Rooted cyclic permutations of lattice paths and uniform partitions

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Abstract

A partition of a given set is said to be uniform if all the partition classes have the same cardinality. In this paper, we will introduce the concepts of rooted \( n \)-lattice path and rooted cyclic permutation and prove some fundamental theorems concerning the actions of rooted cyclic permutations on rooted lattice \( n \)-paths. The main results we obtained have important applications in finding new uniform partitions. Many uniform partitions of combinatorial structures are special cases or consequence of our main theorems.

Keywords: cyclic permutation; fluctuation theory; lattice path; uniform partition

1 Introduction

Let \( \mathbb{N} \) be the set of natural numbers and \( \mathbb{R} \) the set of real numbers. An \( n \)-lattice path is a sequence \( L \) of two-dimensional vectors

\[ L = (x_i, y_i)_{i=1}^n = (x_1, y_1)(x_2, y_2) \cdots (x_n, y_n), \]

where \( (x_i, y_i) \in \mathbb{N} \times \mathbb{R} \) for every \( i \). Let \( a_0 = 0, s_0 = 0, \) and

\[ a_i = \sum_{j=1}^i x_j, \quad s_i = \sum_{j=1}^i y_j, \quad 1 \leq i \leq n, \]

then \( L \) corresponds to the following sequence of points

\[ (a_0, s_0)(a_1, s_1) \cdots (a_n, s_n). \]

We call \( (a_n, s_n) \) the end point of \( L \). Let

\[ P(L) = \{ i \mid s_i > 0 \} \]

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and $p(L) = |P(L)|$. Moreover, we use $m(L)$ to denote the smallest index $i$ with $s_i = \max_{0 \leq k \leq n} s_k$.

Given an $i \in \{1, 2, \cdots, n\}$, the $i$-th cyclic permutation $L_i$ of $L$ is the following sequence

$$L_i = (x_i, y_i) \cdots (x_n, y_n)(x_1, y_1) \cdots (x_{i-1}, y_{i-1}).$$

Let

$$\mathcal{P}(L) = \{p(L_i) \mid i \in \{1, 2, \cdots, n\}\},$$
$$\mathcal{M}(L) = \{m(L_i) \mid i \in \{1, 2, \cdots, n\}\}.$$

Obviously, if $s_n > 0$ then

$$\mathcal{P}(L) \subseteq \{1, 2, \cdots, n\},$$
$$\mathcal{M}(L) \subseteq \{1, 2, \cdots, n\}.$$

For every $k \in \{1, 2, \cdots, n\}$, if

$$\mathcal{P}(L) = \{1, 2, \cdots, n\}$$

then there is exactly one cyclic permutation $L_i$ of $L$ such that $p(L_i) = k$, and if

$$\mathcal{M}(L) = \{1, 2, \cdots, n\}$$

then there is exactly one cyclic permutation $L_j$ of $L$ such that $p(L_j) = k$. Thus, for $s_n > 0$, an interesting problem is:

**Problem 1.1** Determine necessary and sufficient conditions for

$$\mathcal{P}(L) = \{1, 2, \cdots, n\},$$
$$\mathcal{M}(L) = \{1, 2, \cdots, n\},$$

or

$$\mathcal{P}(L) = \mathcal{M}(L) = \{1, 2, \cdots, n\}.$$ 

For $s_n \leq 0$, we have

$$\mathcal{P}(L) \subseteq \{0, 1, \cdots, n - 1\},$$
$$\mathcal{M}(L) \subseteq \{0, 1, \cdots, n - 1\}.$$

Similarly, we are also interested in the following problem:

**Problem 1.2** Determine necessary and sufficient conditions for

$$\mathcal{P}(L) = \{0, 1, \cdots, n - 1\},$$
$$\mathcal{M}(L) = \{0, 1, \cdots, n - 1\},$$

or

$$\mathcal{P}(L) = \mathcal{M}(L) = \{0, 1, \cdots, n - 1\}.$$
These problems have been studied by several authors and partial results obtained. In the case of \( s_n = 0 \), Spitzer [24] gave sufficient conditions for \( \mathcal{P}(L) = \mathcal{M}(L) = \{0, 1, \cdots, n-1\} \).

**Proposition 1.3** (Spitzer combinatorial lemma, [24]) Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path. If \( s_n = 0 \) and no other partial sum of distinct elements vanishes, then

\[
\mathcal{P}(L) = \mathcal{M}(L) = \{0, 1, \cdots, n-1\}.
\]

Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with \( s_n = 1 \), where \( y_i \) is integer for any \( i \in \{1, 2, \cdots, n\} \). Raney [23] discovered a fact: there exists a unique cyclic permutation \( L_i \) of \( L \) such that \( p(L_i) = n \). Narayana [21] gave sufficient conditions for \( \mathcal{P}(L) = \{1, 2, \cdots, n\} \). Graham and Knuth’s book [12] introduced a simple geometric argument of the results obtained by Raney. This geometric argument yields \( \mathcal{P}(L) = \mathcal{M}(L) = \{1, 2, \cdots, n\} \).

**Proposition 1.4** (Graham and Knuth [12]) Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path. If \( y_i \) is integer for any \( i \in \{1, 2, \cdots, n\} \) and \( s_n = 1 \), then

\[
\mathcal{P}(L) = \mathcal{M}(L) = \{1, 2, \cdots, n\}.
\]

Recently, Huang, Ma and Yeh [14] solved Problem 1.1 and Problem 1.2 completely.

**Theorem 1.5** (Huang, Ma and Yeh [14]) Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\).

1. Suppose \( s_n > 0 \). Then

\[
\mathcal{M}(L) = \{1, 2, \cdots, n\} \text{ if and only if } s_j - s_i \geq s_n \text{ for } i \text{ satisfying } 1 \leq i \leq j_0 - 1, \text{ where } j_0 = m(L);
\]

\[
\mathcal{P}(L) = \{1, 2, \cdots, n\} \text{ if and only if } s_j - s_i \notin (0, s_n) \text{ for } i \text{ and } j \text{ satisfying } 1 \leq i < j \leq n, \text{ where } (0, s_n) \text{ denotes the set of all real numbers } x \text{ satisfying } 0 < x < s_n.
\]

2. Suppose \( s_n \leq 0 \). Then

\[
\mathcal{M}(L) = \{0, 1, \cdots, n-1\} \text{ if and only if } s_i - s_j < s_n \text{ for } i \text{ satisfying } j_0 + 1 \leq i \leq n - 1, \text{ where } j_0 = m(L);
\]

\[
\mathcal{P}(L) = \{0, 1, \cdots, n-1\} \text{ if and only if } s_j - s_i \notin [s_n, 0] \text{ for } i \text{ and } j \text{ satisfying } 1 \leq i < j \leq n, \text{ where } [s_n, 0] \text{ denotes the set of all real numbers } x \text{ satisfying } s_n \leq x \leq 0.
\]

Proposition 1.3 and Proposition 1.4 are consequences of Theorem 1.5. Moreover, by Theorem 1.5, for \( s_n > 0 \), we must have \( \mathcal{M}(L) = \{1, 2, \cdots, n\} \) if \( \mathcal{P}(L) = \{1, 2, \cdots, n\} \); for \( s_n \leq 0 \), we must have \( \mathcal{M}(L) = \{0, 1, \cdots, n-1\} \) if \( \mathcal{P}(L) = \{0, 1, \cdots, n-1\} \).

Let

\[
L = (x_i, y_i)_{i=1}^n, \quad \tilde{L} = (\tilde{x}_i, \tilde{y}_i)_{i=1}^n
\]

be two \( n \)-lattice paths. If \( y_i = \tilde{y}_i \) for any \( i \in \{1, 2, \cdots, n\} \) then

\[
\mathcal{M}(L) = \mathcal{M}(\tilde{L}), \quad \mathcal{P}(L) = \mathcal{P}(\tilde{L}).
\]
We observe that 

**Example 1.7** Let \( L = (3,3)(2,-2) \). Then 

\[
\mathcal{H}(L) = \{[L_1,0], [L_1,1], [L_1,2], [L_2,0], [L_2,1]\}, \\
\mathcal{T}(L) = \{[L_1,0], [L_1,1], [L_2,0], [L_2,1], [L_2,2]\}; \\
\mathcal{P}_H(L) = \{5,4,3,2\}, \quad \mathcal{M}_H(L) = \{3,2,1,5,4\}, \\
\mathcal{P}_T(L) = \{5,4,3,2,1\}, \quad \mathcal{M}_T(L) = \{3,2,5,4\}.
\]

We observe that \( \mathcal{M}_H(L) = \mathcal{P}_T(L) = \{1,2,3,4,5\} = \{1,2,\cdots, a_2\} \) since \( a_2 = x_1 + x_2 = 5 \).

Let \( L = (3,-3)(2,2) \). Then 

\[
\mathcal{H}(L) = \{[L_1,0], [L_1,1], [L_1,2], [L_2,0], [L_2,1]\}, \\
\mathcal{T}(L) = \{[L_1,0], [L_1,1], [L_2,0], [L_2,1], [L_2,2]\}; \\
\mathcal{P}_H(L) = \{0,1,2,3\}, \quad \mathcal{M}_H(L) = \{0,1,2,3\}, \\
\mathcal{P}_T(L) = \{0,1,2,3,4\}, \quad \mathcal{M}_T(L) = \{0,1,2,3,4\}.
\]

We observe that \( \mathcal{P}_T(L) = \mathcal{M}_T(L) = \{0,1,2,3,4\} = \{0,1,\cdots, a_2 - 1\} \) since \( a_2 = 5 \).
In general, when $s_n > 0$, we always have

$$
\mathcal{P}_T(L) \subseteq \{1, 2, \cdots, a_n\},
\mathcal{M}_H(L) \subseteq \{1, 2, \cdots, a_n\};
$$

and when $s_n \leq 0$, we always have

$$
\mathcal{P}_T(L) \subseteq \{0, 1, \cdots, a_n - 1\},
\mathcal{M}_T(L) \subseteq \{0, 1, \cdots, a_n - 1\}.
$$

It is natural to consider the following two problems:

**Problem 1.8** For $s_n > 0$, determine necessary and sufficient conditions for

$$
\mathcal{P}_T(L) = \mathcal{M}_H(L) = \{1, 2, \cdots, a_n\},
$$
or

$$
\mathcal{P}_T(L) = \mathcal{M}_T(L) = \{1, 2, \cdots, a_n\}.
$$

**Problem 1.9** For $s_n \leq 0$, determine necessary and sufficient conditions for

$$
\mathcal{P}_T(L) = \mathcal{M}_T(L) = \{0, 1, \cdots, a_n - 1\},
$$
or

$$
\mathcal{P}_T(L) = \mathcal{M}_T(L) = \{0, 1, \cdots, a_n - 1\}.
$$

In this paper, we will solve Problem 1.8 and Problem 1.9 completely by proving the following theorems:

**Theorem 1.10** Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$ and $j_0 = m(L)$.

1. For $s_n > 0$, $\mathcal{M}_H(L) = \{1, 2, \cdots, a_n\}$ if and only if $s_{j_0} - s_i \geq s_n$ for any $i \in \{1, 2, \cdots, j_0 - 1\}$.
2. For $s_n \leq 0$, $\mathcal{M}_T(L) = \{0, 1, \cdots, a_n - 1\}$ if and only if $s_i - s_{j_0} < s_n$ for any $i \in \{j_0 + 1, j_0 + 2, \cdots, n - 1\}$.

**Theorem 1.11** Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$.

1. For $s_n > 0$, $\mathcal{P}_T(L) = \{1, 2, \cdots, a_n\}$ if and only if $s_j - s_i \notin (0, s_n)$ for $i$ satisfying $1 \leq i < j \leq n$, where $(0, s_n)$ denotes the set of all real numbers $y$ satisfying $0 < y < s_n$.
2. For $s_n \leq 0$, $\mathcal{P}_T(L) = \{0, 1, \cdots, a_n - 1\}$ if and only if $s_j - s_i \notin [s_n, 0]$ for $i$ and $j$ satisfying $1 \leq i < j \leq n$, where $[s_n, 0]$ denotes the set of all real numbers $y$ satisfying $s_n \leq y \leq 0$. 
A partition of a given set is called uniform if all partition classes have the same cardinality. Many uniform partitions of combinatorial structures are consequences of Theorem 1.10 and Theorem 1.11. This theorem was proved by several authors using different methods [5–8; 16; 18; 20–22]. It can also be proved by using Theorem 1.11(2).

Use \( S_n \) to denote the set of all the permutations on the set \( \{1, 2, \cdots, n\} \). We write permutations of \( S_n \) in the form \( \sigma = (\sigma(1)\sigma(2)\cdots\sigma(n)) \). Given an \( n \)-lattice path \( L = (x_i, y_i)_{i=1}^n \) and a permutation \( \sigma \), let \( L_\sigma = (x_{\sigma(1)}, y_{\sigma(1)})\cdots(x_{\sigma(n)}, y_{\sigma(n)}) \).

and

\[
S_H(L) = \{[\sigma, j] \mid \sigma \in S_n, j \in \{0, 1, \cdots, x_{\sigma(1)} - 1\}\}
\]

\[
S_T(L) = \{[\sigma, j] \mid \sigma \in S_n, j \in \{0, 1, \cdots, x_{\sigma(n)} - 1\}\}
\]

As applications of Theorem 1.10 and Theorem 1.11, we will give uniform partitions of \( S_T(L) \) and \( S_H(L) \).

The rest of this paper is organized as follows. In Section 2, we study properties of rooted cyclic permutations of \( n \)-lattice paths and prove our main theorems. In Section 3, we use our main theorems in the section 2 to give uniform partitions of the sets \( S_T(L) \) and \( S_H(L) \).

## 2 Rooted cyclic permutations of lattice paths

In this section, we study properties of rooted cyclic permutations of lattice paths and prove our main theorems.

**Lemma 2.1** Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\). Then

\[ |\mathcal{H}(L)| = |\mathcal{T}(L)| = a_n. \]

**Proof.** For every \( i \in \{1, 2, \cdots, n\} \), let \( H(L)_i \) be the set of rooted cyclic permutations \([L_i, j]\) of \( L \) such that \( j \in \{0, 1, \cdots, x_i - 1\} \). Then \( H(L)_i \subseteq H(L) \) and \( |H(L)_i| = x_i \). Hence,

\[ |H(L)| = \sum_{i=1}^n |H(L)_i| = a_n. \]

Similarly, we can prove that \(|T(L)| = a_n. \]

**Lemma 2.2** Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\) and \( j_0 = m(L) \). Suppose \( s_n > 0 \). For any \( i \in \{j_0 + 1, j_0 + 2, \cdots, n\} \) and \( j \in \{0, 1, \cdots, x_i - 1\} \), we have

\[ m(L_i, j) = x_i + \cdots + x_n + x_1 + \cdots + x_{j_0} - j. \]
Proof. Since $j_0 = m(L)$, for any $t \in \{0, 1, \ldots, j_0 - 1\}$ we have
\[
(y_t + \cdots + y_n) + y_1 + \cdots + y_t < (y_t + \cdots + y_n) + y_1 + \cdots + y_{j_0},
\]
and for any $t \in \{j_0, j_0 + 1, \ldots, i - 1\}$ we have
\[
(y_t + \cdots + y_n) + y_1 + \cdots + y_t \leq (y_t + \cdots + y_n) + y_1 + \cdots + y_{j_0}.
\]
Assume that there is an index $t \in \{i, i + 1, \ldots, n - 1\}$ such that
\[
y_i + \cdots + y_t \geq (y_t + \cdots + y_t) + y_{t+1} + \cdots + y_n + y_1 + \cdots + y_{j_0}.
\]
Then we have
\[
0 \geq (y_{t+1} + \cdots + y_n) + (y_1 + \cdots + y_{j_0}).
\]
Note that $y_1 + \cdots + y_{j_0} \geq y_1 + \cdots + y_t$ since $j_0 = m(L)$. So,
\[
0 \geq (y_1 + \cdots + y_{j_0}) + (y_{t+1} + \cdots + y_n) \geq (y_1 + \cdots + y_t) + (y_{t+1} + \cdots + y_n) = s_n > 0,
\]
a contradiction. Thus, we have
\[
y_i + \cdots + y_t < y_i + \cdots + y_n + y_1 + \cdots + y_{j_0}
\]
for any $t \in \{i, i + 1, \ldots, n - 1\}$. Hence, $m(L, j) = x_i + \cdots + x_n + x_1 + \cdots + x_{j_0} - j$. □

Lemma 2.3 $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$ and $j_0 = m(L)$. Suppose $s_n > 0$ and there exists a unique element $[L_i, j] \in \mathcal{H}(L)$ such that $m(L_i, j) = a_n$. Then we have
\[
s_{j_0} - s_i \geq s_n
\]
for any $i \in \{1, 2, \ldots, j_0 - 1\}$.

Proof. Clearly, we have $m(L_{j_0+1}, 0) = a_n$. Let
\[
A = \{i \mid s_{j_0} - s_i < s_n, 1 \leq i \leq j_0 - 1\}.
\]
Assume that $A \neq \emptyset$ and let $\bar{i} = \min A$. Then $\bar{i} + 1 \leq j_0$. We consider the lattice path $L_{\bar{i}+1}$. Since $\bar{i} \in A$, we have
\[
s_{j_0} - s_{\bar{i}} = y_{\bar{i}+1} + \cdots + y_{j_0} < s_n = y_{\bar{i}+1} + \cdots + y_n + y_1 + \cdots + y_{\bar{i}}.
\]
Since $j_0 = m(L)$, for any $i \in \{\bar{i} + 1, \bar{i} + 2, \ldots, j_0 - 1\}$ we have
\[
y_{\bar{i}+1} + \cdots + y_i < y_{\bar{i}+1} + \cdots + y_{j_0},
\]
and for any $i \in \{j_0, j_0 + 1, \ldots, n\}$ we have
\[
y_{\bar{i}+1} + \cdots + y_i \leq y_{\bar{i}+1} + \cdots + y_{j_0}.
\]
For every $i \in \{1, \ldots, \bar{i} - 1\}$, we have $s_{j_0} - s_i = y_{i+1} + \cdots + y_{j_0} \geq s_n$ since $i \notin A$. So,
\[
y_{\bar{i}+1} + \cdots + y_{j_0} \geq y_{\bar{i}+1} + \cdots + y_n + y_1 + \cdots + y_{\bar{i}}.
\]
Hence, $m(L_{\bar{i}+1}, 0) = a_n = m(L_{j_0+1}, 0)$, a contradiction. □
Theorem 2.4 Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\) and \( j_0 = m(L) \). For \( s_n > 0 \), \( \mathcal{M}_H(L) = \{1, 2, \ldots, a_n\} \) if and only if \( s_{j_0} - s_i \geq s_n \) for any \( i \in \{1, 2, \ldots, j_0 - 1\} \).

Proof. For any \( i \in \{1, 2, \ldots, n\} \), we have \( m(L_i) \neq 0 \) since \( s_n > 0 \). Furthermore, for any \( i \in \{j_0 + 1, v + 2, \ldots, n\} \) and \( j \in \{0, 1, \ldots, x_i - 1\} \), Lemma 2.2 tells us that
\[
m(L_i, j) = a_n - (x_{j_0 + 1} + \cdots + x_{i-1}) - j.
\]

Suppose \( s_{j_0} - s_i \geq s_n \) for any \( i \in \{1, 2, \ldots, j_0 - 1\} \). Let \([L_i, j] \in \mathcal{H}(L) \) with \( i \in \{1, 2, \ldots, j_0\} \). Since \( j_0 = m(L) \), for any \( t \in \{i, i+1, \ldots, j_0 - 1\} \) we have
\[
y_t + \cdots + y_{j_0} < y_t + \cdots + y_{j_0},
\]
and for any \( t \in \{j_0, j_0 + 1, \ldots, n\} \) we have
\[
y_t + \cdots + y_{j_0} \leq y_t + \cdots + y_{j_0}.
\]

Assume that there is an index \( t \in \{1, 2, \ldots, i - 1\} \) such that
\[
y_t + \cdots + y_{j_0} < y_t + \cdots + y_{j_0} + y_1 + \cdots + y_t = s_n - (y_{t+1} + \cdots + y_{i-1}).
\]
Thus, \( s_{j_0} - s_i = y_{t+1} + \cdots + y_t + y_{j_0} \leq s_n \), a contradiction. Hence
\[
m(L_i, j) = x_i + \cdots + x_{j_0} - j.
\]

So, \( \mathcal{M}_H(L) = \{1, 2, \ldots, a_n\} \).

Conversely, suppose \( \mathcal{M}_H(L) = \{1, 2, \ldots, a_n\} \). Then there exists a unique element \([L_i, j] \in \mathcal{H}(L)\) such that \( m(L_i, j) = a_n \) since \(|\mathcal{H}(L)| = a_n\). By Lemma 2.3, we have
\[
s_{j_0} - s_i \geq s_n
\]
for any \( i \in \{1, 2, \ldots, j_0 - 1\} \).

\[\square\]

Corollary 2.5 Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\) and \( j_0 = m(L) \). For \( s_n > 0 \), \( \mathcal{M}_H(L) = \{1, 2, \ldots, a_n\} \) if and only if there exists a unique element \([L_i, j] \in \mathcal{H}(L)\) such that \( m(L_i, j) = a_n \).

Lemma 2.6 Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\) and \( j_0 = m(L) \). Suppose \( s_n \leq 0 \) and \( j_0 \geq 1 \). For any \( i \in \{1, 2, \ldots, j_0\} \) and \( j \in \{0, 1, \ldots, x_i - 1\} \), we have
\[
m(L_i, j) = x_i + \cdots + x_{j_0} + j.
\]
Proof. Since \( j_0 = m(L) \), for any \( t \in \{i, i+1, \ldots, j_0 - 1\} \) we have
\[
y_i + \cdots + y_t < y_i + \cdots + y_{j_0},
\]
and for any \( t \in \{j_0, j_0 + 1, \ldots, n\} \) we have
\[
y_i + \cdots + y_t \leq y_i + \cdots + y_{j_0}.
\]
For any \( t \in \{1, 2, \ldots, i-1\} \), we have
\[
y_{t+1} + \cdots + y_{j_0} > 0 \geq s_n
\]
since \( j_0 = m(L) \). This implies
\[
0 > y_{j_0+1} + \cdots + y_n + y_1 + \cdots + y_t
\]
and
\[
y_i + \cdots + y_{j_0} > (y_i + \cdots + y_{j_0}) + y_{j_0+1} + \cdots + y_n + y_1 + \cdots + y_t.
\]
Note that \( y_i + \cdots + y_{j_0} > 0 \). Hence, \( m(L_{i,j}) = x_i + \cdots + x_{j_0} + j \).

Lemma 2.7 Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\) and \( j_0 = m(L) \). Suppose \( s_n \leq 0 \) and there exists a unique element \([L_i,j] \in T(L)\) such that \( m(L_i,j) = 0 \). Then we have
\[
s_i - s_{j_0} < s_n
\]
for any \( i \in \{j_0 + 1, j_0 + 2, \ldots, n-1\} \).

Proof. Clearly, we have \( m(L_{j_0+1,0}) = 0 \). Let
\[
A = \{i \mid s_i - s_{j_0} \geq s_n, j_0 + 1 \leq i \leq n\}.
\]
Note that \( n \notin A \) if \( j_0 \geq 1 \), otherwise \( n \in A \). Assume that \( A \setminus \{n\} \neq \emptyset \) and let \( \bar{t} = \max(A \setminus \{n\}) \).

Clearly \( j_0 + 1 \leq \bar{t} \leq n - 1 \). We consider the lattice path \( L_{\bar{t}+1} \). Since \( j_0 = m(L) \), for any \( i \in \{0, 1, \ldots, j_0 - 1\} \) we have
\[
y_{\bar{t}+1} + \cdots + y_n + y_1 + \cdots + y_i < y_{\bar{t}+1} + \cdots + y_n + y_1 + \cdots + y_{j_0},
\]
and for any \( i \in \{j_0 + 1, \ldots, \bar{t}\} \) we have
\[
y_{\bar{t}+1} + \cdots + y_n + y_1 + \cdots + y_i \leq y_{\bar{t}+1} + \cdots + y_n + y_1 + \cdots + y_{j_0}.
\]
For any \( i \in \{\bar{t} + 1, \bar{t} + 2, \ldots, n-1\} \), since \( i \notin A \), we have
\[
s_i - s_{j_0} = y_{j_0+1} + \cdots + y_i < s_n.
\]
So
\[
y_{\bar{t}+1} + \cdots + y_i < y_{\bar{t}+1} + \cdots + y_n + y_1 + \cdots + y_{j_0}.
\]
Since \( \bar{t} \in A \), we have
\[
y_{\bar{t}+1} + \cdots + y_n + y_1 + \cdots + y_{j_0} \leq 0.
\]
Hence \( m(L_{\bar{t}+1,0}) = 0 \), a contradiction.
Theorem 2.8 Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$ and $j_0 = m(L)$. For $s_n \leq 0$, $\mathcal{M}(L) = \{0, 1, \cdots, a_n - 1\}$ if and only if $s_i - s_{j_0} < s_n$ for any $i \in \{j_0 + 1, j_0 + 2, \cdots, n - 1\}$.

Proof. For any $i \in \{1, 2, \cdots, n\}$ and $j \in \{0, 1, \cdots, x_{i-1} - 1\}$, we have $m(L_{i, j}) \neq a_n$ since $s_n \leq 0$.

Suppose $s_i - s_{j_0} < s_n$ for all $j_0 + 1 \leq i \leq n - 1$. Let $[L_{i, j}] \in T(L)$ with $i \in \{j_0 + 1, j_0 + 2, \cdots, n\}$. Since $j_0 = m(L)$, for any $t \in \{0, 1, \cdots, j_0 - 1\}$ we have

$$(y_i + \cdots + y_n) + y_1 + \cdots + y_t < (y_i + \cdots + y_n) + y_1 + \cdots + y_{j_0},$$

and for any $t \in \{j_0, j_0 + 1, \cdots, i - 1\}$ we have

$$(y_i + \cdots + y_n) + y_1 + \cdots + y_t \leq (y_i + \cdots + y_n) + y_1 + \cdots + y_{j_0}.$$ 

For any $t \in \{i, i + 1, \cdots, n - 1\}$, since

$s_t - s_{j_0} = y_{j_0+1} + \cdots + y_t < s_n,$

we have

$$y_{t+1} + \cdots + y_n + y_1 + \cdots + y_{j_0} > 0$$

and

$$y_i + \cdots + y_t < (y_i + \cdots + y_t) + y_{t+1} + \cdots + y_n + y_1 + \cdots + y_{j_0}.$$ 

Note that $y_i + \cdots + y_n + y_1 + \cdots + y_{j_0} > 0$ since $j_0 = m(L)$ for $i \geq j_0 + 1$. Clearly,

$$y_{j_0+1} + \cdots + y_n + y_1 + \cdots + y_{j_0} = s_n.$$ 

Hence, for any $i \in \{j_0 + 2, \cdots, n\}$, we have

$$m(L_{i, j}) = x_i + \cdots + x_n + x_1 + \cdots + x_{j_0} + j$$

and $m(L_{j_0+1, j}) = j$.

When $j_0 \geq 1$, Lemma 2.6 tells us that

$$m(L_{i, j}) = x_i + \cdots + x_{j_0} + j$$

for any $i \in \{1, 2, \cdots, j_0\}$ and $j \in \{0, 1, \cdots, x_{i-1} - 1\}$. Hence, we have $\mathcal{M}(L) = \{0, 1, \cdots, a_n - 1\}$.

Conversely, suppose $\mathcal{M}(L) = \{0, 1, \cdots, a_n - 1\}$. Then there exists a unique element $[L_{i, j}] \in T(L)$ such that $m(L_{i, j}) = 0$ since $|T(L)| = a_n$. By Lemma 2.7, we have

$s_i - s_k < s_n$

for any $k + 1 \leq i \leq n - 1$. ■

Combining Theorem 2.4 and Theorem 2.8 gives the proof of Theorem 1.10.
Corollary 2.9 Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\) and \( j_0 = m(L)\). For \( s_n \leq 0 \), \( \mathcal{M}_T(L) = \{0, 1, \ldots, a_n - 1\} \) if and only if there exists a unique element \([L_i, j] \in \mathcal{T}(L)\) such that \( m(L_i, j) = 0\).

For any \( n \)-lattice path \( L = (x_i, y_i)_{i=1}^n \) with end point \((a_n, s_n)\), we define a linear order \( \prec_L \) on the set \( \{1, 2, \ldots, n\} \) by the following rules:

For any \( i, j \in \{1, 2, \ldots, n\} \), \( i \prec_L j \) if either (1) \( s_i < s_j \) or (2) \( s_i = s_j \) and \( i > j \).

The sequence formed by writing elements in the set \( \{1, 2, \ldots, n\} \) in the increasing order with respect to \( \prec_L \) is denoted by \( \pi(L) = (\pi_1, \pi_2, \ldots, \pi_n) \). Note that \( \pi(L) \) can also be viewed as a bijection from the set \( \{1, 2, \ldots, n\} \) to itself.

Lemma 2.10 Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end points \((a_n, s_n)\) and \( \pi(L) \) the linear order on the set \( \{1, 2, \ldots, n\} \) with respect to \( \prec_L \). For any \( i \in \{1, 2, \ldots, n\} \), suppose \( \pi_k = i \) for some \( k \). If \( s_n > 0 \), then for any \( j \in \{0, 1, \ldots, x_i - 1\} \), we have

\[
p(L_{i+1}, j) \geq x_{\pi_k} + x_{\pi_{k+1}} + \cdots + x_{\pi_n} - j.
\]

Furthermore, if \( s_i - s_t \neq (0, s_n) \) for any \( t \in \{1, 2, \ldots, i - 1\} \), then

\[
p(L_{i+1}, j) = x_{\pi_k} + x_{\pi_{k+1}} + \cdots + x_{\pi_n} - j.
\]

Proof. Note that \( i \prec_L t \) implies either (I) \( s_i < s_t \) or (II) \( s_i = s_t \) and \( i > t \). So, we consider two cases as follows:

Case I. \( s_i < s_t \).

For \( t > i \), we have \( y_{i+1} + \cdots + y_t > 0 \). For \( t < i \), we have \( y_{i+1} + \cdots + y_t < 0 \). Hence

\[
y_{i+1} + \cdots + y_n + y_1 + \cdots + y_t = s_n - (y_{i+1} + \cdots + y_t) > s_n > 0.
\]

Case II. \( s_i = s_t \) and \( i > t \).

We have \( y_{i+1} + \cdots + y_t = 0 \) and \( y_{i+1} + \cdots + y_n + y_1 + \cdots + y_t = s_n > 0 \).

Clearly, \( y_{i+1} + \cdots + y_n + y_1 + \cdots + y_t = s_n > 0 \). Hence

\[
\{\pi_k, \pi_{k+1}, \ldots, \pi_n\} \subseteq \mathcal{P}(L_{i+1})
\]

and

\[
p(L_{i+1}, j) = \sum_{t \in \mathcal{P}(L_{i+1})} x_t - j \geq x_{\pi_k} + \cdots + x_{\pi_n} - j.
\]

Now suppose \( s_i - s_t \notin (0, s_n) \) for all \( t \in \{1, 2, \ldots, i - 1\} \). For any \( t \prec_L i \), we discuss the following two cases.

Case I. \( s_t < s_i \).

For \( t > i \), we have \( y_{i+1} + \cdots + y_t < 0 \).

For \( t < i \), we have \( s_i - s_t = y_{i+1} + \cdots + y_t \geq s_n \) since \( s_i - s_t > 0 \) and \( s_i - s_t \notin (0, s_n) \). Hence \( y_{i+1} + \cdots + y_n + y_1 + \cdots + y_t = s_n - y_{i+1} - \cdots - y_t \leq 0 \).
Case II. \( s_t = s_i \) and \( t > i \).

Clearly, \( s_t - s_i = y_{i+1} + \cdots + y_t = 0 \). Thus, we have
\[
\{\pi_k, \pi_{k+1}, \cdots, \pi_n\} = P(L_{i+1})
\]
and
\[
p(L_{i+1}, j) = x_{\pi_k} + \cdots x_{\pi_n} - j.
\]

**Theorem 2.11** Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \( (a_n, s_n) \). For \( s_n > 0 \), \( \mathcal{P}_T(L) = \{1, 2, \cdots, a_n\} \) if and only if \( s_j - s_i \notin (0, s_n) \) for \( i \) and \( j \) satisfying \( 1 \leq i < j \leq n \), where \( (0, s_n) \) denotes the set of all real numbers \( y \) satisfying \( 0 < y < s_n \).

**Proof.** Let \( \pi(L) \) be the linear order on the set \( \{1, 2, \cdots, n\} \) with respect to \( \prec_L \). Suppose \( s_j - s_i \notin (0, s_n) \) for any \( 1 \leq i < j \leq n \). Lemma 2.10 implies
\[
p(L_{\pi_k+1}, j) = x_{\pi_k} + \cdots x_{\pi_n} - j
\]
for all \( k \in \{1, 2, \cdots, n\} \) and \( j \in \{0, 1, \cdots, x_{\pi_k} - 1\} \). Hence \( \mathcal{P}_T(L) = \{1, 2, \cdots, a_n\} \).

Conversely, suppose \( \mathcal{P}_T(L) = \{1, 2, \cdots, a_n\} \). Lemma 2.10 tells us that
\[
p(L_{\pi_k+1}, j) \geq x_{\pi_k} + \cdots x_{\pi_n} - j
\]
for all \( k \in \{1, 2, \cdots, n\} \) and \( j \in \{0, 1, \cdots, x_{\pi_k} - 1\} \). Let
\[
A_k = \{t \mid 0 < s_{\pi_k} - s_t < s_n, 1 \leq t < \pi_k\}
\]
for any \( k \in \{1, 2, \cdots, n\} \). Assume that \( A_k \neq \emptyset \) for some \( k \in [n] \). Let \( \bar{k} = \min \{k \mid A_k \neq \emptyset\} \). We consider the rooted lattice path \( [L_{\pi_{\bar{k}}+1}, 0] \). Let \( t \in A_{\bar{k}} \). Since \( s_{\pi_{\bar{k}}} - s_t > 0 \), we have \( s_{\pi_{\bar{k}}} > s_t \) and \( t < L \pi_{\bar{k}} \). Since \( s_{\pi_{\bar{k}}} - s_t < s_n \), we have
\[
y_{\pi_{\bar{k}}+1} + \cdots + y_n + y_1 + \cdots + y_t = s_n - (y_{t+1} + \cdots + y_{\pi_{\bar{k}}}) > 0
\]
and \( t \in P(L_{\pi_{\bar{k}}+1}) \). By Lemma 2.10, we get
\[
p(L_{\pi_{\bar{k}}+1}, 0) \geq x_{\pi_{\bar{k}}} + \cdots x_{\pi_n} + x_t \geq x_{\pi_{\bar{k}}} + \cdots x_{\pi_n} + 1.
\]
Note that \( \bar{k} \geq 2 \) and
\[
p(L_{\pi_{\bar{k}-1}+1, \pi_{\bar{k}-1}} - 1) = x_{\pi_{\bar{k}}} + \cdots x_{\pi_n} + 1.
\]
Hence \( \mathcal{P}_T(L) \neq \{1, 2, \cdots, a_n\} \), a contradiction. \( \blacksquare \)
Lemma 2.12 Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$ and $\pi(L)$ the linear order on the set $\{1, 2, \cdots, n\}$ with respect to $\prec_L$. For any $i \in \{1, 2, \cdots, n\}$, suppose $\pi_k = i$ for some $k$. If $s_n \leq 0$, then for any $j \in \{0, 1, \cdots, x_i - 1\}$, we have

$$p(L_{i+1}, j) \leq a_n - (x_{\pi_1} + \cdots + x_{\pi_k}) + j.$$ 

Furthermore, if $s_i - s_t \notin [s_n, 0]$ for any $t \in \{1, 2, \cdots, i - 1\}$, then

$$p(L_{i+1}, j) = a_n - (x_{\pi_1} + \cdots + x_{\pi_k}) + j.$$ 

Proof. Note that $t \prec_L i$ implies either (I) $s_t < s_i$ or (II) $s_t = s_i$ and $t > i$. So, we consider two cases as follows.

Case I. $s_t < s_i$

For $t > i$, we have $y_{i+1} + \cdots + y_t < 0$. For $t < i$, we have $y_{t+1} + \cdots + y_i > 0$. Hence $y_{i+1} + \cdots + y_n + y_1 + \cdots y_i = s_n - (y_{t+1} + \cdots + y_i) < 0$.

Case II. $s_t = s_i$ and $t > i$.

We have $y_{i+1} + \cdots + y_t = 0$.

Clearly, $y_{i+1} + \cdots + y_n + y_1 + \cdots y_i = s_n \leq 0$. So,

$$P(L_{i+1}) \subseteq \{\pi_{k+1}, \pi_{k+2}, \cdots, \pi_n\}$$

and

$$p(L_{i+1}, j) \leq x_{\pi_{k+1}} + x_{\pi_{k+2}} + \cdots x_{\pi_n} + j = a_n - (x_{\pi_1} + \cdots x_{\pi_k}) + j.$$ 

Now, suppose $s_i - s_t \notin [s_n, 0]$ for any $t \in \{1, 2, \cdots, i - 1\}$. For any $i \prec_L t$, if $s_t = s_i$, then $t < i$ and $s_t - s_i = 0$, a contradiction. Hence, $s_t > s_i$.

For $t > i$, we have $y_{i+1} + \cdots + y_t > 0$. For $t < i$, we have $s_t - s_i < s_n$ since $s_t - s_t < 0$ and $s_i - s_t \notin [s_n, 0]$. So $y_{i+1} + \cdots + y_n + y_1 + \cdots y_t = s_n - (y_{t+1} + \cdots + y_i) > 0$. Hence,

$$P(L_{i+1}) = \{\pi_{k+1}, \pi_{k+2}, \cdots, \pi_n\}$$

and

$$p(L_{i+1}, j) = x_{\pi_{k+1}} + x_{\pi_{k+2}} + \cdots x_{\pi_n} + j = a_n - (x_{\pi_1} + \cdots x_{\pi_k}) + j.$$ 

\[\square\]

Theorem 2.13 Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$. For $s_n \leq 0$, $\mathcal{P}_L(L) = \{0, 1, \cdots, a_n - 1\}$ if and only if $s_j - s_i \notin [s_n, 0]$ for $i$ and $j$ satisfying $1 \leq i < j \leq n$, where $[s_n, 0]$ denotes the set of all real numbers $y$ satisfying $s_n \leq y \leq 0$.

Proof. Let $\pi(L)$ be the linear order on the set $\{1, 2, \cdots, n\}$ with respect to $\prec_L$. Suppose $s_j - s_i \notin [s_n, 0]$. 

\[s_j - s_i \notin [s_n, 0]\]
for any $1 \leq i < j \leq n$. Lemma 2.12 implies
\[ p(L_{\pi_k+1}, j) = a_n - (x_{\pi_1} + \cdots + x_{\pi_k}) + j \]
for any $k \in \{1, 2, \cdots, n\}$ and $j \in \{0, 1, \cdots, x_{\pi_k} - 1\}$. Hence $\mathcal{P}_T(L) = \{0, 1, \cdots, a_n - 1\}$.

Conversely, suppose $\mathcal{P}_T(L) = \{0, 1, \cdots, a_n - 1\}$. Lemma 2.12 tells us that
\[ p(L_{\pi_k+1}, j) \leq a_n - (x_{\pi_1} + \cdots + x_{\pi_k}) + j \]
for any $k \in \{1, 2, \cdots, n\}$ and $j \in \{0, 1, \cdots, x_{\pi_k} - 1\}$.

Let
\[ A_k = \{ t \mid s_n \leq s_{\pi_k} - s_t \leq 0, 1 \leq t \leq \pi_k - 1 \} \]
for any $k \in \{1, 2, \cdots, n\}$. Assume that $A_k \neq \emptyset$ for some $k \in \{1, 2, \cdots, n\}$. Let $\bar{k} = \max\{k \mid A_k \neq \emptyset\}$.

By Lemma 2.12, we have
\[ p(L_{\pi_{\bar{k}}+1}, j) = a_n - (x_{\pi_1} + \cdots + x_{\pi_{\bar{k}}}) + j \]
for any $k > \bar{k}$ and $j \in \{0, 1, \cdots, x_{\pi_k} - 1\}$. We consider the rooted lattice path $[L_{\pi_{\bar{k}}+1}, 0]$. Let $t \in A_{\bar{k}}$. Since $s_{\pi_{\bar{k}}} - s_t \leq 0$, we have $s_{\pi_{\bar{k}}} \leq s_t$ and $\pi_{\bar{k}} \prec_L t$. Since $s_{\pi_{\bar{k}}} - s_t \geq s_n$, we have
\[ y_{\pi_{\bar{k}}+1} + \cdots + y_n + y_1 + \cdots + y_t = s_n - (y_{t+1} + \cdots + y_{\pi_{\bar{k}}}) \leq 0 \]
and $t \notin P(L_{\pi_{\bar{k}}+1})$. By Lemma 2.12, we get
\[ p(L_{\pi_{\bar{k}}+1}, 0) \leq a_n - (x_{\pi_1} + \cdots x_{\pi_{\bar{k}}}) - x_t \leq a_n - (x_{\pi_1} + \cdots x_{\pi_{\bar{k}}}) - 1. \]

Note that $\bar{k} \leq n - 1$ and
\[ p(L_{\pi_{\bar{k}}+1}, x_{\pi_{\bar{k}}+1} - 1) = a_n - (x_{\pi_1} + \cdots x_{\pi_{\bar{k}}+1}) + x_{\pi_{\bar{k}}+1} - 1 = a_n - (x_{\pi_1} + \cdots x_{\pi_{\bar{k}}}) - 1. \]

Hence $\mathcal{P}_T(L) \neq \{0, 1, \cdots, a_n - 1\}$, a contradiction. □

Combining Theorem 2.11 and Theorem 2.13 gives the proof of Theorem 1.11.

By Theorem 1.10 and Theorem 1.11, we obtain the following corollaries:

**Corollary 2.14** Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$.

1. For $s_n > 0$, if $\mathcal{P}_T(L) = \{1, 2, \cdots, a_n\}$, then $\mathcal{M}_H(L) = \{1, 2, \cdots, a_n\}$.
2. For $s_n \leq 0$, if $\mathcal{P}_T(L) = \{0, 1, \cdots, a_n - 1\}$, then $\mathcal{M}_T(L) = \{0, 1, \cdots, a_n - 1\}$.

**Corollary 2.15** Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, 1)$. If $y_i$ is an integer for any $i \in \{1, 2, \cdots, n\}$, then $\mathcal{P}_T(L) = \mathcal{M}_H(L) = \{1, 2, \cdots, a_n\}$.
Let \( L = (1, y_i)_{i=1}^n \) be an \( n \)-lattice path. Then \( a_n = n \) and we must have \( j = 0 \) in any rooted cyclic permutation \([L_i, j]\) of \( L \). Hence,

\[
p(L_i, j) = \sum_{i \in P(L)} x_i = |P(L_i)|
\]

and

\[
m(L_i, j) = \sum_{i=1}^{m(L_i)} x_i = m(L_i).
\]

So, for \( s_n > 0 \), we have \( \mathcal{P}_T(L) = \mathcal{P}(L) \) and \( \mathcal{M}_H(L) = \mathcal{M}(L) \); for \( s_n \leq 0 \), we have \( \mathcal{P}_T(L) = \mathcal{P}(L) \) and \( \mathcal{M}_T(L) = \mathcal{M}(L) \). Thus, Theorem 1.5 is a corollary of the main theorems of this paper.

### 3 Applications of main theorems

Fix an \( n \)-lattice path \( L = (x_i, y_i)_{i=1}^n \). For any \( i \in \{0, 1, \cdots, n\} \), let

\[
N_i(L) = |\{\sigma \in \mathfrak{S}_n \mid p(L_\sigma) = i\}|
\]

\[
W_i(L) = |\{\sigma \in \mathfrak{S}_n \mid m(L_\sigma) = i\}|
\]

Andersen [2] first proved the following fundamental theorem in the fluctuation theory:

**Theorem 3.1 (Andersen [2])** Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path. Then we have

\[
N_i(L) = W_i(L)
\]

for any \( i \in \{0, 1, \cdots, n\} \).


Note that the sets \( N_i(L) \) and \( W_i(L) \) are independent of the \( x \)-coordinates \( x_i \) for any \( i \in \{1, 2, \cdots, n\} \) by their definitions. Let

\[
N_i(L) = |\{\sigma \in \mathfrak{S}_n \mid \sum_{j \in P(L_\sigma)} x_{\sigma(j)} = i\}|
\]

and

\[
W_i(L) = |\{\sigma \in \mathfrak{S}_n \mid \sum_{j=1}^{m(L_\sigma)} x_{\sigma(j)} = i\}|
\]

Using Baxter’s bijection method [3], we can obtain the following results.
**Theorem 3.2** Let \( L = (x_i, y_i)_{i=1}^n \) be an \( n \)-lattice path with end point \((a_n, s_n)\). Then we have 
\[ N_i(L) = W_i(L) \]
for any \( i \in \{0, 1, \ldots, a_n\} \).

**Proof.** Baxter’s bijection \( \phi = \phi_L : \mathfrak{S}_n \leftrightarrow \mathfrak{S}_n \) is defined as follows: for any \( \sigma \in \mathfrak{S}_n \), suppose that 
\[ P(L_\sigma) = \{i_1, i_2, \ldots, i_k\} \]
with \( i_1 < i_2 < \cdots < i_k \) and 
\[ \{1, 2, \ldots, n\} \setminus P(L_\sigma) = \{j_1, j_2, \ldots, j_{n-k}\} \]
with \( j_1 < j_2 < \cdots < j_{n-k} \). Let 
\[ \phi(\sigma) = (\sigma(i_k)\sigma(i_{k-1})\cdots\sigma(i_1)\sigma(j_1)\cdots\sigma(j_{n-k})). \]
Then \( m(L_{\phi(\sigma)}) = |P(L_\sigma)| = k \). Furthermore, we have 
\[ \sum_{s \in P(L_\sigma)} x_{\sigma(s)} = x_{\sigma(i_1)} + \cdots + x_{\sigma(i_k)} = \sum_{s=1}^{m(L_{\phi(\sigma)})} x_{\phi(\sigma)(s)}. \]
Hence, we have \( N_i(L) = W_i(L) \) for any \( i \in \{0, 1, \ldots, a_n\} \).

**Lemma 3.3** \( |\mathfrak{S}_H(L)| = |\mathfrak{S}_T(L)| = (n-1)!a_n \).

**Proof.** Let 
\[ \mathfrak{S}_H^i(L) = \{[\sigma,j] \in \mathfrak{S}_H(L) \mid \sigma(1) = i\} \]
Note that 
\[ |\mathfrak{S}_H^i(L)| = \sum_{\sigma : \sigma(1) = i} x_i = (n-1)!x_i \]
Hence, 
\[ |\mathfrak{S}_H(L)| = \sum_{i=1}^n |\mathfrak{S}_H^i(L)| = \sum_{i=1}^n (n-1)!x_i = (n-1)!a_n. \]
Similarly, we can prove that \( |\mathfrak{S}_T(L)| = (n-1)!a_n \).

A partition of a given set is called uniform if all partition classes have the same cardinality. Many uniform partitions of combinatorial structures are consequences of Theorem 1.10 and Theorem 1.11. As applications of Theorem 1.10 and Theorem 1.11, we give uniform partitions of \( \mathfrak{S}_T(L) \) and \( \mathfrak{S}_H(L) \).

Let \( \sigma, \tau \in \mathfrak{S}_n \). We say that \( \sigma \) and \( \tau \) are cyclically equivalent, denoted by \( \sigma \sim \tau \), if there is an index \( i \in \{1, 2, \ldots, n\} \) such that 
\[ \tau = (\sigma(i) \cdots \sigma(n)\sigma(1) \cdots \sigma(i-1)). \]
So, for a given pair \([\sigma, j] \in S_T(L)\), let
\[
EQ_T(\sigma, j) = \{[\tau, k] \in S_T(L) \mid \tau \sim \sigma\}
\]
and for a given pair \([\sigma, j] \in S_H(L)\), let
\[
EQ_H(\sigma, j) = \{[\tau, k] \in S_H(L) \mid \tau \sim \sigma\}.
\]

Then we have the following lemma:

**Lemma 3.4**

(1) \(|EQ_T(\sigma, j)| = a_n\) for any \([\sigma, j] \in S_T(L)\).

(2) \(|EQ_H(\sigma, j)| = a_n\) for any \([\sigma, j] \in S_H(L)\).

Furthermore, for any \(i \in \{0, 1, \ldots, a_n\}\), let
\[
\mathcal{N}_{i,H}(L) = \{[\sigma, j] \in S_H(L) \mid p(L_{\sigma, j}) = i\},
\]
\[
\mathcal{W}_{i,H}(L) = \{[\sigma, j] \in S_H(L) \mid m(L_{\sigma, j}) = i\},
\]
\[
\mathcal{N}_{i,T}(L) = \{[\sigma, j] \in S_T(L) \mid p(L_{\sigma, j}) = i\},
\]
\[
\mathcal{W}_{i,T}(L) = \{[\sigma, j] \in S_T(L) \mid m(L_{\sigma, j}) = i\}.
\]

**Lemma 3.5**

Let \(L = (x_i, y_i)_{i=1}^n\) be an n-lattice path with end point \((a_n, s_n)\). Suppose \(s_n > 0\) and
\[
\mathcal{N}_{a_n,T}(L) = \mathcal{W}_{a_n,H}(L) = (n - 1)!.
\]
Then for any nonempty proper subset \(I\) of \(\{1, 2, \ldots, n\}\), we have
\[
\sum_{i \in I} y_i \notin (0, s_n).
\]

**Proof.** Let \([\sigma, j] \in S_T(L)\). It is well known that there are at least one element \([\tau, k] \in EQ_T(\sigma, j)\) such that \(p(L_\tau, k) = a_n\). Thus \(\mathcal{N}_{a_n,T}(L) \geq (n - 1)!\). Similarly, we have \(\mathcal{W}_{a_n,H}(L) \geq (n - 1)!\).

Assume that there exists a nonempty proper subset \(I\) of \(\{1, 2, \cdots, n\}\) such that \(0 < \sum_{i \in I} y_i < s_n\).

Let
\[
A = \{i \in I \mid y_i \leq 0\},
\]
\(a = |A|\) and \(j = |I|\). Suppose
\[
I = \{i_1, \ldots, i_a, i_{a+1}, \ldots, i_j\},
\]
where \(i_k \in A\) for every \(k \in \{1, 2, \cdots, a\}\). Let \(J = \{1, 2, \cdots, n\} \setminus I\), \(B = \{i \in J \mid y_i \leq 0\}\) and \(b = |B|\). Suppose
\[
J = \{i_{j+1}, \ldots, i_{j+b}, i_{j+b+1}, \ldots, i_n\},
\]
where \(i_{j+k} \in B\) for every \(k \in \{1, 2, \cdots, b\}\). Let \(\sigma\) be a permutation in \(S_n\) such that
\[
\sigma(n + 1 - k) = i_k
\]
for any $k \in \{1, 2, \ldots, n\}$ and

$$\tau = \sigma(n-j+1) \cdots \sigma(n) \sigma(1) \cdots \sigma(n-j).$$

So, $[\sigma, 0] \sim [\tau, 0]$. Note that

$$0 < \sum_{k=1}^{j} y_{\sigma(n+1-k)} = \sum_{i \in I} y_i < s_n$$

Thus we have $p(L_\sigma, 0) = p(L_\tau, 0) = a_n$, a contradiction.

Let $\sigma'$ be a permutation in $S_n$ such that

$$\sigma'(k) = i_k$$

for any $k \in \{1, 2, \ldots, n\}$ and

$$\tau' = \sigma'(j+1) \cdots \sigma'(n) \sigma'(1) \cdots \sigma'(j).$$

So, $\sigma' \sim \tau'$. Note that

$$0 < \sum_{k=1}^{j} y_{\sigma'(k)} = \sum_{i \in I} y_i < s_n$$

Thus we have $m(L_{\sigma'}, 0) = m(L_{\tau'}, 0) = a_n$, a contradiction. \hfill $\blacksquare$

**Theorem 3.6** Let $L = (x_i, y_i)_{i=1}^{n}$ be an $n$-lattice path with end point $(a_n, s_n)$. Suppose $s_n > 0$. Then for any $i \in \{1, 2, \ldots, a_n\}$, the necessary and sufficient conditions for

$$\mathcal{N}_{i,T}(L) = \mathcal{W}_{i,H}(L) = (n-1)!$$

are $\sum_{i \in I} y_i \notin (0, s_n)$ for any nonempty proper subset $I$ of $\{1, 2, \ldots, n\}$.

**Proof.** Suppose

$$\sum_{i \in I} y_i \notin (0, s_n)$$

for any nonempty proper subset $I$ of $\{1, 2, \ldots, n\}$. For any a pair $[\sigma, j] \in \mathcal{G}_T(L)$, by Theorems 2.11, every set $EQ_T(\sigma, j)$ contains exactly one pair $[\tau, k]$ such that $p(L_\tau, k) = i$. Hence,

$$\mathcal{N}_{i,T}(L) = \frac{(n-1)!a_n}{a_n} = (n-1)!.$$ 

For any a pair $[\sigma, j] \in \mathcal{G}_H(L)$, by Theorems 2.4, every set $EQ_H(\sigma, j)$ contains exactly one pair $[\tau, k]$ such that $m(L_\tau, k) = i$. Hence,

$$\mathcal{W}_{i,H}(L) = \frac{(n-1)!a_n}{a_n} = (n-1)!.$$ 

Suppose

$$\mathcal{N}_{i,T}(L) = \mathcal{W}_{i,H}(L) = (n-1)!$$

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for any $i \in \{1, 2, \ldots, a_n\}$. Particularly,

$$
\mathcal{N}_{a_n,T}(L) = \mathcal{W}_{a_n,H}(L) = (n-1)!
$$

By Lemma 3.5, we have

$$
\sum_{i \in I} y_i \notin (0, s_n)
$$

for any nonempty proper subset $I$ of $\{1, 2, \ldots, n\}$.

As a consequence of Theorem 3.6, we have proved the following theorem:

**Theorem 3.7** Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$. Suppose $s_n > 0$. Then

$$
\mathcal{N}_{a_n,T}(L) = \mathcal{W}_{a_n,H}(L) = (n-1)!
$$

if and only if

$$
\mathcal{N}_{i,T}(L) = \mathcal{W}_{i,H}(L) = (n-1)!
$$

for any $i \in \{1, 2, \ldots, a_n - 1\}$.

**Lemma 3.8** Let $L = (x_i, y_i)_{i=1}^n$ be an $n$-lattice path with end point $(a_n, s_n)$. Suppose $s_n \leq 0$ and

$$
\mathcal{N}_{0,T}(L) = \mathcal{W}_{0,T}(L) = (n-1)!
$$

Then we have

$$
\sum_{i \in I} y_i \notin [s_n, 0]
$$

for any nonempty proper subset $I$ of $\{1, 2, \ldots, n\}$.

**Proof.** Let $[\sigma, j] \in \mathcal{G}_T(L)$. It is well known that there are at least one element $[\tau, k] \in EQ_T(\sigma, j)$ such that $p(L_\tau, k) = 0$. Thus $\mathcal{N}_{0,T}(L) \geq (n-1)!$. Similarly, we have $\mathcal{W}_{0,T}(L) \geq (n-1)!$.

Assume that there exists a nonempty proper subset $I$ of $\{1, 2, \ldots, n\}$ such that $s_n \leq \sum_{i \in I} y_i \leq 0$. Let

$$
A = \{i \in I \mid y_i \leq 0\},
$$

$a = |A|$ and $j = |I|$. Suppose

$$
I = \{i_1, \ldots, i_a, i_{a+1}, \ldots, i_j\},
$$

where $i_k \in A$ for every $k \in \{1, 2, \ldots, a\}$. Let $J = \{1, 2, \ldots, n\} \setminus I$, $B = \{i \in J \mid y_i \leq 0\}$ and $b = |B|$. Suppose

$$
J = \{i_{j+1}, \ldots, i_{j+b}, i_{j+b+1}, \ldots, i_n\},
$$

where $i_{j+k} \in B$ for every $k \in \{1, 2, \ldots, b\}$. Let $\sigma$ be a permutation in $\mathcal{G}_n$ such that

$$
\sigma(k) = i_k
$$
for any \( k \in \{1, 2, \cdots, n\} \) and
\[
\tau = \sigma(j + 1) \cdots \sigma(n) \sigma(1) \cdots \sigma(j).
\]
So, \( \sigma \sim \tau \). Note that
\[
s_n \leq \sum_{k=1}^{j} y_{\sigma(k)} = \sum_{i \in I} y_i \leq 0.
\]
Thus we have \( m(L_{\sigma}, 0) = m(L_{\tau}, 0) = 0 \) and \( p(L_{\sigma}, 0) = p(L_{\tau}, 0) = 0 \), a contradiction. \( \blacksquare \)

**Theorem 3.9** Let \( L = (x_i, y_i)_{i=1}^{n} \) be an \( n \)-lattice path with end point \( (a_n, s_n) \). Suppose \( s_n \leq 0 \). Then for any \( i \in \{0, 1, \cdots, a_n - 1\} \), the necessary and sufficient conditions for
\[
\mathcal{N}_{i,T}(L) = \mathcal{W}_{i,T}(L) = (n-1)!
\]
determine
\[
\sum_{i \in I} y_i \notin [s_n, 0]
\]
for any nonempty proper subset \( I \) of \( \{1, 2, \cdots, n\} \).

**Proof.** Suppose \( \sum_{i \in I} y_i \notin [s_n, 0] \) for any nonempty proper subset \( I \) of \( \{1, 2, \cdots, n\} \). For any a pair \([\sigma, j] \in \mathcal{S}_T(L)\), by Theorems 2.13, every set \( \mathcal{E}_{T}(\sigma, j) \) contains exactly one pair \([\tau, k]\) such that \( p(L_{\tau}, k) = i \). Hence,
\[
\mathcal{N}_{i,T}(L) = \frac{(n-1)!a_n}{a_n} = (n-1)!
\]
By Theorems 2.8, there exists a unique element \([\tau, k]\) in \( \mathcal{E}_{T}(\sigma, j) \) such that \( m(L_{\tau}, k) = i \). Hence,
\[
\mathcal{W}_{i,T}(L) = \frac{(n-1)!a_n}{a_n} = (n-1)!
\]
Suppose
\[
\mathcal{N}_{i,T}(L) = \mathcal{W}_{i,T}(L) = (n-1)!
\]
for any \( i \in \{0, 1, \cdots, a_n - 1\} \). Particularly,
\[
\mathcal{N}_{0,T}(L) = \mathcal{W}_{0,T}(L) = (n-1)!
\]
By Lemma 3.8, we have\[ \sum_{i \in I} y_i \notin [s_n, 0] \]
for any nonempty proper subset \( I \) of \( \{1, 2, \cdots, n\} \).

As a consequence of Theorem 3.9, we have proved the following theorem:

**Theorem 3.10** Let \( L = (x_i, y_i)_{i=1}^{n} \) be an \( n \)-lattice path with end point \( (a_n, s_n) \). Suppose \( s_n \leq 0 \). Then
\[
\mathcal{N}_{0,T}(L) = \mathcal{W}_{0,T}(L) = (n-1)!
\]
if and only if for any \( i \in \{1, \cdots, a_n - 1\} \), we have
\[
\mathcal{N}_{i,T}(L) = \mathcal{W}_{i,T}(L) = (n-1)!
\]
Similar to the proofs of Theorem 3.6 and Theorem 3.9, Chung-Feller Theorem proved by using Theorem 1.11(2).

**Theorem 3.11** (Chung and Feller [11]) Let \( \mathcal{D} \) be the set of \((2n + 1)\)-lattice paths

\[ L = (1, y_1)(1, y_2) \cdots (1, y_{2n+1}) \]

such that \( s_{2n+1} = 1 \) and \( y_i \in \{1, -1\} \) for all \( i \in \{1, 2, \cdots, 2n + 1\} \). Let

\[ \mathcal{D}_i = \{ L \in \mathcal{D} \mid p(L) = i \} \]

for any \( i \in \{1, 2, \cdots, 2n + 1\} \). Then

\[ |\mathcal{D}_i| = \frac{1}{2n + 1} \binom{2n + 1}{n} \]

for any \( i \in \{1, 2, \cdots, 2n + 1\} \).


Moreover, Woan [25] presented another uniform partition of the set \( \mathcal{D} \).

**Theorem 3.12** (Woan [25]) Let \( \mathcal{D} \) be the set of \((2n + 1)\)-lattice paths

\[ L = (1, y_1)(1, y_2) \cdots (1, y_{2n+1}) \]

such that \( s_{2n+1} = 1 \) and \( y_i \in \{1, -1\} \) for all \( i \in \{1, 2, \cdots, 2n + 1\} \). Let

\[ \mathcal{D}'_i = \{ L \in \mathcal{D} \mid m(L) = i \} \]

for any \( i \in \{0, 1, \cdots, 2n\} \). Then

\[ |\mathcal{D}'_i| = \frac{1}{2n + 1} \binom{2n + 1}{n} \]

for any \( i \in \{0, 1, \cdots, 2n\} \).

Eu, Liu and Yeh [10] obtained the following results:
Theorem 3.13 \textit{(Eu, Liu and Yeh [10])} Let $B$ be the set of $(n+1)$-lattice paths

$$L = (1, y_1)(1, y_2) \cdots (1, y_{n+1})$$

such that $s_{n+1} = 1$ and $y_i \in \{1, 0, -1\}$ for all $i \in \{1, 2, \cdots , n+1\}$. Let

$$B_i = \{ L \in B \mid m(L) = i \}$$

for any $i \in \{0, 1, 2, \cdots , n\}$. Then

$$B_i = \frac{1}{n+1} |B|$$

for any $i \in \{0, 1, \cdots , n\}$.

Theorem 3.12 and Theorem 3.13 can be proved by using Theorem 1.10(1) and methods similar to those in the proofs of Theorem 3.6 and Theorem 3.9.

References


[6] D. Callan, Why are these equal? \texttt{http://www.stat.wisc.edu/~callan/notes/}


