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Refinements of (n, m)-Dyck paths

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ABSTRACT

The classical Chung–Feller theorem tells us that the number of (n, m)-Dyck paths is the nth Catalan number and independent of m. In this paper, we consider refinements of (n, m)-Dyck paths by using four parameters, namely the peak, valley, double descent and double ascent. Let $p_{n,m,k}$ be the total number of (n, m)-Dyck paths with k peaks. First, we derive the reciprocity theorem for the polynomial $P_{n,m}(x) = \sum_{k=1}^{n} p_{n,m,k}x^k$. In particular, we prove that the number of (n, m)-Dyck paths with k peaks is equal to the number of $(n, n-m)$-Dyck paths with $n-k$ peaks. Then we find the Chung–Feller properties for the sum of $p_{n,m,k}$ and $p_{n,m,n-k}$, i.e., the number of (n, m)-Dyck paths which have k or $n-k$ peaks is $\frac{2(n+2)}{n(n-1)}\binom{n}{k-1}\binom{n}{k+1}$ for $1 \le m \le n-1$ and independent of m. Finally, we provide a Chung–Feller type theorem for Dyck paths of semilength n with k double ascents: the total number of (n, m)-Dyck paths with k double ascents is equal to the total number of n-Dyck paths that have k double ascents and never pass below the x-axis, which is counted by the Narayana number. Let $v_{n,m,k}$ (resp. $d_{n,m,k}$) be the total number of (n, m)-Dyck paths with k valleys (resp. double descents). Some similar results are derived.

1. Introduction

Let $\mathbb{Z}$ denote the set of integers. We consider n-Dyck paths in the plane $\mathbb{Z} \times \mathbb{Z}$ using *up* $(1, 1)$ and *down* $(1, -1)$ steps that go from the origin to the point $(2n, 0)$. We say that n is the *semilength* because there are $2n$ steps. Define $\mathcal{L}_n$ as the set of all n-Dyck paths. Let $\mathcal{L} = \bigcup_{n \ge 0} \mathcal{L}_n$. The number of n-Dyck paths that never pass below the x-axis is the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. Such paths are

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Table 1
The polynomials $P_{n,m}(x)$ for small values of n and m.

(n, m)	$P_{n,m}(x)$	(n, m)	$P_{n,m}(x)$
$(1, 0)$	x	$(5, 0)$	$x^5 + 10x^4 + 20x^3 + 10x^2 + x$
$(1, 1)$	1	$(5, 1)$	$5x^4 + 20x^3 + 15x^2 + 2x$
$(2, 0)$	$x^2 + x$	$(5, 2)$	$4x^4 + 18x^3 + 17x^2 + 3x$
$(2, 1)$	$2x$	$(5, 3)$	$3x^4 + 17x^3 + 18x^2 + 4x$
$(2, 2)$	$x + 1$	$(5, 4)$	$2x^4 + 15x^3 + 20x^2 + 5x$
$(3, 0)$	$x^3 + 3x^2 + x$	$(5, 5)$	$x^4 + 10x^3 + 20x^2 + 10x + 1$
$(3, 1)$	$3x^2 + 2x$	$(6, 0)$	$x^6+15x^5+50x^4+50x^3+15x^2+x$
$(3, 2)$	$2x^2 + 3x$	$(6, 1)$	$6x^5 + 40x^4 + 60x^3 + 24x^2 + 2x$
$(3, 3)$	$x^2 + 3x + 1$	$(6, 2)$	$5x^5 + 35x^4 + 60x^3 + 29x^2 + 3x$
$(4, 0)$	$x^4 + 6x^3 + 6x^2 + x$	$(6, 3)$	$4x^5 + 32x^4 + 60x^3 + 32x^2 + 4x$
$(4, 1)$	$4x^3 + 8x^2 + 2x$	$(6, 4)$	$3x^5 + 29x^4 + 60x^3 + 35x^2 + 5x$
$(4, 2)$	$3x^3 + 8x^2 + 3x$	$(6, 5)$	$2x^5 + 24x^4 + 60x^3 + 40x^2 + 6x$
$(4, 3)$	$2x^3 + 8x^2 + 4x$	$(6, 6)$	$x^5+15x^4+50x^3+50x^2+15x+1$
$(4, 4)$	$x^3 + 6x^2 + 6x + 1$		

called the *Catalan paths of length n*. The generating function $C(z) := \sum_{n\geq 0} c_n z^n$ satisfies the functional equation $C(z) = 1 + zC(z)^2$ and $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ explicitly.

A Dyck path is called an (n, m)-Dyck path if it contains m up steps under the x-axis and its semilength is n. Clearly, $0 \leq m \leq n$. Let $\mathcal{L}_{n,m}$ denote the set of all (n, m)-Dyck paths and $l_{n,m} = |\mathcal{L}_{n,m}|$. The classical Chung–Feller theorem [2] says that $l_{n,m} = c_n$ for $0 \leq m \leq n$.

We can consider an (n, m)-Dyck path P as a word of $2n$ letters using only U and D. In this word, let P_i denote the ith $(1 \leq i \leq 2n)$ letter from the left. If a joint node in the Dyck path is formed by an up step followed by a down step, then this node is called a *peak*; if a joint node in the Dyck path is formed by a down step followed by an up step, then this node is called a *valley*; if a joint node in the Dyck path is formed by an up step followed by an up step, then this node is called a *double ascent*; if a joint node in the Dyck path is formed by a down step followed by a down step, then this node is called a *double descent*.

Define $\mathcal{P}_{n,m,k}$ (resp. $\mathcal{V}_{n,m,k}$) as the set of all (n, m)-Dyck paths with k peaks (resp. valleys). Let $p_{n,m,k} = |\mathcal{P}_{n,m,k}|$ and $v_{n,m,k} = |\mathcal{V}_{n,m,k}|$. We also define $\mathcal{A}_{n,m,k}$ (resp. $\mathcal{D}_{n,m,k}$) as the set of (n, m)-Dyck paths with k double ascents (resp. k double descents). Let $a_{n,m,k} = |\mathcal{A}_{n,m,k}|$ and $d_{n,m,k} = |\mathcal{D}_{n,m,k}|$. Let ε be a mapping from the set $\{U, D\}$ to itself such that $\varepsilon(U) = D$ and $\varepsilon(D) = U$. Furthermore, for any path $P = P_1P_2 \ldots P_{2n} \in \mathcal{P}_{n,m,k}$, let $\phi(P) = \varepsilon(P_1)\varepsilon(P_2) \ldots \varepsilon(P_{2n})$. It is easy to see that ϕ is a bijection between the sets $\mathcal{P}_{n,m,k}$ and $\mathcal{V}_{n,n-m,k}$. For any $P = P_1P_2 \ldots P_{2n} \in \mathcal{A}_{n,m,k}$, let $\psi(P) = \varepsilon(P_{2n})\varepsilon(P_{2n-1}) \ldots \varepsilon(P_1)$. Clearly, ψ is a bijection from the set $\mathcal{A}_{n,m,k}$ to the set $\mathcal{D}_{n,m,k}$. Hence, in this paper, we focus on the polynomials $P_{n,m}(x) = \sum_{k=1}^{n} p_{n,m,k}x^k$ and $A_{n,m}(x) = \sum_{k=0}^{n-1} a_{n,m,k}x^k$. Table 1 shows the polynomials $P_{n,m}(x)$ for small values of n and m. From the classical Chung–Feller theorem, we have $P_{n,m}(1) = A_{n,m}(1) = c_n$ for $0 \leq m \leq n$. The classical Chung–Feller theorem was proved by using an analytic method in [2]. Narayana [6] proved the theorem by combinatorial methods. Eu et al. studied the theorem by using the Taylor expansions of generating functions in [4] and obtained a refinement of this theorem in [3]. Chen [1] revisited the theorem by establishing a bijection. Recently, Shu-Chung Liu et al. [5] use a unified algebra approach to prove Chung–Feller theorems for Dyck paths and Motzkin paths and develop a new method for finding some combinatorial structures which have the Chung–Feller property. However, the macroscopic structures should be supported by some microcosmic structures. We want to find the Chung–Feller phenomena in more exquisite structures.

Richard Stanley's book [7], in the context of rational generating functions, devotes an entire section exploring the relationships (called reciprocity relationships) between positively and nonpositively indexed terms of a sequence. First, we give the reciprocity theorem for the polynomial $P_{n,m}(x)$. In particular, we prove that the number of (n, m)-Dyck paths with k peaks is equal to the number of $(n, n-m)$-Dyck paths with $n-k$ peaks.

One observes that the sum of $p_{n,m,k}$ and $p_{n,m,n-k}$ is independent of m for any $1 \leq m \leq n-1$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, in Table 1. This is proved in Theorem 3.2 by using algebraic methods. Given n

and k, we also show that the polynomials $A_{n,m}(x)$ have the Chung–Feller property on m. In particular, we conclude that the total number of (n, m)-Dyck paths with k double ascents is equal to the total number of n-Dyck paths that have k double ascents and never pass below the x-axis, which is counted by the Narayana number. So, the classical Chung–Feller theorem can be viewed as the direct corollary of this result.

This paper is organized as follows. In Section 2, we will prove the reciprocity theorem for the polynomial $P_{n,m}(x)$. In Section 3, we will show that $p_{n,m,k} + p_{n,m,n-k}$ has the Chung–Feller property on m for any $1 \le m \le n-1$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$. In Section 4, we will prove that the polynomials $A_{n,m}(x)$ have the Chung–Feller property on m.

2. The reciprocity theorem for the polynomial $P_{n,m}(x)$

In this section, first, define the generating functions $P_m(x,z) = \sum_{n\ge m} P_{n,m}(x)z^n$. When $m = 0$, $p_{n,0,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}$ gives the Narayana numbers. It is well known that

$$P_0(x,z) = 1 + P_0(x,z)z\left[x + P_0(x,z) - 1\right],$$

or equivalently,

$$P_0(x,z) = \frac{1 + (1-x)z - \sqrt{1 - 2(1+x)z + (1-x)^2z^2}}{2z}.$$

Similarly, let $V_m(x,z) = \sum_{n,k\ge 0} v_{n,m,k}x^kz^n$. It is easy to obtain

$$V_0(x,z) = 1 + z + z[V_0(x,z) - 1](1 + xV_0(x,z)),$$

or equivalently,

$$V_0(x,z) = \frac{1 - (1-x)z - \sqrt{1 - 2(1+x)z + (1-x)^2z^2}}{2zx}.$$

In fact, we have $v_{n,0,k} = p_{n,0,k+1}$, since the number of valleys is equal to the number of peaks minus 1 for each Catalan path. So, $P_n(x,z) = V_0(x,z)$.

Now, let $P(x,y,z) = \sum_{n\ge 0}\sum_{m=0}^{n}\sum_{k=1}^{n} p_{n,m,k}x^ky^mz^n$. Let $P \in \mathcal{L}$ contain some step over the x-axis. We decompose P into $P_1UP_2\bar{D}P_3$, where U and D are the first up and down steps leaving and returning to the x-axis and on the x-axis respectively. Note that all the steps of P_1 are below the x-axis, P_2 is a Catalan path and $P_3 \in \mathcal{L}$. If $P_2 = \emptyset$, then we get a peak UD. So, we obtain the following lemma.

Lemma 2.1.

$$P(x,y,z) = V_0(x,yz)\left\{1 + z[x + P_0(x,z) - 1]P(x,y,z)\right\}.$$

Equivalently,

$$P(x,y,z) = \frac{2}{\sqrt{f(x,z)} + \sqrt{f(x,yz)} + (1-x)(1-y)z}$$

where $f(x,y) = 1 - 2(1+x)y + (1-x)^2y^2$.

We state the reciprocity relationships for the polynomials $P_{n,m}(x)$ as the following theorem.

Theorem 2.2. *Let* $n \ge 1$. $P_{n,m}(x) = x^nP_{n,n-m}(\frac{1}{x})$ *for all* $0 \le m \le n$. *Equivalently,* $p_{n,m,k} = p_{n,n-m,n-k}$.

Proof. Let $f(x,y) = 1 - 2(1+x)y + (1-x)^2y^2$. Note that: (1) $f(x^{-1}, xyz) = f(x,yz)$; (2) $f(x^{-1}, xz) = f(x,z)$; and (3) $(1-x^{-1})(1-y^{-1})xyz = (1-x)(1-y)z$.

By Lemma 2.1, we have

$$P(x,y,z) = P(x^{-1}, y^{-1}, xyz).$$

Since $P(x, y, z) = 1 + \sum_{n\geq 1}\sum_{m=0}^{n} P_{n,m}(x)y^m z^n$, we have

$$\begin{aligned} P(x^{-1}, y^{-1}, xyz) &= 1 + \sum_{n\geq 1}\sum_{m=0}^{n} P_{n,m}\left(\frac{1}{x}\right) y^{-m}(xyz)^n \\ &= 1 + \sum_{n\geq 1}\sum_{m=0}^{n} x^n P_{n,m}\left(\frac{1}{x}\right) y^{n-m} z^n \\ &= 1 + \sum_{n\geq 1}\sum_{m=0}^{n} x^n P_{n,n-m}\left(\frac{1}{x}\right) y^{m} z^n. \end{aligned}$$

This implies $P_{n,m}(x) = x^n P_{n,n-m}(\frac{1}{x})$ for all $0 \leq m \leq n$. Comparing the coefficients on either side of the identity, we derive $p_{n,m,k} = p_{n,n-m,n-k}$. □

Recall that $v_{n,m,k}$ is the number of (n, m)-Dyck paths with k valleys and $v_{n,m,k} = p_{n,n-m,k}$.

Corollary 2.3. *Let $n \geq 1$. Then $v_{n,m,k} = v_{n,n-m,n-k}$.*

3. The refinement of (n, m)-Dyck paths obtained by using the peak

In this section, we will consider the refinement of (n, m)-Dyck paths obtained by using the peak and prove that the values of $p_{n,m,k} + p_{n,m,n-k}$ have the Chung–Feller property on m for any $1 \leq m \leq n-1$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Lemma 3.1.

$$P_1(x, z) = (1 + z - xz)P_0(x, z) - 1.$$

Furthermore, we have

$$p_{n,1,k} = \frac{2(n-k)}{n(n-1)}\binom{n}{k-1}\binom{n}{k}$$

for any $n \geq 2$.

Proof. Let P be a Dyck path containing exactly one up step under the x-axis. Then we can decompose the path P into $P_1 DUP_2$, where P_1 and P_2 are both Catalan paths. So, $P_1(x, z) = z[P_0(x, z)]^2$. Hence, we have $P_1(x, z) = (1 + z - xz)P_0(x, z) - 1$ since $P_0(x, z) = 1 + P_0(x, z)z\,[x + P_0(x, z) - 1]$.

Note that $P_0(x, z) = 1 + \sum_{n\geq 1}\sum_{k=1}^{n} p_{n,0,k}x^k z^n$, where $p_{n,0,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}$. Therefore,

$$\begin{aligned} p_{n,1,k} &= \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} + \frac{1}{k}\binom{n-2}{k-1}\binom{n-1}{k-1} - \frac{1}{k-1}\binom{n-2}{k-2}\binom{n-1}{k-2} \\ &= \frac{2(n-k)}{n(n-1)}\binom{n}{k-1}\binom{n}{k}. \quad \square \end{aligned}$$

Theorem 3.2. *Let n be an integer with $n \geq 1$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$p_{n,m,k} + p_{n,m,n-k} = p_{n,m,k} + p_{n,n-m,k} = \frac{2(n+2)}{n(n-1)}\binom{n}{k-1}\binom{n}{k+1}$$

for any $1 \leq m \leq n-1$.

Proof. Theorem 2.2 implies that $p_{n,m,k} + p_{n,m,n-k} = p_{n,m,k} + p_{n,n-m,k}$. We consider the generating function $R(x, y, z) = \sum_{n\geq 1} \sum_{m=1}^{n-1} \sum_{k=1}^{n-1} (p_{n,m,k} + p_{n,n-m,k}) x^k y^m z^n$. It is easy to see that

$$R(x, y, z) = P(x, y, z) + P(x, y^{-1}, yz) + 2 - [V_0(x, z) + V_0(x, yz)] - [P_0(x, z) + P_0(x, yz)]$$

Let $\alpha(x, z) = \frac{1+x-(1-x)z}{x} P_0(x, z) - \frac{P_0(x,z)}{V_0(x,z)} - \frac{1}{x}$.

Then

$$R(x, y, z) = \frac{y\alpha(x, z) - \alpha(x, yz)}{1 - y}.$$

Suppose $\alpha(x, z) = \sum_{n\geq 1} \sum_{k=1}^{n-1} a_{k,n} x^k z^n$. Then

$$\begin{aligned} R(x, y, z) &= \sum_{n\geq 1} \sum_{k=1}^{n-1} a_{k,n} x^k z^n \frac{y(1 - y^{n-1})}{1 - y} \\ &= \sum_{n\geq 1} \sum_{m=1}^{n-1} \sum_{k=1}^{n-1} a_{k,n} x^k y^m z^n. \end{aligned}$$

Hence, given $n \geq 1$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we have $p_{n,m,k} + p_{n,n-m,k} = p_{n,m,k} + p_{n,m,n-k} = a_{k,n}$ for all $1 \leq m \leq n - 1$. By Lemma 3.1, we have

$$\begin{aligned} p_{n,m,k} + p_{n,n-m,k} &= p_{n,1,k} + p_{n,1,n-k} \\ &= \frac{2(n + 2)}{n(n - 1)} \binom{n}{k - 1} \binom{n}{k + 1}. \quad \square \end{aligned}$$

Corollary 3.3. *Let n be an integer with n ≥ 1. Then*

$$p_{2n,m,n} = \frac{1}{2n - 1} \binom{2n}{n - 1} \binom{2n}{n}$$

for any $1 \leq m \leq 2n - 1$.

Note that $v_{n,m,k} = p_{n,n-m,k}$. We obtain the following corollaries.

Corollary 3.4. *Let n be an integer with n ≥ 1 and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. *Then*

$$v_{n,m,k} + v_{n,m,n-k} = v_{n,m,k} + v_{n,n-m,k} = \frac{2(n + 2)}{n(n - 1)} \binom{n}{k - 1} \binom{n}{k + 1}$$

for any $1 \leq m \leq n - 1$.

Corollary 3.5. *Let n be an integer with n ≥ 1. Then*

$$v_{2n,m,n} = \frac{1}{2n - 1} \binom{2n}{n - 1} \binom{2n}{n}$$

for any $1 \leq m \leq 2n - 1$.

In the following theorem, we derive a recurrence relation for the polynomial $P_{n,m}(x)$.

Theorem 3.6. *For any m, r ≥ 0, we have*

$$P_{m+r,m}(x) = \begin{cases} 1 & \text{if } (m, r) = (0, 0) \\ \sum_{k=1}^{m} \frac{1}{k} \binom{m - 1}{k - 1} \binom{m}{k - 1} x^{k-1} & \text{if } r = 0 \text{ and } m \geq 1 \\ x \sum_{i=0}^{m} \sum_{j=0}^{r-1} P_{m-i,m-i}(x) P_{r-j-1,r-j-1}(x) P_{j+i,i}(x) & \text{if } r \geq 1. \end{cases}$$

Proof. It is trivial for the case with $r = 0$. We only consider the case with $r \geq 1$. Note that $x + P_0(x, z) - 1 = xV_0(x, z)$. Lemma 2.1 tells us that

$$P(x, y, z) = V_0(x, yz) + xzV_0(x, z)V_0(x, yz)P(x, y, z). \tag{1}$$

It is well known that $V_0(x, z) = \sum_{n\geq 0} b_n(x)z^n$, where $b_0(x) = 1$ and $b_n(x) = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} x^{k-1}$ for all $n \geq 1$. Comparing the coefficients of y^m on either side of the identity (1), we get

$$P_m(x, z) = b_m(x)z^m + xzV_0(x, z)\sum_{i=0}^{m} P_i(x, z)b_{m-i}(x)z^{m-i}. \tag{2}$$

Finally, since $P_m(x, z) = \sum_{n\geq m} P_{n,m}(x)z^n$, comparing the coefficients of z^n on either side of the identity (2), we obtain

$$P_{m,m}(x) = b_m(x), \quad \text{and}$$
$$P_{n,m}(x) = x\sum_{i=0}^{m}\sum_{j=i}^{n-m+i-1} b_{m-i}(x)b_{n-m+i-j-1}(x)P_{j,i}(x).$$

This completes the proof. □

4. The refinement of (n, m)-Dyck paths obtained by using the double ascent

In this section, we will consider the refinement of (n, m)-Dyck paths obtained by using the double ascent and prove that the values of $a_{n,m,k}$ have the Chung–Feller property on m. Define the generating functions $A_m(x, z) = \sum_{n\geq m} A_{n,m}(x)z^n$. When $m = 0$, $a_{n,0,k} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$. It is well known that

$$A_0(x, z) = 1 + \frac{zA_0(x, z)}{1 - xzA_0(x, z)},$$

or equivalently, $A_0(x, z) = \frac{1+(x-1)z-\sqrt{(1+zx-z)^2-4xz}}{2xz}$.

Define a generating function $A(x, y, z) = \sum_{n\geq 0}\sum_{m=0}^{n}\sum_{k=1}^{n} a_{n,m,k}x^k y^m z^n$.

Lemma 4.1.

$$A(x, y, z) = \frac{A_0(x, z)A_0(x, yz)}{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]}.$$

Proof. Let the mapping ϕ be defined as that in the Introduction. An alternating Catalan path is a Dyck path which can be decomposed into RT, where $R \neq \emptyset$ and $T \neq \emptyset$, such that $\phi(R)$ and T are both Catalan paths.

Now, let $P \in \mathcal{L}$. We can uniquely decompose P into $\hat{P}Q_1 \dots Q_m\hat{R}$ such that $\hat{P}$ and $\phi(\hat{R})$ are Catalan paths and Q_i is the alternating Catalan path for all i. Hence,

$$\begin{aligned} A(x, y, z) &= A_0(x, z)\left(\sum_{m\geq 0} x[A_0(x, z) - 1][A_0(x, yz) - 1]\right)A_0(x, yz) \\ &= \frac{A_0(x, z)A_0(x, yz)}{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]}. \quad \square \end{aligned}$$

Theorem 4.2. *Let n be an integer with $n \geq 0$ and $0 \leq k \leq n - 1$. Then*

$$a_{n,m,k} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$$

for any $0 \leq m \leq n$.

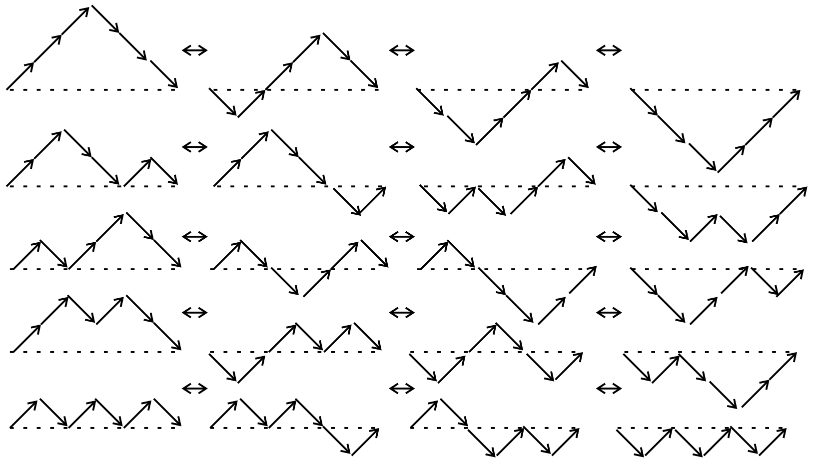

Fig. 1. $(3, m)$-Dyck path P with k double ascents and $\varphi(P)$.

Proof. First, we give an algebraic proof of this theorem. Since $xz[A_0(x, z)]^2 = A_0(x, z)[1 + xz - z] - 1$, simple calculations tell us that

$$\begin{aligned} & z\,\{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]\}\,[yA_0(x, yz) - A_0(x, z)] \\ = {} & z(y - 1)A_0(x, z)A_0(x, yz). \end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned} A(x, y, z) &= \frac{A_0(x, z)A_0(x, yz)}{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]} \\ &= \frac{yA_0(x, yz) - A_0(x, z)}{y - 1} \\ &= \sum_{n\geq 0}\sum_{m=0}^{n} A_{n,0}(x)y^m z^n. \end{aligned}$$

This implies that $A_{n,m}(x) = A_{n,0}(x)$ for any $0 \leq m \leq n$. Therefore, $a_{n,m,k} = a_{n,0,k} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$.

Now, we give a bijection proof of this theorem. Let P be an (n, m)-Dyck path with k double ascents, where $0 \leq m \leq n-1$. We say that a Catalan path is prime if the path touches the x-axis exactly twice. We can decompose P into $SRUQDT$ such that:

(1) UQD is the rightmost prime Catalan path in P;

(2) $\phi(R)$ is a Catalan path, where ϕ is defined as that in the Introduction; and

(3) the final step of S is D on the x-axis or $S = \emptyset$.

It is easy to see that $\phi(T)$ is a Catalan path. We define a path $\varphi(P)$ as

$$\varphi(P) = STDRUQ.$$

Clearly, the number of double ascents in $\varphi(P)$ is equal to the number of double ascents in P and the number of flaws in $\varphi(P)$ is $m + 1$ (see Fig. 1).

To prove that the mapping φ is a bijection, we describe the inverse φ^{-1} of the mapping φ as follows: Let P' be an $(n, m + 1)$-Dyck path with k double ascents, where $0 \leq m \leq n - 1$. We can decompose P' into $STDRUQ$ such that:

(1) D and U are the rightmost steps leaving and returning to the x-axis steps and under the x-axis in P' respectively;

(2) $\phi(T)$ is a Catalan path, where ϕ is defined as that in the Introduction; and

(3) the final step of S is D on the x-axis or $S = \emptyset$.

Clearly, Q and $\phi(DRU)$ are both Catalan paths. We define a path $\varphi^{-1}(P')$ as $\varphi^{-1}(P') = SRUQDT$. □

Corollary 4.3 (*Chung–Feller*)**.** *The number of* (n, m)*-Dyck paths is the Catalan number* c_n *for any* $0 \leq m \leq n$*.*

Recall that $d_{n,m,k}$ is the number of (n, m)-Dyck paths with k double descents and $d_{n,m,k} = a_{n,m,k}$.

Corollary 4.4. *Let* n *be an integer with* $n \geq 0$ *and* $0 \leq k \leq n - 1$*. Then*

$$d_{n,m,k} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$$

for any $0 \leq m \leq n$*.*

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