminus e, \phi(f)}(i) = \pi_{G,f}(i + 1)$ for all $i \in \{1, \dots, |V(G)| - 2\}$. For any $f \in \mathcal{P}_{G,1}$, it is easy to see that v is a critical vertex of $\phi(f)$ in $G \setminus e$ if and only if it is critical vertex of f in G . Write $G \setminus e$ as \tilde{G} for short. Then for any $v \notin \{0, u\}$, we have $G(f, v) = \tilde{G}(\phi(f), v)$. Hence, $CB_G(f) \setminus \{u\} = CB_{G \setminus e}(\phi(f))$ for any $f \in \mathcal{P}_{G,0}$. \square

For any $f \in \mathcal{P}_{G,1}$, let $\varphi(f)$ be a function from the set $V(G)$ to $\mathbb{N} \cup \{-1\}$ such that $\varphi(f)(w) = f(w)$ for any $w \in V(G) \setminus \{u\}$ and $\varphi(f)(u) = f(u) - 1$.

Lemma 3.2.

- (1) The mapping φ is a bijection from $\mathcal{P}_{G,1}$ to \mathcal{P}_{G-e} with $w_{G-e}(\varphi(f)) = w_G(f)$.
- (2) For any $f \in \mathcal{P}_{G,1}$, we have $CB_G(f) = CB_{G-e}(\varphi(f))$.

Proof. For any $f \in \mathcal{P}_{G,1}$, since $f(u) \geq 1$, we have the edge $\{0, u\}$ is not a bridge. So, $G - e$ is a connected graph.

(1) For any $I \subset V(G - e)$ with $0 \notin I$ and $w \in I$, we have $\text{outdeg}_{I, G-e}(w) = \text{outdeg}_{I, G}(w)$ if $w \neq u$; $\text{outdeg}_{I, G-e}(w) = \text{outdeg}_{I, G}(w) - 1$ if $w = u$. Note that $g(u) = f(u) - 1$. Hence, $\varphi(f)$ is a $(G - e)$ -parking function. Conversely, for any a $(G - e)$ -parking function g , we consider the function $f : V(G) \rightarrow \mathbb{N} \cup \{-1\}$ such that $f(w) = g(w)$ for any $w \neq u$ and $f(u) = g(u) + 1$. For any $I \subset V(G)$ with $0 \notin I$, we have $\text{outdeg}_{I, G}(w) = \text{outdeg}_{I, G-e}(w)$ if $w \neq u$; $\text{outdeg}_{I, G}(w) = \text{outdeg}_{I, G-e}(w) + 1$ if $w = u$. Note that $f(u) = g(u) + 1$. This implies $0 \leq f(w) < \text{outdeg}_{I, G}(w)$ for some $w \in I$ since g is a $(G - e)$ -parking function. Hence, f is a G -parking function and $\varphi^{-1}(g) = f$. Clearly, $w_{G-e}(g) = w_G(f)$.

(2) Note that $\pi_{G,f} = \pi_{G-e,\varphi(f)}$ for all $f \in \mathcal{P}_{G,1}$. Write $G - e$ as \tilde{G} for short. Then for any $v \notin \{0\}$, we have $G(f, v) = \tilde{G}(\varphi(f), v)$. For any $f \in \mathcal{P}_{G,1}$, it is easy to see that v is a critical vertex of $\varphi(f)$ in $G - e$ if and only if it is a critical vertex of f in G . Hence, $CB_G(f) = CB_{G-e}(\varphi(f))$ for any $f \in \mathcal{P}_{G,1}$. \square

It is well known that the Tutte polynomial $T_G(x, y)$ satisfies the following deletion and contraction recurrence:

$$T_G(x, y) = \begin{cases} yT_{G-e}(x, y) & \text{if } e \text{ is a loop,} \\ xT_{G \setminus e}(x, y) & \text{if } e \text{ is a bridge,} \\ T_{G \setminus e}(x, y) + T_{G-e}(x, y) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

We define a bivariate polynomial $P_G(x, y) = \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)}$. We will prove that $P_G(x, y)$ obeys the deletion and contraction recurrence. We state the main theorem of this paper as follows.

Theorem 3.3. Let $T_G(x, y)$ be the Tutte polynomial of G . Then $T_G(x, y) = \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)}$.

Proof. Consider the polynomial $P_G(x, y) = \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)}$. Let e be an edge of G connecting the vertex u to 0 , where $u \leq w$ for all $w \in I_G(0)$. We consider the following three cases.

Case 1. e is a loop of G .

For any $f \in \mathcal{P}_G$, it is easy to see that f is a $(G - e)$ -parking function as well. Note that

$$\begin{aligned} w_{G-e}(f) &= |E(G - e)| - |V(G - e)| - \sum_{v \in V(G-e)} f(v) \\ &= |E(G)| - 1 - |V(G)| - \sum_{v \in V(G)} f(v) \\ &= w_G(f) - 1. \end{aligned}$$

Hence,

$$\begin{aligned} P_G(x, y) &= \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)} \\ &= \sum_{g \in \mathcal{P}_{G-e}} x^{cb_{G-e}(g)} y^{w_{G-e}(g)+1} \\ &= yP_{G-e}(x, y). \end{aligned}$$

Case 2. e is a bridge of G .

For any $f \in \mathcal{P}_G$, we have $f(u) = 0$ since e is a bridge. So, $\mathcal{P}_G = \mathcal{P}_{G,0}$. Let ϕ be defined as that in Lemma 3.1. Clearly, $u \notin CB_G(f)$ for all $f \in \mathcal{P}_G$. Lemma 3.1 (2) tells us that $cb_G(f) = cb_{G \setminus e}(\phi(f)) + 1$.

$$\begin{aligned} P_G(x, y) &= \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)} \\ &= \sum_{g \in \mathcal{P}_{G \setminus e}} x^{cb_{G \setminus e}(g)+1} y^{w_{G \setminus e}(g)} \\ &= x P_{G \setminus e}(x, y). \end{aligned}$$

Case 3. e is neither loop nor bridge of G .

It is easy to see that $u \notin CB_G(f)$ for any $f \in \mathcal{P}_{G,0}$. By Lemmas 3.1 and 3.2, we have

$$\begin{aligned} P_G(x, y) &= \sum_{f \in \mathcal{P}_G} x^{cb_G(f)} y^{w_G(f)} \\ &= \sum_{f \in \mathcal{P}_{G,0}} x^{cb_G(f)} y^{w_G(f)} + \sum_{f \in \mathcal{P}_{G,1}} x^{cb_G(f)} y^{w_G(f)} \\ &= \sum_{g \in \mathcal{P}_{G \setminus e}} x^{cb_{G \setminus e}(g)} y^{w_{G \setminus e}(g)} + \sum_{g \in \mathcal{P}_{G-e}} x^{cb_{G-e}(g)} y^{w_{G-e}(g)} \\ &= P_{G \setminus e}(x, y) + P_{G-e}(x, y). \end{aligned}$$

Finally, we consider initial conditions. Let G be a graph with vertex set $\{0\}$ and $E(G) = \emptyset$. There is a unique G -parking function $f(0) = -1$. Clearly, $w_G(f) = 0$ and $CB_G(f) = \emptyset$. So, $T_G(x, y) = 1$. Next, let G be a graph with vertex set $\{0, 1\}$ and $E(G) = \{\{0, 1\}\}$. There is a unique G -parking function $f(0) = -1$ and $f(1) = 0$. It is easy to see $CB_G(f) = \{1\}$ and $w_G(f) = 0$. Hence, $P_G(x, y) = x$. This complete the proof. \square

In [6], Kostić and Yan gave an expression for the polynomial $T_{K_{n+1}}(x, y)$ in terms of statistics of classical parking functions. It counts the classical parking functions by the number of critical left-to-right maximum subscript. Given a classical parking function $\alpha = (a_1, a_2, \dots, a_n)$ and a subscript $v \in \{1, 2, \dots, n\}$, we suppose $a_v = j$. The subscript v is critical if $|\{w \mid a_w < j\}| = j$ and $|\{w \mid a_w > j\}| = n - 1 - j$. We say that the subscript v is a left-to-right maximum if $a_w < j$ for all $w < v$. For example, let $\alpha = (a_1, a_2, a_3, a_4, a_5) = (3, 0, 0, 2, 4)$. Then in α , the subscripts 1, 4 and 5 are critical. The subscripts 1 and 5 are left-to-right maximum. Let $cm(\alpha)$ be the number of critical left-to-right maximum subscripts in a classical parking function α . As an application of our main theorem, we prove the following corollary.

Corollary 3.4. (See Kostić and Yan [6].)

$$T_{K_{n+1}}(x, y) = \sum_{\alpha \in \mathcal{P}_n} x^{cm(\alpha)} y^{\binom{n}{2} - \sum_{i=1}^n a_i},$$

where \mathcal{P}_n is the set of classical parking functions of length n .

Proof. For any $\alpha \in \mathcal{P}_n$, let $\bar{\alpha}$ be a function on the vertex set $V(K_{n+1}) = \{0, 1, 2, \dots, n\}$ such that $\bar{\alpha}(0) = -1$ and $\bar{\alpha}(v) = a_{n+1-v}$ for all $v \neq 0$. Then $\bar{\alpha}$ is a K_{n+1} -parking function.

First, we prove that v is a critical left-to-right maximum subscript in α if and only if $n + 1 - v$ is a critical-bridge vertex of $\bar{\alpha}$ in K_{n+1} .

Let v be a critical left-to-right maximum subscript in α . Suppose $a_v = j$. By Algorithm A, it is easy to see $n + 1 - v$ is a critical vertex of $\bar{\alpha}$ since there are exactly j terms less than j and exactly $n - 1 - j$ terms larger than j . Let $\pi_{K_{n+1}, \bar{\alpha}}$ be the sequence obtained by Algorithm A. Then $\pi_{K_{n+1}, \bar{\alpha}}(j + 1) = n + 1 - v$. Let w be a vertex of K_{n+1} such that $w = \pi_{K_{n+1}, \bar{\alpha}}(m)$ for some $m > j + 1$. Then $\bar{\alpha}(w) > \bar{\alpha}(n + 1 - v)$. Hence, $w < n + 1 - v$ since v is a left-to-right maximum subscript in α . Write K_{n+1} as \tilde{G} for short. Then the edge $\{0, n + 1 - v\}$ is a bridge in the graph $\tilde{G}(\bar{\alpha}, n + 1 - v)$. Hence, the vertex $n + 1 - v$ is a critical-bridge vertex $\bar{\alpha}$.

Conversely, suppose the vertex $n + 1 - v$ is a critical-bridge vertex of $\bar{\alpha}$ and $\bar{\alpha}(n + 1 - v) = j$. We consider the sequence $\pi_{K_{n+1}, \bar{\alpha}}$. Then $\pi_{K_{n+1}, \bar{\alpha}}(j + 1) = n + 1 - v$ since the vertex $n + 1 - v$ is a critical vertex of $\bar{\alpha}$ and $\bar{\alpha}(n + 1 - v) = j$. For any $w \in \{\pi_{K_{n+1}, \bar{\alpha}}(m) \mid 0 \leq m \leq j\}$, Algorithm A tells us $a_{n+1-w} = \bar{\alpha}(w) < j$. For any $w \in \{\pi_{K_{n+1}, \bar{\alpha}}(m) \mid j + 2 \leq m\}$, we must have $w < n + 1 - v$ since the vertex $n + 1 - v$ is a bridge vertex of $\bar{\alpha}$. Furthermore, by Algorithm A, we have $a_{n+1-w} = \bar{\alpha}(w) > j + 1$. Hence, v is a critical left-to-right maximum subscript in α .

By Theorem 3.3,

$$\begin{aligned} T_{K_{n+1}}(x, y) &= \sum_{\bar{\alpha} \in \mathcal{P}_{K_{n+1}}} x^{cb(\bar{\alpha})} y^{\binom{n+1}{2} - (n+1) - \sum_{i=0}^n \bar{\alpha}(i)} \\ &= \sum_{\alpha \in \mathcal{P}_n} x^{cm(\alpha)} y^{\binom{n}{2} - \sum_{i=1}^n a_i}. \quad \square \end{aligned}$$

4. Conclusions

Consider the original definition of the Tutte polynomial based on the notions of internally and externally active edges in [10]. Suppose we are given a total ordering of edges in G . A sub-tree of G is a connected subgraph of G without cycles. A spanning tree of G is a sub-tree of G containing all the vertices of G . Fix a spanning tree T of G . Given an edge $e \in E(G) \setminus E(T)$, we call the edge e an externally active edge of T if it is the smallest edge in the unique cycle contained in $T \cup e$. Define $EA_G(T)$ as the set of externally active edges for T and let $ea_G(T) = |EA_G(T)|$. Given an edge $e \in E(T)$, define $U_T(e)$ as the set of edges \tilde{e} in $E(G)$ such that $(T - e) \cup \tilde{e}$ is a spanning tree. An edge e in $E(T)$ is internally active if it is the smallest edge in $U_T(e)$. Define $IA_G(T)$ as the set of internally active edges in T and let $ia_G(T) = |IA_G(T)|$. In [10], the Tutte polynomial was defined as follows.

$$T_G(x, y) = \sum_{T \in \mathcal{T}_G} x^{ia_G(T)} y^{ea_G(T)},$$

where \mathcal{T}_G denote the set of spanning trees of G . By Theorem 3.3, we immediately obtain the following corollary.

Corollary 4.1. *The number of G -parking functions f such that $cb_G(f) = i$ and $w_G(f) = j$ is equal to the number of spanning trees T in G such that $ia_G(T) = i$ and $ea_G(T) = j$.*

Chebikin and Pylyavskyy [2] established a family of bijections from the set of G -parking functions to the set of spanning trees of G . Naturally, the following problem arises:

- Is there a bijection Θ from the set of G -parking functions f to the set of spanning trees T of G and a total ordering of edges in G such that $cb_G(f) = ia_G(\Theta(f))$ and $w_G(f) = ea_G(\Theta(f))$?

Let H be a subgraph of G . Let $c(H)$ denote the number of components of H . Define two parameters associated with H as $\sigma(H) = c(H) - 1$ and $\sigma^*(H) = |E(H)| - |V(G)| + c(H)$. Then we have the following theorem

Theorem 4.2. (See Biggs [1].) $T_G(1+x, 1+y) = \sum_{H \subseteq G} x^{\sigma(H)} y^{\sigma^*(H)}$, where the sum is over all spanning subgraphs H of G .

Given a G -parking function f , another interesting problem is:

- How is $cb_G(f)$ related to the number of components if one goes from spanning trees to spanning subgraphs?

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