Combinatorial interpretations for $T_G(1,-1)$

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Abstract

Let G be a connected and simple graph with vertex set $\{1,2,\ldots,n+1\}$ and $T_G(x,y)$ the Tutte polynomial of G. In this paper, we give combinatorial interpretations for $T_G(1,-1)$. In particular, we give the definitions of even spanning tree and left spanning tree. We prove $T_G(1,-1)$ is the number of even-left spanning trees of G. We associate a permutation with a spanning forest of G and give the definition of odd G-permutations. We show $T_G(1,-1)$ is the number of odd G-permutations. We give a bijection from the set of odd K_{n+1}-permutations to the set of alternating permutations on the set $\{1,2,\ldots,n\}$.

Keywords: alternating permutation; spanning tree; Tutte polynomial

1 Introduction

Let G be a connected graph with vertex $V(G)$ and edge set $E(G)$. Suppose we are given a total ordering of edges in G. Fix a spanning tree T of G. Given an edge $e\in E(G)\setminus E(T)$, we call the edge e an externally active edge of T if it is the smallest edge in the unique cycle contained in $T\cup e$. Define $EA_G(T)$ as the set of externally active edges for T and let $ea_G(T)=|EA_G(T)|$. Given an edge $e\in E(T)$, define $U_T(e)$ as the set of edges $\tilde{e}$ in $E(G)$ such that $(T-e)\cup\tilde{e}$ is a spanning tree. An edge e in $E(T)$ is internally active if it is the smallest edge in $U_T(e)$. Define $IA_G(T)$ as the set of internally active edges in T and let $ia_G(T)=|IA_G(T)|$. In [9], the Tutte polynomial was defined as follows

$$T_G(x,y)=\sum_{T\in\mathcal{T}_G}x^{ia_G(T)}y^{ea_G(T)},$$

where $\mathcal{T}_G$ denotes the set of spanning trees of G.

In general, the Tutte polynomial encodes information about subgraphs of G. Following [1], we list some special values of the Tutte polynomial in Table 1.

$T_G(1,1)$	is the number of spanning trees of G.
$T_G(2,1)$	is the number of spanning forests of G.
$T_G(1,2)$	is the number of connected spanning subgraphs of G.
$T_G(2,2)$	is the number of spanning subgraphs of G.

Table 1. Some special values of the Tutte polynomial.

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Let $\mathfrak{S}_n$ denote the set of permutations on the set $\{1, 2, \ldots, n\}$. A permutation $\sigma = (\sigma(1) \ldots \sigma(n)) \in \mathfrak{S}_n$ is said to be *alternating* if $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \sigma(5) > \sigma(6) < \cdots \sigma(n)$. Let $\mathscr{A}_n$ be the set of alternating permutations in $\mathfrak{S}_n$. We put $a_n = |\mathscr{A}_n|$. The sequence $\{a_n\}_{n\geq 0}$ satisfies the following recurrence relation

$$a_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i} a_{2i} a_{n-2i-1}$$

with initial condition $a_0 = 1$, see [6] and [8].

Let K_{n+1} be the complete graph on $n+1$ vertices. In [6] and [8], it is proved that $T_{K_{n+1}}(1, -1)$ is equal to the number of alternating permutations on the set $\{1, 2, \ldots, n\}$ follows from the fact that satisfies the same recurrence relation as the sequence $\{a_n\}_{n\geq 0}$. We are interested in combinatorial interpretations of $T_{K_{n+1}}(1, -1)$. A tree on vertex set $\{1, 2, \ldots, n\}$ and rooted at 1 is said to be *increasing* if its vertices increase along the paths away from the root. A increasing tree is even if every non-root vertex has an even number of children. Kuznetsov, Pak and Postnikov [8] constructed an involution that is fixed on even increasing trees and changes the parity of the number of inversions of the remaining trees and proved $T_{K_{n+1}}(1, -1)$ is equal to the number of even increasing trees in K_{n+1}. Moreover, A 0-1-2 increasing tree is an increasing tree where all the vertices including the root have at most 2 children. An important fact is $T_{K_{n+1}}(1, -1)$ is equal to the number of 0-1-2 increasing trees in K_n. This was first formulated by Foata [4] and was proved by Foata and Strehl [5]. Donaghey [3] gave a simple bijective proof for this fact.

Let G be a connected and simple graph with vertex set $\{1, 2, \ldots, n+1\}$. In this paper, we give combinatorial interpretations for $T_G(1, -1)$. In particular, for any a spanning tree T of G, let π_T be a sequence on vertices of G formed by the vertex-adding order, which is the third example of a proper tree order given in Chebikin and Pylyavskyy [2], and described in Algorithm A below. With respect to the sequence π_T, we define N_T as the set of T-redundant edges of G, in the sense of Kostic and Yan [7,§4.1] and as defined below after Algorithm A. In [7, p.91] it is shown that $T_G(1, y) = \sum_{T \in \mathcal{T}_G} y^{|N_T|}$. We then proceed to define even spanning trees and left spanning trees. Let $\mathscr{T}_G$ be all even-left spanning trees of G and $\bar{\mathscr{T}}_G = \mathcal{T}_G \setminus \mathscr{T}_G$. By establishing an involution on the set $\bar{\mathscr{T}}_G$, we show $T_G(1, -1)$ is the number of even-left spanning trees of G.

We associate a permutation in $\mathfrak{S}_n$ with a spanning forest of the graph G and give the definition of odd G-permutations. Let $\mathfrak{S}_{G,odd}$ be the set of odd G-permutations. We give a bijection from $\mathscr{T}_G$ to $\mathfrak{S}_{G,odd}$. Hence $T_G(1, -1)$ is also the number of odd G-permutations in $\mathfrak{S}_n$. We consider the case in which the graph G is the complete graph K_{n+1}. We give a bijection from the set $\mathfrak{S}_{K_{n+1},odd}$ to the set $\mathscr{A}_n$, where $\mathscr{A}_n$ denotes the set of alternating permutations of $\mathfrak{S}_n$.

We organize this paper as follows. In Section 2, we prove $T_G(1, -1)$ is the number of even-left spanning trees of G. In Section 3, we give the definition of odd G-permutations and prove $T_G(1, -1)$ is the number of odd G-permutations in $\mathfrak{S}_n$. Let $G = K_{n+1}$. We give a bijection from the set $\mathfrak{S}_{K_{n+1},odd}$ to the set $\mathscr{A}_n$.

2 Even-left spanning trees of G

Let $[n] := \{1, 2, \ldots, n\}$ and G be a simple and connected graph with vertex set $[n+1]$ and edge set $E(G)$. In this section, we give the definition of even-left spanning trees of G and prove $T_G(1, -1)$ is the number of even-left spanning trees of G.

A subtree of G is a connected subgraph of G without cycles. A spanning tree of G is a subtree of G containing all the vertices of G. Let $\mathcal{T}_G$ be the set of the spanning trees of G. Given disjoint subsets U and

W of the vertex set of G, we write $E_G(U,W)$ for the set of U-W edges, that is, for the set of edges joining a vertex in U to a vertex in W.

Given a spanning tree T of G, we associate T with a sequence

$$\pi_T = (\pi_T(0), \pi_T(1), \ldots, \pi_T(n))$$

on vertices of G by the following algorithm:

Algorithm A.
Step 1. Let $\pi_T(0) = n+1$.
Step 2. Assume that $\pi_T(0), \pi_T(1), \ldots, \pi_T(i)$ are determined. Let $U_i = \{\pi_T(j) \mid j = 0, 1, \ldots, i\}$. Let $v = \min\{w \in V(G) \setminus U_i \mid E_T(U_i, \{w\}) \neq \emptyset\}$ and set $\pi_T(i+1) = v$.

The sequence π_T obtained by Algorithm A is the same as the vertex-adding order on vertices of G rooted at $n+1$ defined by Chebikin and Pylyavskyy [2]. Note that π_T can be viewed as a bijection from the set $\{0, 1, \ldots, n\}$ to $[n+1] = V(G)$. Furthermore, for any vertex $v \in V(G) \setminus \{n+1\}$, there exists a unique vertex u such that $\{u, v\} \in E(T)$ and $\pi_T^{-1}(u) < \pi_T^{-1}(v)$. We say u is the predecessor of v and write $u = p_T(v)$. Fixing a vertex v, suppose u is its predecessor. Define $N_{T,v}$ as the set of vertices w such that $\{w, v\} \in E(G) \setminus E(T)$ and $\pi_T^{-1}(u) < \pi_T^{-1}(w) < \pi_T^{-1}(v)$. For any $w \in N_{T,v}$, the edge $\{w, v\}$ is called a T-redundant edge in [7]. Let N_T be the set of T-redundant edges in G, i.e., $N_T = \bigcup_{v \in [n]} \{\{w, v\} \mid w \in N_{T,v}\}$.

Lemma 2.1 (Kostić and Yan [7]) $T_G(1, y) = \sum_{T \in \mathcal{T}_G} y^{|N_T|}$.

Fix a spanning tree T. Let π_T be the sequence of vertices of G obtained by Algorithm A. We say a vertex v in $[n]$ is *T-even* if the number of T-redundant edges incident with v in G is even, i.e., $|N_{T,v}| \equiv 0 (\mod 2)$. If every vertex in $[n]$ is T-even, then T is called an *even spanning tree*.

Given a vertex $v \in [n]$, for any $w \in [n+1]$, we say (w, v) is a inversion of the sequence π_T if $w > v$ and $\pi_T^{-1}(w) < \pi_T^{-1}(v)$. Since $\pi_T(0) = n+1$, there exists a unique vertex u such that (u, v) is a inversion of π_T and $\pi_T(j) < v$ for any $\pi_T^{-1}(u) < j < \pi_T^{-1}(v)$. We say the vertex u is a *$(T; v)$-rightmost inversion vertex.* Let $w = p_T(v)$. It is easy to see $\pi_T^{-1}(u) \leq \pi_T^{-1}(w)$ by Algorithm A. Define $M_{T,v}$ as the set of vertices s such that $\{s, v\} \in E(G) \setminus E(T)$ and $\pi_T^{-1}(u) \leq \pi_T^{-1}(s) < \pi_T^{-1}(w)$. If $M_{T,v} = \emptyset$, then we say the vertex v is T-left. If every vertex in $[n]$ is T-left, then we say T is a *left spanning tree.*

Example 2.2 Let us consider the complete graph K_{n+1}. Let T and π_T be a spanning tree and the sequence of vertices of K_{n+1} obtained by Algorithm A respectively. For any vertex $v \in [n]$, let w and u be the predecessor of v and the $(T; v)$-rightmost inversion vertex, respectively. Then v is T-even if and only if the number of vertices between w and v in the sequence π_T, not including w and v, is even; v is T-left if and only if $u = w$.

Example 2.3 We consider the graph G in Fig. 1.

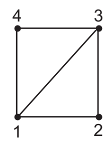

Fig. 1. A graph G.

We give all the spanning trees of the graph G in Fig. 2.

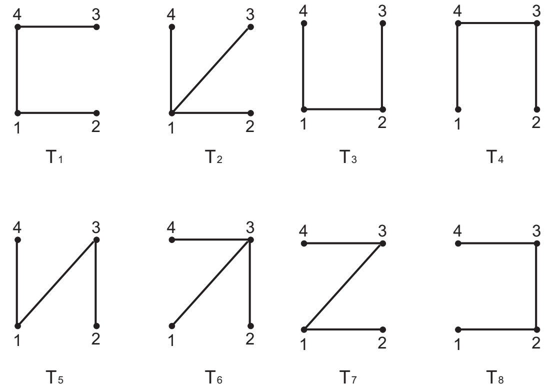

Fig. 2. All the spanning trees of G.

See Table 2. For the spanning tree T_1 in Fig. 2., by Algorithm A, we have $\pi_{T_1} = (4, 1, 2, 3)$. It is easy to see $N_{T_1,1} = \emptyset$, $N_{T_1,2} = \emptyset$ and $N_{T_1,3} = \{1, 2\}$. Hence, T_1 is an even spanning tree in G. Note that $M_{T_1,v} = \emptyset$ for all $v = 1, 2, 3$. So T_1 is a left spanning tree in G.

	π_{T_i}	$N_{T_i,1}$	$N_{T_i,2}$	$N_{T_i,3}$		$M_{T_i,1}$	$M_{T_i,2}$	$M_{T_i,3}$	
T_1	$(4,1,2,3)$	$\emptyset$	$\emptyset$	$\{1,2\}$	*even*	$\emptyset$	$\emptyset$	$\emptyset$	*left*
T_2	$(4,1,2,3)$	$\emptyset$	$\emptyset$	$\{2\}$		$\emptyset$	$\emptyset$	$\{4\}$	
T_3	$(4,1,2,3)$	$\emptyset$	$\emptyset$	$\emptyset$	*even*	$\emptyset$	$\emptyset$	$\{1,4\}$	
T_4	$(4,1,3,2)$	$\emptyset$	$\emptyset$	$\{1\}$		$\emptyset$	$\emptyset$	$\emptyset$	*left*
T_5	$(4,1,3,2)$	$\emptyset$	$\emptyset$	$\emptyset$	*even*	$\emptyset$	$\emptyset$	$\{4\}$	
T_6	$(4,3,1,2)$	$\emptyset$	$\{1\}$	$\emptyset$		$\emptyset$	$\emptyset$	$\emptyset$	*left*
T_7	$(4,3,1,2)$	$\emptyset$	$\emptyset$	$\emptyset$	*even*	$\emptyset$	$\{3\}$	$\emptyset$	
T_8	$(4,3,2,1)$	$\emptyset$	$\emptyset$	$\emptyset$	*even*	$\emptyset$	$\emptyset$	$\emptyset$	*left*

Table 2. Even and left spanning trees of G.

We say T is *even-left* if T is a spanning tree of G that is both even and left. Define $\mathscr{T}_G$ as the set of even-left spanning trees of G. Let $\bar{\mathscr{T}}_G = \mathcal{T}_G \setminus \mathscr{T}_G$.

Lemma 2.4 *There is an involution $\phi = \phi_G$ from the set $\bar{\mathscr{T}}_G$ to itself such that $||N_T| - |N_{\phi(T)}|| = 1$ for any $T \in \bar{\mathscr{T}}_G$.*

Proof. For any $T \in \bar{\mathscr{T}}_G$, let π_T be the sequence of vertices of G obtained by Algorithm A. Since T is not an even-left spanning tree, there are some vertices w such that either $|N_{T,w}| \equiv 1 (\mod 2)$ or $M_{T,w} \neq \emptyset$. Let v be the first such vertex in the sequence π_T. We discuss the following two cases.

Case 1. $|N_{T,v}| \equiv 1 (\mod 2)$.

Clearly, $N_{T,v} \neq \emptyset$. Let u be the vertex in $N_{T,v}$ such that $\pi_T^{-1}(u) \leq \pi_T^{-1}(w)$ for any $w \in N_{T,v}$. We construct a new spanning tree of G, denoted $\phi(T)$, by adding the edge $\{u, v\}$ and deleting the edge $\{p_T(v), v\}$. It is easy to see $\pi_T = \pi_{\phi(T)}$, $|N_{\phi(T),v}| \equiv 0 (\mod 2)$, $M_{\phi(T),v} \neq \emptyset$ and $|N_T| - |N_{\phi(T)}| = 1$.

Case 2. $|N_{T,v}| \equiv 0 (\mod 2)$ and $M_{T,v} \neq \emptyset$.

Let u be the vertex in $M_{T,v}$ such that $\pi_T^{-1}(u) \geq \pi_T^{-1}(w)$ for any $w \in M_{T,v}$ since $M_{T,v} \neq \emptyset$. Adding the edge $\{u, v\}$ and deleting the edge $\{p_T(v), v\}$, we denote the obtained spanning tree by $\phi(T)$. Then $\pi_T = \pi_{\phi(T)}$, $|N_{\phi(T),v}| \equiv 1 (\mod 2)$ and $|N_T| - |N_{\phi(T)}| = -1$. ∎

Example 2.5 Let G be the graph in Fig. 1. See Table 2. We have $\mathscr{T}_G = \{T_1, T_8\}$ and $\bar{\mathscr{T}}_G = \{T_i \mid 2 \leq i \leq 7\}$. We easily obtain $\phi(T_2) = T_3$, $\phi(T_4) = T_5$ and $\phi(T_6) = T_7$.

Theorem 2.6 *The number of even-left spanning trees of G is $T_G(1,-1)$, i.e., $T_G(1,-1) = |\mathscr{T}_G|$.*

Proof. Note that $|N_T| \equiv 0(\mod 2)$ for any $T \in \mathscr{T}_G$. By Lemma 2.4, we have

$$\begin{aligned} T_G(1,-1) &= \sum_{T \in \mathcal{T}_G} (-1)^{|N_T|} \\ &= \sum_{T \in \mathscr{T}_G} (-1)^{|N_T|} + \sum_{T \in \bar{\mathscr{T}}_G} (-1)^{|N_T|} \\ &= \sum_{T \in \mathscr{T}_G} 1 \\ &= |\mathscr{T}_G|. \end{aligned}$$

∎

3 Odd G-permutations

In this Section, we give the definition of odd G-permutations and prove $T_G(1,-1)$ is the number of odd G-permutations in $\mathfrak{S}_n$. We conclude with a bijection from the set of odd G-permutations to the set of alternating permutations on the set $[n]$.

Fix a simple graph G with vertex set $[n+1]$. Given a permutation $\sigma = (\sigma(1) \ldots \sigma(n)) \in \mathfrak{S}_n$, we associate σ with a spanning forest of G as follows.

Algorithm B.
Step 1. Set $\sigma(0) = n+1$. Let $E_0 = \emptyset$.
Step 2. At time $i \geq 1$, let $j = \max\{k \mid 0 \leq k < i, \sigma(k) > \sigma(i)\}$. Define $\tilde{N}_{\sigma,i}$ as the set of indexes k such that $j \leq k < i$ and $\{\sigma(k), \sigma(i)\} \in E(G)$. If $\tilde{N}_{\sigma,i} = \emptyset$, then let $E_i = E_{i-1}$. Otherwise let $E_i = E_{i-1} \cup \{\sigma(m), \sigma(i)\}$, where $m = \min \tilde{N}_{\sigma,i}$.

Iterating Step 2 until $i = n$, we obtain a spanning forest of G with edge set E_n. Denote this spanning forest by F_σ. If F_σ is a spanning tree, then we say σ is a *G-permutation*. Clearly, a permutation σ is a G-permutation if and only if $\tilde{N}_{\sigma,i} \neq \emptyset$ for all $i \in [n]$. Let $\mathfrak{S}_G$ be the set of all G-permutations. Furthermore, we say a G-permutation σ is *odd* if $|\tilde{N}_{\sigma,i}| \equiv 1(\mod 2)$ for each $i \in [n]$. Let $\mathfrak{S}_{G,odd}$ be the set of odd G-permutations.

Example 3.1 Let G be the graph in Fig. 1. We consider all the permutations in $\mathfrak{S}_3$ and give the following table.

	σ	$\tilde{N}_{\sigma,1}$	$\tilde{N}_{\sigma,2}$	$\tilde{N}_{\sigma,3}$
σ_1	$(1,2,3)$	$\{0\}$	$\{1\}$	$\{0,1,2\}$
σ_2	$(1,3,2)$	$\{0\}$	$\{0,1\}$	$\{2\}$
σ_3	$(2,1,3)$	$\emptyset$	$\{1\}$	$\{0,1,2\}$
σ_4	$(2,3,1)$	$\emptyset$	$\{0,1\}$	$\{2\}$
σ_5	$(3,1,2)$	$\{0\}$	$\{1\}$	$\{1,2\}$
σ_6	$(3,2,1)$	$\{0\}$	$\{1\}$	$\{2\}$

Table 3. All the permutations in $\mathfrak{S}_3$.

It is easy to see $\mathfrak{S}_G = \{\sigma_1, \sigma_2, \sigma_5, \sigma_6\}$ and $\mathfrak{S}_{G,odd} = \{\sigma_1, \sigma_6\}$.

Lemma 3.2 *There is a bijection from $\mathscr{T}_G$ to $\mathfrak{S}_{G,odd}$.*

Proof. For any $T \in \mathscr{T}_G$, let π_T be the sequence of vertices of G obtained by Algorithm A. Consider a permutation σ such that $\sigma(i) = \pi_T(i)$ for any $i \in [n]$. It is easy to see $\sigma \in \mathfrak{S}_{G,odd}$.

Conversely, for any $\sigma = (\sigma(1) \dots \sigma(n)) \in \mathfrak{S}_{G,odd}$, let T be the spanning tree of G obtained by Algorithm B. Let π_T be the sequence of vertices of G obtained by Algorithm A. To prove $T \in \mathscr{T}_G$, it is sufficient to show $\pi_T(i) = \sigma(i)$ for all $i \in \{0, 1, \dots, n\}$. Note that $\pi_T(0) = \sigma(0) = n + 1$. By Algorithm A, for the spanning tree T, we assume $\pi_T(0), \dots, \pi_T(i)$ are determined and $\pi_T(j) = \sigma(j)$ for all $0 \le j \le i$. Let $U_i = \{\pi_T(j) \mid j = 0, \dots, i\}$ and $W = \{w \mid w \notin U_i, E_T(U_i, \{w\}) \neq \emptyset\}$. By Algorithm B, we have $\sigma(i+1) \in W$ since $|\tilde{N}_{\sigma,i+1}| \equiv 1(\mod 2)$. We claim $\sigma(i+1) = \min W$. Otherwise, there exists a vertex $u = \sigma(k)$ for some $k \ge i+2$ such that $u < \sigma(i+1)$ and $p_T(u) \in V_i$. $u = \sigma(k) < \sigma(i+1)$ implies $\tilde{N}_{\sigma,k} \subseteq \{\sigma(s) \mid i+1 \le s \le k-1\}$. Hence, by Algorithm B, $p_T(u) \notin V_i$, a contradiction. So $\pi_T(i) = \sigma(i)$ for all $0 \le i \le n$. By Algorithm B, we have $|N_{T,\sigma(i)}| = |\tilde{N}_{\sigma,i}| - 1$ and $M_{T,\sigma(i)} = \emptyset$ for all $i \in [n]$. Hence, $T \in \mathscr{T}_G$. ∎

Combining Theorem 2.6 and Lemma 3.2, we obtain the following corollary.

Corollary 3.3 *The number of odd G-permutations is $T_G(1, -1)$, i.e., $T_G(1, -1) = |\mathfrak{S}_{G,odd}|$.*

We now consider the case when G is the complete graph K_{n+1}. Let op_n be the number of odd K_{n+1}-permutations, i.e., $op_n = |\mathfrak{S}_{K_{n+1},odd}|$. By Corollary 3.3, we immediately obtain the following corollary.

Corollary 3.4 $T_{K_{n+1}}(1, -1) = op_n$.

Recall $\mathscr{A}_n$ is the set of alternating permutations in $\mathfrak{S}_n$ and $a_n = |\mathscr{A}_n|$. By Corollary 3.4, we have $op_n = a_n$ since $T_{K_{n+1}}(1, -1) = a_n$. It is easy to see the sequence $\{op_n\}_{n\ge0}$ satisfies the same recurrence relation and the initial condition as the sequence $\{a_n\}_{n\ge0}$. In particular, for any $\sigma = (\sigma(1) \dots \sigma(n)) \in \mathfrak{S}_n$, let $\tau = (\sigma(i_1)\sigma(i_2) \dots \sigma(i_k))$ be a subsequence of σ and "*red*" the increasing bijection of $\{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\}$ onto $[k]$. Define the reduction of the subsequence τ, denoted by $red(\tau)$, as $(red(\sigma(i_1))red(\sigma(i_2)) \dots red(\sigma(i_k)))$. For any $\sigma = (\sigma(1) \dots \sigma(n)) \in \mathfrak{S}_n$, suppose $\sigma(2i + 1) = n$ for some $0 \le i \le \lfloor \frac{n-1}{2} \rfloor$. We obtain two subsequences $\tau_1 = (\sigma(1) \dots \sigma(2i))$ and $\tau_2 = (\sigma(2i + 2) \dots \sigma(n))$ of σ. Clearly, $red(\tau_1) \in \mathfrak{S}_{K_{2i+1},odd}$ and $red(\tau_2) \in \mathfrak{S}_{K_{n-2i},odd}$. There are $\binom{n-1}{2i}$ ways to form the set $\{\sigma(1), \dots, \sigma(2i)\}$. Hence, we have $op_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i} op_{2i} op_{n-2i-1}$.

For any $\sigma \in \mathfrak{S}_n$, suppose $\tau = \sigma(i_1)\sigma(i_2) \dots \sigma(i_k)$ be a subsequence of σ. Let $\bar{\tau}$ denote the sequence $(n + 1 - \sigma(i_1))(n + 1 - \sigma(i_2)) \dots (n + 1 - \sigma(i_k))$.

Lemma 3.5 *There is a a bijection Φ_n from $\mathfrak{S}_{K_{n+1},odd}$ to $\mathscr{A}_n$.*

Proof. By induction. Clearly, $\Phi_1(1) = 1$. Now, we assume that Φ_k is the bijection from $\mathfrak{S}_{K_{k+1},odd}$ to $\mathscr{A}_k$ for all $k \le n$. For any $\sigma = (\sigma(1) \dots \sigma(n+1)) \in \mathfrak{S}_{n+1}$, suppose $\sigma(2k+1) = n+1$, $\tau_1 = (\sigma(1) \dots \sigma(2k))$ and $\tau_2 = (\sigma(2k + 2) \dots \sigma(n + 1))$ for some $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Note that $red(\tau_1) \in \mathfrak{S}_{K_{2k+1},odd}$ and $red(\tau_2) \in \mathfrak{S}_{K_{n-2k+1},odd}$. It is easy to see that $\Phi_{2k}(red(\tau_1)) \in \mathscr{A}_{2k}$ and $\Phi_{n-2k}(red(\tau_2)) \in \mathscr{A}_{n-2k}$. Let $\phi_m = red^{-1} \circ \Phi_m \circ red$ and $\psi_m = red^{-1} \circ (\overline{\Phi_m \circ red})$ for short. Let $\Phi_n(\pi) = \phi_{2k}(\tau_1)(n + 1)\psi_{n-2k}(\tau_2)$. Then $\Phi_n(\pi) \in \mathscr{A}_n$. ∎

Example 3.6 Given $\sigma = (42531) \in \mathfrak{S}_{K_6,odd}$, we have $\tau_1 = 42$ and $\tau_2 = 31$. So, $\phi_2(42) = red^{-1} \circ \Phi_2 \circ red(42) = red^{-1} \circ \Phi_2(21) = red^{-1}(21) = 42$ and $\psi_2(31) = red^{-1} \circ (\overline{\Phi_2 \circ red(31)}) = red^{-1} \circ (\overline{\Phi_2(21)}) = red^{-1} \circ \overline{21} = red^{-1}(12) = 13$. Then $\Phi_5(\sigma) = 42513$.

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