ENUMERATION OF PERMUTATIONS BY NUMBER OF ALTERNATING DESCENTS

SHI-MEI MA AND YEONG-NAN YEH

Abstract. In this paper we present an explicit formula for the number of permutations with a given number of alternating descents. As an application, we obtain an interlacing property for the zeros of alternating Eulerian polynomials.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. For $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$, we define a *descent* to be an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. Let $\operatorname{des}(\pi)$ be the number of descents of π. Then the equations

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{des}(\pi)} = \sum_{k=0}^{n-1} A(n,k)x^k.$$

define the *Eulerian polynomials* $A_n(x)$ and *Eulerian numbers* $A(n,k)$. The exponential generating function for $A_n(x)$ is

$$A(x,z) = \sum_{n\geq 1} A_n(x)\frac{z^n}{n!} = \frac{e^{(1-x)z}-1}{1-xe^{(1-x)z}}.$$

The numbers $A(n,k)$ satisfy the recurrence relation

$$A(n+1,k) = (k+1)A(n,k) + (n-k+1)A(n,k-1).$$

with the initial conditions $A(1,0) = 1$ and $A(1,k) = 0$ for $k \geq 1$ (see [13, A008292]). There is a large literature devoted to the descent statistic and its variations (see [1, 7, 11] for instance).

As a variation of the descent statistic, the number of *alternating descents* of a permutation $\pi \in \mathfrak{S}_n$ is defined by

$$\operatorname{altdes}(\pi) = |\{2i : \pi(2i) < \pi(2i+1)\} \cup \{2i+1 : \pi(2i+1) > \pi(2i+2)\}|.$$

We say that π has a *3-descent* at index i if $\pi(i)\pi(i+1)\pi(i+2)$ has one of the patterns: 132, 213, or 321. Chebikin [3] showed that the alternating descent statistic of permutations in $\mathfrak{S}_n$ is equidistributed with the 3-descent statistic of permutations in $\{\pi \in \mathfrak{S}_{n+1} : \pi_1 = 1\}$. Then the equations

$$\widehat{A}_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{altdes}(\pi)} = \sum_{k=0}^{n-1} \widehat{A}(n,k)x^k$$

2010 *Mathematics Subject Classification.* Primary 05A15; Secondary 26C10.

Key words and phrases. Alternating Eulerian polynomials; Derivative polynomials; Zeros.

define the *alternating Eulerian polynomials* $\widehat{A}_n(x)$ and the *alternating Eulerian numbers* $\widehat{A}(n,k)$. The first few $\widehat{A}_n(x)$ are given as follows:

$$\begin{aligned}
\widehat{A}_1(x) &= 1,\\
\widehat{A}_2(x) &= 1+x,\\
\widehat{A}_3(x) &= 2+2x+2x^2,\\
\widehat{A}_4(x) &= 5+7x+7x^2+5x^3,\\
\widehat{A}_5(x) &= 16+26x+36x^2+26x^3+16x^4.
\end{aligned}$$

The bijection $\pi \mapsto \pi^c$ on $\mathfrak{S}_n$ defined by $\pi^c(i) = n+1-\pi(i)$ shows that $\widehat{A}_n(x)$ is symmetric. Chebikin [3] proved the following formulas:

$$\sum_{n\geq 1} \widehat{A}_n(x)\frac{z^n}{n!} = \frac{\sec(1-x)z + \tan(1-x)z - 1}{1 - x(\sec(1-x)z + \tan(1-x)z)}; \tag{1}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\widehat{A}(i,j+1)\widehat{A}(n-i,k-j+1) = (n+1-k)\widehat{A}(n,k+1) + (k+1)\widehat{A}(n,k+2).$$

In recent years, several authors pay attention to the alternating descent statistic and its associated permutation statistics. The reader is referred to [2, 4, 12] for recent progress on this subject. For example, Gessel and Zhuang [4] defined an alternating run to be a maximal consecutive subsequence with no alternating descents. This paper is a continuation of [9]. In Section 2, we express the polynomials $\widehat{A}_n(x)$ in terms of the *derivative polynomials* $P_n(x)$ defined by Hoffman [6]:

$$P_n(\tan\theta) = \frac{d^n}{d\theta^n}\tan\theta.$$

2. An explicit formula

Let D denote the differential operator $d/d\theta$. Set $x = \tan\theta$. Then $D(x^n) = nx^{n-1}(1+x^2)$ for $n \geq 1$. Thus $D^n(x)$ is a polynomial in x. Let $P_n(x) = D^n(x)$. Then $P_0(x) = x$ and

$$P_{n+1}(x) = (1+x^2)P_n'(x). \tag{2}$$

Clearly, $\deg P_n(x) = n+1$. By definition, we have

$$\tan(\theta+z) = \sum_{n\geq 0} P_n(x)\frac{z^n}{n!} = \frac{x+\tan z}{1-x\tan z}, \tag{3}$$

Let $P_n(x) = \sum_{k=0}^{n+1} p(n,k)x^k$. It is easy to verify that

$$p(n,k) = (k+1)p(n-1,k+1) + (k-1)p(n-1,k-1).$$

Note that $P_n(-x) = (-1)^{n+1}P_n(x)$ and $x \| P_{2n}(x)$. Thus

$$P_n(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n,n-2k+1)x^{n-2k+1}.$$

There is an explicit formula for the numbers $p(n,n-2k+1)$.

Proposition 1 ([9, Proposition 1])**.** *For $n \geq 1$ and $0 \leq k \leq \lfloor (n+1)/2 \rfloor$, we have*

$$p(n, n-2k+1) = (-1)^k \sum_{i \geq 1} i! \left\{ {n \atop i} \right\} (-2)^{n-i} \left[\binom{i}{n-2k} - \binom{i}{n-2k+1} \right],$$

where $\left\{ {n \atop i} \right\}$ is the Stirling number of the second kind.

Now we present the first main result of this paper.

Theorem 2. *For $n \geq 1$, we have*

$$2^n (1+x^2) \widehat{A}_n(x) = (1-x)^{n+1} P_n \left(\frac{1+x}{1-x} \right). \tag{4}$$

Proof. It follows from (3) that

$$\begin{aligned} \sum_{n \geq 1} (1-x)^{n+1} P_n \left(\frac{1+x}{1-x} \right) \frac{z^n}{n!} &= (1-x) \sum_{n \geq 1} P_n \left(\frac{1+x}{1-x} \right) \frac{(z-xz)^n}{n!} \\ &= (1+x^2) \frac{2 \tan(z-xz)}{1-x-(1+x)\tan(z-xz)}. \end{aligned}$$

Comparing with (1), it suffices to show the following

$$\frac{\sec(2z-2xz) + \tan(2z-2xz) - 1}{1 - x(\sec(2z-2xz) + \tan(2z-2xz))} = \frac{2\tan(z-xz)}{1-x-(1+x)\tan(z-xz)}. \tag{5}$$

Set $t = \tan(z - xz)$. Using the tangent half-angle substitution, we have

$$\sec(2z-2xz) = \frac{1+t^2}{1-t^2}, \ \tan(2z-2xz) = \frac{2t}{1-t^2}.$$

Then the left hand side of (5) equals

$$\frac{2t(1+t)}{1-t^2-x(1+t)^2} = \frac{2t}{1-x-(1+x)t}.$$

This completes the proof. □

It follows from (2) that

$$2^n \widehat{A}_{n+1}(x) = (1-x)^n P_n' \left(\frac{1+x}{1-x} \right) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (n-2k+1) p(n, n-2k+1) (1-x)^{2k} (1+x)^{n-2k}. \tag{6}$$

Denote by $E(n,k,s)$ the coefficients x^s of $(1-x)^{2k}(1+x)^{n-2k}$. Clearly,

$$E(n,k,s) = \sum_{j=0}^{\min(\lfloor \frac{k}{2} \rfloor, s)} (-1)^j \binom{2k}{j} \binom{n-2k}{s-j}.$$

Then we get the following result.

Corollary 3. *For $n \geq 2$ and $1 \leq s \leq n$, we have*

$$\widehat{A}(n+1, s) = \frac{1}{2^n} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (n-2k+1) p(n, n-2k+1) E(n,k,s).$$

It follows from (2) and (4) that

$$2\widehat{A}_{n+1}(x)=(1+n+2x+nx^2-x^2)\widehat{A}_n(x)+(1-x)(1+x^2)\widehat{A}_n'(x)$$

for $n\geq 1$. Therefore, the numbers $\widehat{A}(n,k)$ satisfy the recurrence relation

$$2\widehat{A}(n+1,k)=(k+1)(\widehat{A}(n,k+1)+\widehat{A}(n,k-1))+(n-k+1)(\widehat{A}(n,k)+\widehat{A}(n,k-2)),$$

with initial conditions $\widehat{A}(1,0)=1$ and $\widehat{A}(1,k)=0$ for $k\geq 1$.

Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$. An *interior peak* in π is an index $i\in\{2,3,\ldots,n-1\}$ such that $\pi(i-1)<\pi(i)>\pi(i+1)$. Let $\mathrm{pk}\,(\pi)$ denote the number of interior peaks of π. Let $W_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{pk}\,(\pi)}$. Clearly, we have $\deg W_n(x)=\lfloor (n-1)/2\rfloor$. It is well known that the polynomials $W_n(x)$ satisfy the recurrence relation

$$W_{n+1}(x)=(nx-x+2)W_n(x)+2x(1-x)W_n'(x),$$

with initial conditions $W_1(x)=1, W_2(x)=2$ and $W_3(x)=4+2x$. By the theory of *enriched P-partitions*, Stembridge [14, Remark 4.8] showed that

$$W_n\left(\frac{4x}{(1+x)^2}\right)=\frac{2^{n-1}}{(1+x)^{n-1}}A_n(x). \tag{7}$$

From [8, Theorem 2], we have

$$P_n(x)=x^{n-1}(1+x^2)W_n(1+x^{-2}). \tag{8}$$

Therefore, by (4) and (8), we get a counterpart of (7):

$$W_n\left(\frac{2+2x^2}{(1+x)^2}\right)=\frac{2^{n-1}}{(1+x)^{n-1}}\widehat{A}_n(x).$$

Thus

$$\widehat{A}_n(x)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1}{2^{n-k-1}}W(n,k)(1+x)^{n-2k-1}(1+x^2)^k,$$

which implies that $(x+1)\|\widehat{A}_{2n}(x)$ for $n\geq 1$.

3. Zeros of the alternating Eulerian polynomials

It is well known that the Eulerian polynomial $A_n(x)$ has only real zeros, and the zeros of $A_n(x)$ separates that of $A_{n+1}(x)$ (see Bóna [1, p. 24] for instance). Now we present a corresponding result for $\widehat{A}_n(x)$.

Theorem 4. *For any $n\geq 1$, all the zeros of $\widehat{A}_{2n+1}(x)$ and $\widehat{A}_{2n+2}(x)/(1+x)$ are non-real complexes with multiplicity 1, and the moduli of the zeros of $\widehat{A}_n(x)$ are all equal to 1. Furthermore, the sequence of real parts of the zeros of $\widehat{A}_n(x)$ separates that of $\widehat{A}_{n+1}(x)$. More precisely, let $\{r_j\pm\ell_j\mathrm{i}\}_{j=1}^{n-1}$ be zeros of $\widehat{A}_{2n}(x)/(1+x)$, let $\{s_j\pm t_j\mathrm{i}\}_{j=1}^{n}$ be zeros of $\widehat{A}_{2n+1}(x)$ and let $\{p_j\pm q_j\mathrm{i}\}_{j=1}^{n}$ be zeros of $\widehat{A}_{2n+2}(x)/(1+x)$, where $-1<r_1<r_2<\cdots<r_{n-1}<0$, $-1<s_1<s_2<\cdots<s_n<0$ and $-1<p_1<p_2<\cdots<p_n<0$. Then*

$$-1<s_1<r_1<s_2<r_2<\cdots<r_{n-1}<s_n, \tag{9}$$

$$-1<s_1<p_1<s_2<p_2<\cdots<s_n<p_n. \tag{10}$$

Moreover, the sequence of imaginary parts of the zeros of $\widehat{A}_n(x)$ also separates that of $\widehat{A}_{n+1}(x)$.

Proof. Set $\widetilde{P}_n(x) = \mathrm{i}^{n-1}P_n(\mathrm{i}x)$. Then

$$\widetilde{P}_{n+1}(x) = (1-x^2)\widetilde{P}_n'(x).$$

Using [10, Theorem 2], we get that the polynomials $\widetilde{P}_n(x)$ have only real zeros, belong to $[-1,1]$ and the sequence of zeros of $\widetilde{P}_n(x)$ separates that of $\widetilde{P}_{n+1}(x)$. From [5, Corollary 8.7], we see that the zeros of the derivative polynomials $P_n(x)$ are pure imaginary with multiplicity 1, belong to the line segment $[-\mathrm{i},\mathrm{i}]$. In particular, $(1+x^2)\|P_n(x)$. Therefore, the polynomials $P_{2n+1}(x)$ and $P_{2n+2}(x)$ have the following expressions:

$$P_{2n+1}(x) = (1+x^2)\prod_{i=1}^{n}(x^2+a_i), P_{2n+2}(x) = x(1+x^2)\prod_{i=1}^{n}(x^2+b_i),$$

where

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < 1. \tag{11}$$

By (4), we obtain

$$2^{2n}\widehat{A}_{2n+1}(x) = \prod_{i=1}^{n}((1+x)^2 + a_i(1-x)^2),$$
$$2^{2n+1}\widehat{A}_{2n+2}(x) = (1+x)\prod_{i=1}^{n}((1+x)^2 + b_i(1-x)^2).$$

Hence

$$2^{2n}\widehat{A}_{2n+1}(x) = \prod_{i=1}^{n}(1+a_i)\left(x + \frac{1-a_i}{1+a_i} + \frac{2\mathrm{i}\sqrt{a_i}}{1+a_i}\right)\left(x + \frac{1-a_i}{1+a_i} - \frac{2\mathrm{i}\sqrt{a_i}}{1+a_i}\right),$$
$$2^{2n+1}\widehat{A}_{2n+2}(x) = (1+x)\prod_{i=1}^{n}(1+b_i)\left(x + \frac{1-b_i}{1+b_i} + \frac{2\mathrm{i}\sqrt{b_i}}{1+b_i}\right)\left(x + \frac{1-b_i}{1+b_i} - \frac{2\mathrm{i}\sqrt{b_i}}{1+b_i}\right).$$

Set

$$s_j = -\frac{1-a_j}{1+a_j},\ p_j = -\frac{1-b_j}{1+b_j},$$

By (11), we get (10). Along the same lines, we get (9). Note that

$$\left(\frac{1-a_i}{1+a_i}\right)^2 + \left(\frac{2\sqrt{a_i}}{1+a_i}\right)^2 = \left(\frac{1-b_i}{1+b_i}\right)^2 + \left(\frac{2\sqrt{b_i}}{1+b_i}\right)^2 = 1.$$

Then the moduli of zeros of $\widehat{A}_n(x)$ are all equal to 1, which also implies the interlacing property of the imaginary parts of these zeros. □

Acknowledgements.

This work was supported by NSFC (11401083) and the Fundamental Research Funds for the Central Universities (N130423010).

References

[1] M. Bóna, Combinatorics of Permutations, second edition, CRC Press, Boca Raton, FL, 2012.

[2] D. Callan, S.-M. Ma and T. Mansour, Some combinatorial arrays related to the Lotka-Volterra system, Electron. J. Combin. 22(2) (2015) #P2.22.

[3] D. Chebikin, Variations on descents and inversions in permutations, Electron. J. Combin. 15 (2008) #R132.

[4] I.M. Gessel, Y. Zhuang, Counting Permutations by alternating descents, Electron. J. Combin. 21(4) (2014) #P4.23.

[5] G. Hetyei, Tchebyshev triangulations of stable simplicial complexes, J. Combin. Theory Ser. A 115 (2008) 569–592.

[6] M.E. Hoffman, Derivative polynomials for tangent and secant, Amer. Math. Monthly 102 (1995) 23–30.

[7] K. Kurşungöz, A.J. Yee, Alternating permutations and the mth descents, Discrete Math. 311 (2011) 2610–2622.

[8] S.-M. Ma, Derivative polynomials and enumeration of permutations by number of interior and left peaks, Discrete Math. 312 (2012) 405–412.

[9] S.-M. Ma, An explicit formula for the number of permutations with a given number of alternating runs, J. Combin. Theory Ser. A 119 (2012) 1660–1664.

[10] S.-M. Ma, Y. Wang, q-Eulerian polynomials and polynomials with only real zeros, Electron. J. Combin. 15 (2008) #R17.

[11] I. Mezö, A kind of Eulerian numbers connected to Whitney numbers of Dowling lattices, Discrete Math. 328 (2014) 88–95.

[12] J.B. Remmel, Generating Functions for Alternating Descents and Alternating Major Index, Ann. Comb. 16 (2012) 625–650.

[13] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

[14] J. Stembridge, Enriched P-partitions, Trans. Amer. Math. Soc. 349(2): (1997) 763–788.

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P.R. China

E-mail address: shimeimapapers@gmail.com (S.-M. Ma)

Institute of Mathematics, Academia Sinica, Taipei, Taiwan

E-mail address: mayeh@math.sinica.edu.tw (Y.-N. Yeh)