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Journal Name	Journal of Combinatorial Optimization	
Corresponding Author	Family Name	**Zhang**
	Particle	
	Given Name	**Xiao-Dong**
	Suffix	
	Division	Department of Mathematics, MOE-LSC, and SHL-MAC
	Organization	Shanghai Jiao Tong University
	Address	800 Dongchuan Road, Shanghai, 200240, China
	Email	xiaodong@sjtu.edu.cn
Author	Family Name	**Jin**
	Particle	
	Given Name	**Ya-Lei**
	Suffix	
	Division	Department of Mathematics, MOE-LSC, and SHL-MAC
	Organization	Shanghai Jiao Tong University
	Address	800 Dongchuan Road, Shanghai, 200240, China
	Email	
Author	Family Name	**Yeh**
	Particle	
	Given Name	**Yeong-Nan**
	Suffix	
	Division	Institute of Mathematics
	Organization	Academia Sinica
	Address	Taipei, 11529, Taiwan
	Email	

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Abstract

Let $\mathcal{L}(T,\lambda)=\sum_{k=0}^{n}(-1)^k c_k(T)\lambda^{n-k}$ be the characteristic polynomial of its Laplacian matrix of a tree *T*. This paper studied some properties of the generating function of the coefficients sequence $(c_0,\ldots,c_n)$ which are related with the matching polynomials of division tree of *T*. These results, in turn, are used to characterize all extremal trees having the minimum Laplacian coefficient generation function and the minimum incidence energy of trees with described maximum degree, respectively.

Keywords (separated by '-') Laplacian coefficient - Matching polynomial - Incidence energy - Tree - Subdivision tree

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Laplacian coefficient, matching polynomial and incidence energy of trees with described maximum degree

Ya-Lei Jin[1] · Yeong-Nan Yeh[2] · Xiao-Dong Zhang[1]

Abstract Let $\mathcal{L}(T,\lambda)=\sum_{k=0}^{n}(-1)^k c_k(T)\lambda^{n-k}$ be the characteristic polynomial of its Laplacian matrix of a tree T. This paper studied some properties of the generating function of the coefficients sequence $(c_0,\ldots,c_n)$ which are related with the matching polynomials of division tree of T. These results, in turn, are used to characterize all extremal trees having the minimum Laplacian coefficient generation function and the minimum incidence energy of trees with described maximum degree, respectively.

Keywords Laplacian coefficient · Matching polynomial · Incidence energy · Tree · Subdivision tree

Mathematics Subject Classification 05C25 · 05C50

1 Introduction

Let $G=(V(G),E(G))$ be a simple graph with vertex set $V(G)=\{v_1,\ldots,v_n\}$ and edge set $E(G)$. Let $A(G)=(a_{ij})$ and $D(G)=(d(v_1),\ldots,d(v_n))$ be its adjacency and degree diagonal matrices, respectively. Then the *Laplacian matrix* of G is defined to be $L(G)=D(G)-A(G)$. *The Laplacian polynomial* $\mathcal{L}(G,\lambda)$ of G is the characteristic polynomial of its Laplacian matrix $L(G)$, i.e.,

✉ Xiao-Dong Zhang
xiaodong@sjtu.edu.cn

1 Department of Mathematics, MOE-LSC, and SHL-MAC, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, China

2 Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan

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$$\mathcal{L}(G,\lambda) = det(\lambda I_n - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)\lambda^{n-k}. \quad (1)$$

It is well known that $c_0(G) = 1$, $c_n(G) = 0$, $c_1(G) = 2|E(G)|$ and $c_{n-1} = n\tau(G)$, where $\tau(G)$ is the number of the spanning trees. In addition,

$$\varphi(T,x) = c_0 + c_1 x + \cdots c_{n-1}x^{n-1} \quad (2)$$

is called the *Laplacian coefficient generation function* of T. Mohar (2007) proposed a new notation of poset consisting of all trees with Laplacian coefficients. Let $(\mathcal{T}_n, \preceq)$ be a poset consisting of all trees of order n with $\preceq$, where $T_1 \preceq T_2$, if $(c_0(T_1), \ldots, c_{n-1}(T_1)) \leq (c_0(T_2), \ldots, c_{n-1}(T_2))$, i.e., $c_i(T_1) \leq c_i(T_2)$ for $i = 0, \ldots, n-1$. Moreover, write $T_1 \prec T_2$ if $T_1 \preceq T_2$ and there exists a k with $c_k(T_1) < c_k(T_2)$. Further, he established the monotone relations under two graph operations, which presents a strengthening of Zhou and Gutman's result (Zhou and Gutman 2008) that $(\mathcal{T}_n, \preceq)$ has a unique maximal element the path P_n and a unique minimal element the star $K_{1,n-1}$. Besides, Mohar (2007) also proposed some problems on how to order trees with the Laplacian coefficients. In addition, Ilić (2010) determined the extremal tree which has minimal Laplacian coefficients in all $n-$vertex trees with a fixed matching number. Stevanović and Ilić (2009) characterized the minimum and maximum elements in the poset of unicyclic graphs of order n with $\preceq$. Tan (2011) proved that the poset of unicyclic graphs of order n and fixed matching number with $\preceq$ has only one minimal element. The study on the Laplacian coefficients has attracted more and more attention. The readers are referred to Heuberger and Wagner (2008), Heuberger and Wagner (2009), Zhang et al. (2009) and references therein.

Let $I(G)$ be the vertex-edge incidence matrix, i.e., an $(n \times m)$-matrix whose (i,j)-entry is 1 if the vertex v_i is incident to the edge e_j , and 0 otherwise. Then *incidence energy* $IE(G)$ (see Nikiforov 2007, Gutman et al. 2009 or Jooyandeh et al. 2009) of G is defined to be the sum of the singular values of $I(G)$, i.e., the sum of the square roots of all eigenvalues of $I(G)I(G)^T$. On the other hand, the extremal trees with the minimal Wiener index of trees with maximum degree Δ has attracted considerable attention. Liu et al. (2000), Fischermann et al. (2002) and Jelen and Triesch (2003) independently determined all trees which have the minimum Wiener indices among all trees of order n and maximum degree Δ by different approaches. Kirk and Wang (2008) studied the number of subtrees of a tree with the maximum degree. Zhang (2008) characterized the extremal tree with the maximum Laplacian spectral radius among all trees of order n with the maximum degree. These results motivate us to consider the following problem in this paper.

Problem 1.1 Characterize all minimal elements in the poset $(\mathcal{T}_{n,d+1}, \preceq)$, where $\mathcal{T}_{n,d+1}$ is the set of all trees with the maximum degree $d+1$.

In order to analyze this problem, some more notations are introduced. *A rooted $d-$ary tree* is a rooted tree of which every vertex has 0 or d children. The (rooted) complete $d-$ary tree of height $h-1$, denoted by C_h, is a rooted $d-$ary tree such that the height of each pendent vertex is $h-1$. Then C_1 consists of a single vertex and the

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root of C_h has d branches which are C_{h-1}. Moreover, the degree of the root in rooted complete $d-$ary tree C_h is d.

Definition 1.2 A rooted tree T with the root v_0 and the maximum degree $d+1$ is called $(d+1)$-*greedy tree*, denoted by T^*_{d+1}, if the following properties have been satisfied:

(1) the degree of v_0 is $d+1$, i.e., $deg(v_0) = d+1$.
(2) The height of any two pendent vertices of T differs by at most 1, where the height of a vertex v in T is equal to the distance between v and v_0.
(3) For any vertex v in T, there is at most one $T(u)$ is incomplete $d-$ary tree, where u is the children of v and $T(u)$ is the rooted subtree of T that is induced by u and all of its successors in T, the root of which is u.

Let $T = (V(T), E(T))$ be a tree with $V(T) = \{v_1, \ldots, v_n\}$ and $E(T) = \{e_1, \ldots, e_{n-1}\}$. The *subdivision tree* of T is defined to be a tree $S(T) = (V(S(T)), E(S(T)))$ with vertex set $V(S(T)) = V(T) \bigcup E(T)$, and v_i and e_j are adjacency in $S(T)$ if and only if v_i is incidence with e_j in T. In other words, $S(T)$ is the tree obtained from T by inserting a new vertex in each edge in T.

The main results of this paper can be stated as follows.

Theorem 1.3 *T^*_{d+1} is the unique tree with the minimum Laplacian coefficient generation function in $\mathcal{T}_{n,d+1}$, i.e. for any tree $T \in \mathcal{T}_{n,d+1}$ and $x > 0$,*

$$\varphi(T^*_{d+1}, x) \le \varphi(T, x)$$

with equality if and only if $T = T^*_{d+1}$.

Theorem 1.4 *T^*_{d+1} is the unique tree with the minimum incidence energy in $\mathcal{T}_{n,d+1}$, i.e., for any tree $T \in \mathcal{T}_{n,d+1}$,*

$$IE(T^*_{d+1}) \le IE(T)$$

with equality if and only if $T = T^*_{d+1}$.

The approach to the proof of Theorem 1.3 is different from some known technique, although the extremal trees for different graph variants such as the Wiener index, the Laplacian spectral radius, the number of subtrees among all trees of order n and the maximum degree Δ are greedy trees. The rest of this paper is organized as follows. In Sect. 2, some preliminary on the matching polynomials of tree are presented. In Sect. 3, the proof of Theorem 1.3 is presented. In Sect. 4, we give the proof of Theorem 1.4 and propose a conjecture.

2 Matching generating function

For a tree T, let $m(T, k)$ be the number of matchings of T containing precisely k edges (shortly k-matchings). It is convenient to define $m(T, 0) = 1$. Then the *matching generating function* of T is defined to

$$M(T,x)=\sum_{k\geq 0} m(T,k)x^k. \tag{3}$$

If T is a rooted tree, let $m_1(T,k)$ be the number of $k-$matchings of T saturating the root and $m_0(T,k)$ be the number of $k-$matchings of T not saturating the root. Denote by

$$M_i(T,x)=\sum_{k\geq 0} m_i(T,k)x^k, \text{ for } i\in\{0,1\}. \tag{4}$$

and

$$\tau(T,x)=\frac{M_0(T,x)}{M(T,x)}. \tag{5}$$

Clearly $M(T,x)=M_0(T,x)+M_1(T,x)$. It follows from Heuberger and Wagner (2009) that

Lemma 2.1 (Heuberger and Wagner 2009) *Let T be a rooted tree with root v and the branches $T_1,\ldots,T_k$. Then*

$$M_0(T,x)=\prod_{j=1}^{k} M(T_j,x), \tag{6}$$

$$M_1(T,x)=x\sum_{j=1}^{k}\frac{M_0(T_j,x)}{M(T_j)}\prod_{i=1}^{k} M(T_i), \tag{7}$$

$$\tau(T,x)=\frac{1}{1+x\sum_{j=1}^{k}\tau(T_j,x)}. \tag{8}$$

Lemma 2.2 *Let T_1 and T_2 be two vertex disjoint trees with roots u and v, respectively. If T is the tree obtained from T_1 and T_2 by identifying u and v, then*

$$M(T,x)=M(T_1,x)M_0(T_2,x)+M_0(T_1,x)M_1(T_2,x).$$

Proof It is easy to see that

$$m(T,k)=\sum_{i=0}^{k}\left(m(T_1,i)m_0(T_2,k-i)+m_0(T_1,i)m_1(T_2,k-i)\right).$$

Hence

$$\begin{aligned}M(T,x)&=\sum_{k\geq 0}\sum_{i=0}^{k}(m(T_1,i)m_0(T_2,k-i)+m_0(T_1,i)m_1(T_2,k-i))x^k\\&=\sum_{k\geq 0}\sum_{i=0}^{k}(m(T_1,i)m_0(T_2,k-i)x^k)+\sum_{k\geq 0}\sum_{i=0}^{k}(m_0(T_1,i)m_1(T_2,k-i)x^k)\\&=M(T_1,x)M_0(T_2,x)+M_0(T_1,x)M_1(T_2,x).\end{aligned}$$

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This completes the proof. □

Lemma 2.3 *Let T be a rooted tree with root v and the branches $T_1, \ldots, T_k$. If $S(T), S(T_1), \ldots, S(T_k)$ are the subdivision trees of $T, T_1, \ldots, T_k$, respectively, then*

$$M_0(S(T), x) = \prod_{j=1}^{k} (xM_0(S(T_j), x) + M(S(T_j), x)), \tag{9}$$

$$M_1(S(T), x) = x \sum_{j=1}^{k} \frac{M(S(T_j), x)}{xM_0(S(T_j), x) + M(S(T_j), x)} M_0(S(T), x), \tag{10}$$

$$\tau(S(T), x) = \frac{1}{1 + \sum_{j=1}^{k} \frac{x}{1+x\tau(S(T_j),x)}}. \tag{11}$$

Proof Let $v, v_1, \ldots, v_k$ be the roots of $T, T_1, \ldots, T_k$, respectively and let $u_1, \ldots, u_k$ be new vertices in the edges $vv_1, \ldots, vv_k$ in $S(T)$, respectively. By (6) and (7),

$$M_0(S(T), x) = \prod_{j=1}^{k} M(S(T_j) + v_j u_j, x) = \prod_{j=1}^{k} \left[xM_0(S(T_j), x) + M(S(T_j), x) \right].$$

On the other hand, by (6) and (7),

$$M_1(S(T), x) = x \sum_{j=1}^{k} \frac{M_0(S(T_j) + v_j u_j, x)}{M(S(T_j) + v_j u_j, x))} \prod_{i=1}^{k} M(S(T_i) + v_j u_j, x)$$

$$= x \sum_{j=1}^{k} \frac{M(S(T_j), x)}{xM_0(S(T_j), x) + M(S(T_j), x)} M_0(S(T), x)$$

Hence it follows from (9) and (10) that

$$\tau(S(T), x) = \frac{M_0(S(T), x)}{M(S(T), x)} = \frac{1}{1 + \sum_{j=1}^{k} \frac{x}{1+x\tau(S(T_j),x)}}.$$

Hence the assertions hold. □

Lemma 2.4 *Let T be a tree with root v and the branches $T_1, \ldots, T_k$. If T' is a proper subtree of T with root v and the branches $T'_1, \ldots, T'_r$, then $\tau(S(T'), x) > \tau(S(T), x)$.*

Proof Denote $|V(T')| = n'$. If $n' = 1$, then $\tau(S(T'), x) = 1$ and the assertion holds immediately. Assume that the assertion holds for $n' < t$. Since T' is a proper subgraph of T with root v and the branches $T'_1, \ldots, T'_r$, it is easy to see that, without loss of generality, T'_i, $1 \le i \le r$, are subtrees of $T_1, \ldots, T_r$, respectively. Moreover, there is at least one i such that T'_i is a proper subtree of T_i or $k > r$. By (8), we have

$$\tau(S(T), x) = \frac{1}{1 + \sum_{i=1}^{k} \frac{x}{1+x\tau(S(T_i),x)}}, \quad \tau(S(T'), x) = \frac{1}{1 + \sum_{i=1}^{r} \frac{x}{1+x\tau(S(T'_i),x)}}.$$

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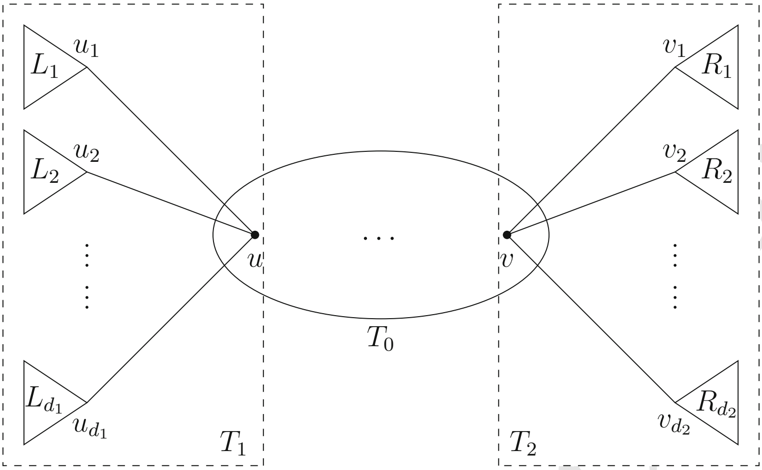

Fig. 1

By the induction hypothesis, $\tau(S(T_i'), x) \geq \tau(S(T_i), x),\ 1 \leq i \leq r$ with at least one strict inequality or $k > r$. Hence, $\tau(S(T'), x) > \tau(S(T), x)$. □

Now we are ready to prove the following exchange theorem which plays a key role in this paper.

Theorem 2.5 *Let T_1 be a tree with root v_1 and the branches $L_1, \ldots, L_{d_1}$ and T_2 be a rooted tree with root v_2 and the branches $R_1, \ldots, R_{d_2}$. Let T_0 be any tree with two vertices u and v. Let T be a tree obtained from T_1, T_2, T_0 by identifying v_1 and u, v_2 and v, respectively. Let $d \geq \max\{d_1, d_2\}$ be a positive integer and rearrange $L_1, \ldots, L_{d_1}, R_1, \ldots, R_{d_2}$ as $L_1', \ldots, L_{d_1+d_2-d}', R_1', \ldots, R_d'$ ($R_1', \ldots, R_{d_1+d_2}'$ for $d_1 + d_2 \leq d$) such that $\tau(S(L_1'), x) \geq \cdots \geq \tau(S(L_{d_1+d_2-d}'), x) \geq \tau(S(R_1'), x) \geq \cdots \geq \tau(S(R_d'), x)$. Let T_1' be the tree with root v_1 and the branches $L_1', \ldots, L_{d_1+d_2-d}'$ for $d_1 + d_2 > d$ (T_1' is an isolate vertex for $d_1 + d_2 \leq d$) and T_2' be the tree with root v_2 and the branches $R_1', \ldots, R_d'$ for $d_1 + d_2 > d$ (the branches $R_1', \ldots, R_{d_1+d_2}'$ for $d_1+d_2 \leq d$). Let T' be the tree obtained from T_1', T_2', T_0 by identifying v_1 and u, v_2 and v, respectively (see Fig. 1). If $M_{10}(S(T_0), x) \leq M_{01}(S(T_0), x)$, where $M_{10}(S(T_0), x)$ and $M_{01}(S(T_0), x)$ are matching generating functions of saturate u but not v, saturate v but not u, respectively, then*

$$M(S(T), x) \geq M(S(T'), x). \tag{12}$$

Further (12) becomes equality if and only if

$$\max\{\tau(S(R_j), x) : 1 \leq j \leq d_2\} \leq \min\{\tau(S(L_j), x) : 1 \leq j \leq d_1\}$$

with $d_2 = d$, or $M_{10}(S(T_0), x) = M_{01}(S(T_0), x)$ and

$$\max\{\tau(S(L_j), x) : 1 \leq j \leq d_1\} \leq \min\{\tau(S(R_j), x) : 1 \leq j \leq d_2\}$$

with $d_1 = d$.

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Proof Let

- $m_{00}(S(T_0), k)$ be the number of matchings of T_0 of cardinality k which saturate neither u nor v;
- $m_{10}(S(T_0), k)$ be the number of matchings of T_0 of cardinality k which saturate u, but not v;
- $m_{01}(S(T_0), k)$ be the number of matchings of T_0 of cardinality k which saturate v, but not u;
- $m_{11}(S(T_0), k)$ be the number of matchings of T_0 of cardinality k which saturate both u and v.

By Lemma 2.3, we have

$$\begin{aligned}
M(S(T),x) &= M_{00}(S(T_0),x)M(S(T_1),x)M(S(T_2),x) + M_{10}(S(T_0),x)M_0(S(T_1),x)M(S(T_2),x)\\
&\quad + M_{01}(S(T_0))M(S(T_1),x)M_0(S(T_2)) + M_{11}(S(T_0))M_0(S(T_1),x)M_0(S(T_2),x)\\
&= \prod_{j=1}^{d_1}(M(S(L_j),x) + xM_0(S(L_j),x)) \prod_{j=1}^{d_2}(M(S(R_j),x) + xM_0(S(R_j),x))\\
&\quad \times \left\{ M_{00}(S(T_0),x)\left(1+\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}\right)\left(1+\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\right)\right.\\
&\quad + M_{11}(S(T_0)) + M_{10}(S(T_0),x)\left(1+\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\right)\\
&\quad \left. + M_{01}(S(T_0),x)\left(1+\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}\right)\right\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
M(S(T'),x) &= M_{00}(S(T_0),x)M(S(T_1'),x)M(S(T_2'),x) + M_{10}(S(T_0),x)M_0(S(T_1'),x)M(S(T_2'),x)\\
&\quad + M_{01}(S(T_0))M(S(T_1'),x)M_0(S(T_2'),x) + M_{11}(S(T_0),x)M_0(S(T_1'),x)M(S(T_2'),x)\\
&= \prod_{j=1}^{d_1+d_2-d}(M(S(L_j'),x) + xM_0(S(L_j'),x)) \prod_{j=1}^{d}(M(S(R_j'),x) + xM_0(S(R_j'),x))\\
&\quad \times \left\{ M_{00}(S(T_0),x)\left(1+\sum_{j=1}^{d_1+d_2-d}\frac{x}{1+x\tau(S(L_j'),x)}\right)\left(1+\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R_j'),x)}\right)\right.\\
&\quad + M_{11}(S(T_0),x) + M_{10}(S(T_0),x)\left(1+\sum_{j=1}^{d}\frac{1}{1+\tau(S(R_j'),x)}\right)\\
&\quad \left. + M_{01}(S(T_0),x)\left(1+\sum_{j=1}^{d_1+d_2-d}\frac{x}{1+x\tau(S(L_j'),x)}\right)\right\}.
\end{aligned}$$

Since $L_1', \ldots, L_{d_1+d_2-d}', R_1', \ldots, R_d'$ is a rearrangement of $L_1, \ldots, L_{d_1}, R_1, \ldots, R_{d_2}$ and $\tau(S(L_1'),x) \geq \cdots \geq \tau(S(L_{d_1+d_2-d}'),x) \geq \tau(S(R_1'),x) \geq \cdots \geq \tau(S(R_d'),x)$, we have

$$\prod_{j=1}^{d_1+d_2-d}(M(S(L_j'),x) + xM_0(S(L_j'),x)) \prod_{j=1}^{d}(M(S(R_j'),x) + xM_0(S(R_j'),x))$$

$$= \prod_{j=1}^{d_1}(M(S(L_j),x)+xM_0(S(L_j),x))\prod_{j=1}^{d_2}(M(S(R_j),x)+xM_0(S(R_j),x)) \tag{13}$$

and

$$\sum_{j=1}^{d_1+d_2-d}\frac{x}{1+x\tau(S(L'_j),x)}+\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R'_j),x)}=\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}$$
$$+\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}. \tag{14}$$

Further

$$\frac{x}{1+x\tau(S(L'_1),x)}\le\cdots\le\frac{x}{1+x\tau(S(L'_{d_1+d_2-d}),x)}\le\frac{x}{1+x\tau(S(R'_1),x)}\le\cdots$$
$$\le\frac{x}{1+x\tau(S(R'_d),x)},$$

yields

$$\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}\ge\sum_{j=1}^{d_1+d_2-d}\frac{x}{1+x\tau(S(L'_j),x)} \tag{15}$$

and

$$\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\le\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R'_j),x)}. \tag{16}$$

Moreover, (16) becomes equality if and only if $d_2=d$ and

$$\max\{\tau(S(R_j),x):1\le j\le d_2\}\le\min\{\tau(S(L_j),x):1\le j\le d_1\}.$$

Then (14), (15) and (16) yield

$$\left(1+\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}\right)\left(1+\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\right)$$
$$-\left(1+\sum_{j=1}^{d_1+d_2-d}\frac{x}{1+x\tau(S(L'_j),x)}\right)\left(1+\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R'_j),x)}\right)$$
$$=\left(\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R'_j),x)}-\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}\right)$$
$$\left(\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R'_j),x)}-\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\right)\ge 0 \tag{17}$$

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Moreover,

$$M_{10}(S(T_0),x)\left(1+\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\right)+M_{01}(S(T_0),x)\left(1+\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L_j),x)}\right)$$

$$-\left\{M_{10}(S(T_0),x)\left(1+\sum_{j=1}^{d}\frac{1}{1+\tau(S(R'_j),x)}\right)+M_{01}(S(T_0),x)\left(1+\sum_{j=1}^{d_1+d_2-d}\frac{x}{1+x\tau(S(L'_j),x)}\right)\right\}$$

$$=(M_{01}(S(T_0),x)-M_{10}(S(T_0),x))\left(\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R'_j),x)}-\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R_j),x)}\right)\geq 0. \quad (18)$$

Hence by (13), (17) and (18), we have

$$M(S(T),x)-M(S(T'),x)\geq 0.$$

Further, if $M(S(T),x)-M(S(T'),x)=0$, then (17) and (18) become equalities. Hence

$$\sum_{j=1}^{d}\frac{x}{1+x\tau(S(R_j),x)}=\sum_{j=1}^{d_2}\frac{x}{1+x\tau(S(R'_j),x)},$$

or

$$M_{10}(S(T_0),x)=M_{01}(S(T_0),x) \text{ and } \sum_{j=1}^{d}\frac{x}{1+x\tau(S(R_j),x)}=\sum_{j=1}^{d_1}\frac{x}{1+x\tau(S(L'_j),x)}.$$

Therefore,

$$\max\left\{\tau(S(R_j),x):1\leq j\leq d_2\right\}\leq\min\left\{\tau(S(L_j),x):1\leq j\leq d_1\right\} \text{ with } d_2=d,$$

or $M_{10}(S(T_0),x)=M_{01}(S(T_0),x)$ and

$$\max\left\{\tau(S(L_j),x):1\leq j\leq d_1\right\}\leq\min\left\{\tau(S(R_j),x):1\leq j\leq d_2\right\} \text{ with } d_1=d.$$

This completes the proof. □

Let $\mathcal{T}_{n,d+1}$ be the set of all trees of order n with the maximum degree $d+1$ and $\mathcal{S}(\mathcal{T})_{n,d+1}$ be the set of the subdivision trees of any tree T in $\mathcal{T}_{n,d+1}$. A tree $S(\widetilde{T})$ in $\mathcal{S}(\mathcal{T})_{n,d+1}$ is called an *optimal tree* if $M(S(T),x)\geq M(S(\widetilde{T}),x)$ for any $S(T)\in\mathcal{S}(\mathcal{T})_{n,d+1}$ and $x>0$.

Corollary 2.6 *Let $S(\widetilde{T})$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If $\widetilde{T}$ can be decomposed as T_1, T_2 and T_0 as Fig. 1. If u and v are non-pendent vertices and $\tau(S(L_1),x)>\tau(S(R_1),x)$, then $d_2=d$ and*

$$\min\left\{\tau(S(L_i),x):1\leq i\leq d_1\right\}\geq\max\left\{\tau(S(R_i),x):1\leq i\leq d_2\right\}. \quad (19)$$

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Proof We follows the symbols in Theorem 2.5. First we have the following claim

$$M_{10}(S(T_0), x) \le M_{01}(S(T_0), x) \tag{20}$$

In fact, suppose $M_{10}(S(T_0), x) > M_{01}(S(T_0), x)$. Rearrange $L_1, \dots, L_{d_1}, R_1, \dots, R_{d_2}$ as $L'_1, \dots, L'_{d_1+d_2-d}, R'_1, \dots, R'_d (R'_1, \dots, R'_{d_1+d_2}$ for $d_1 + d_2 \le d)$ such that $\tau(S(L'_1), x) \ge \dots \ge \tau(S(L'_{d_1+d_2-d}), x) \ge \tau(S(R'_1), x) \ge \dots \ge \tau(S(R'_d), x)$. Let T''_1 be the tree with root v_1 and the branches $R'_1, \dots, R'_d$ for $d_1+d_2 > d$ (the branches $R'_1, \dots, R'_{d_1+d_2}$ for $d_1 + d_2 \le d$) and T''_2 be the tree with root v_2 and the branches $L'_1, \dots, L'_{d_1+d_2-d}$ for $d_1+d_2 > d$ (no branches for $d_1+d_2 \le d$). Then Let T'' be the tree obtained from T''_1, T''_2, T_0 by identifying v_1 and u, v_2 and v, respectively. By Theorem 2.5, $M(S(\widetilde{T}), x) \ge M(S(T''), x)$. On the other hand, since $S(T'') \in \mathcal{S}(\mathcal{T})_{n,d+1}$ and $S(\widetilde{T})$ is an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$, we have $M(S(T''), x) \ge M(S(\widetilde{T}), x)$. Hence $M(S(\widetilde{T}), x) = M(S(T''), x)$. By Theorem 2.5 again, we have

$$\max\{\tau(S(L_j), x) : 1 \le j \le d_1\} \le \min\{\tau(S(R_j), x) : 1 \le j \le d_2\},$$

which contradicts to the condition $\tau(S(L_1), x) > \tau(S(R_1), x)$,. Hence the claim holds. Hence by Theorem 2.5, the ~~coro~~ holds. □

The following Corollary 2.7 is easily obtained from Theorem 2.5.

Corollary 2.7 *Let $S(\widetilde{T})$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. Then there exists at most one vertex u with degree $2 \le deg(u) \le d$.*

3 Proof of Theorem 1.3

In order to prove Theorem 1.3, we need some Lemmas.

Lemma 3.1

$$\tau(S(C_1), x) = 1, \quad \tau(S(C_2), x) = \frac{1+x}{(d+1)x+1},$$

$$\tau(S(C_h), x) = \frac{1}{1 + \frac{dx}{1+x\tau(S(C_{h-1}),x)}}. \tag{21}$$

Further $\tau(S(C_h), x) < \tau(S(C_{h-1}), x)$ for $h \ge 2$.

Proof Clearly, $\tau(S(C_1), x) = \tau(C_1, x) = 1$. For $h \ge 2$, by (11), it is easy to see that

$$\tau(S(C_h), x) = \frac{1}{1 + \frac{dx}{1+x\tau(S(C_{h-1}),x)}}.$$

Further,

$$\tau(S(C_2), x) - \tau(S(C_1), x) = \frac{1+x}{1+(d+1)x} - 1 = -\frac{dx}{1+(d+1)x} < 0.$$

Then, for $h > 2$,

$$\begin{aligned}
&\tau(S(C_h),x) - \tau(S(C_{h-1}),x) \\
&= \frac{1}{1+\frac{dx}{1+x\tau(S(C_{h-1}),x)}} - \frac{1}{1+\frac{dx}{1+x\tau(S(C_{h-2}),x)}} \\
&= \frac{dx^2\left(\tau(S(C_{h-1}),x) - \tau(S(C_{h-2}),x)\right)}{(1+x\tau(S(C_{h-1}),x)(1+x\tau(S(C_{h-2}),x)\left(1+\frac{dx}{1+x\tau(S(C_{h-1}),x)}\right)\left(1+\frac{dx}{1+x\tau(S(C_{h-2}),x)}\right)} \\
&< 0.
\end{aligned}$$

This completes the proof. □

Let u be any vertex in a tree $T = (V(T), E(T))$. Denote by $N(u)$ the set of all vertices adjacent to u, i.e., $N(u) := \{v \in V(T) \mid uv \in E(T)\}$.

Lemma 3.2 *Let $S(\widetilde{T})$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If there exists a vertex u with degree $2 \le deg(u) \le d$ in $\widetilde{T}$, then there are $deg(u) - 1$ pendent vertices in $N(u)$. Further, if there exists a vertex $v \ne u$ such that there are $1 \le k \le d-1$ pendent vertices in $N(v)$, then $uv \in E(\widetilde{T})$, and there are d pendent vertices or no pendent vertices in $N(w)$ for any $w \ne u, v$.*

Proof By Corollary 2.7 that u is a unique vertex. Let y be the farthest non-pendent vertex from u in $\widetilde{T}$. Then there is exact one non-pendent vertex in $N(y)$ and $\widetilde{T}$ can be decomposed as T_0, T_1 and T_2 (see Fig.1), where T_1 has root y with the branches $C_1, \ldots, C_1$ and T_2 has root u with the branches $L_1, \ldots, L_{deg(u)-1}$. By Lemma 2.4, $\tau(S(C_1),x) \ge \tau(S(L_j),x)$ with equality if and only if $L_j = C_1$ for $1 \le j \le deg(u)-1$. By Corollary 2.6 and $deg(u)-1 < d$, we have $\tau(S(C_1),x) = \tau(S(L_j),x)$. Hence $L_j = C_1$ for $1 \le j \le deg(u)-1$. Hence there are $deg(u)-1$ pendent vertices in $N(u)$.

Suppose that $uv \notin E(\widetilde{T})$. Then there exists a vertex z with $uz \in E(\widetilde{T})$ such that $\widetilde{T}$ can be decomposed as T_0, T_1 and T_2, where T_0 contains vertices z, v, not u, T_1 has root z and the branches L_1 containing $u, L_2, \ldots, L_d$, and T_2 has root v and the branches R_1 containing C_2, $R_2 = C_1, R_3, \ldots, R_d$. By Lemma 2.4, $\tau(S(L_1),x) > \tau(S(C_2),x) \ge \tau(S(R_1),x)$. Hence by Corollary 2.6,

$$1 > \tau(S(L_1),x) \ge \min\{\tau(S(L_i),x),\ 1 \le i \le d\} \ge \max\{\tau(S(R_i),x),\ 1 \le i \le d\} = 1,$$

which is a contradiction. So $uv \in E(\widetilde{T})$. Suppose that there exists another vertex $w \ne u, v$ such that there are $1 \le p \le d$ pendent vertices in $N(w)$. Then $\widetilde{T}$ can be decomposed as trees T_0, T_1 and T_2, where T_0 contains vertices v, w, T_1 has root v and the branches $L_1 = C_1, L_2 \ne C_1, L_3, \ldots, L_d$, and T_2 has root w and the branches $R_1 = C_1, R_2 \ne C_1, R_3, \ldots, R_d$. Clearly $\tau(S(L_1),x) > \tau(S(R_2),x)$. Hence

$$1 > \min\{\tau(S(L_i),x),\ 1 \le i \le d\} \ge \max\{\tau(S(R_i),x),\ 1 \le i \le d\} = 1,$$

which is a contradiction. Hence for any $w \ne u, v$, there are d pendent vertices or no pendent vertices in $N(w)$. □

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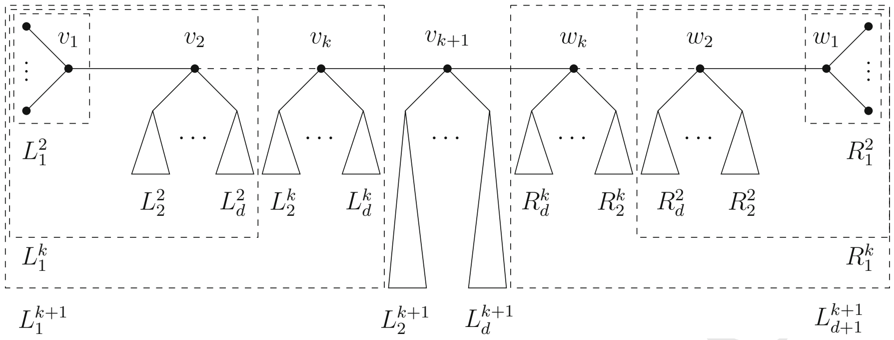

Fig. 2 Tree $\widetilde{T}$

Lemma 3.3 *Let $S(\widetilde{T})$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If there is a vertex v of $V(\widetilde{T})$ with degree $2 \le deg(v) \le d$, then $\widetilde{T}$ is greedy tree T^*_{d+1}.*

Proof By Corollary 2.7, the degree of any vertex $u \neq v$ is $d+1$ or 1. Let P be a longest path with end vertex v in $\widetilde{T}$. If the length of P is odd, denote $P = v_1 v_2 \cdots v_k v_{k+1} w_k \cdots w_1 w_0$ with $v_1 = v$. Moreover, let $L_1^t, \ldots, L_d^t, L_{d+1}^t$ be the $d+1$ branches of $\widetilde{T} - v_t$ such that L_1^t contains v_1 but no w_1; $L_2^t, \ldots, L_d^t$ ~~does~~ not contain v_1 and w_1, L_{d+1}^t contains w_1 but no v_1 for $t = 2, \ldots, k+1$. Similarly, let $R_1^t, \ldots, R_d^t, R_{d+1}^t$ be the $d+1$ branches of $\widetilde{T} - w_t$ such that R_1^t contains w_1 but no v_1; $R_2^t, \ldots, R_d^t$ ~~does~~ not contain v_1 and w_1, R_{d+1}^t contains v_1 but no w_1 for $t = 2, \ldots, k$ (See Fig. 2). Then we have the following Claim. □

Claim

- (i) $\tau(S(C_t), x) < \tau(S(L_1^t), x) < \tau(S(C_{t-1}), x),\ 2 \le t \le k+1$.
- (ii) $L_i^t = C_{t-1}$ or $L_i^t = C_t,\ 2 \le t \le k+1,\ 2 \le i \le d$;
- (iii) $R_1^t = R_2^t = \cdots = R_d^t = C_t,\ 2 \le t \le k$.

We prove the claim by the induction on t. For $t = 2$, by Lemma 3.1, we have

$$\tau(S(C_1), x) = 1 > \tau(S(L_1^2), x) = \frac{1}{1 + \frac{x deg(v)}{1+x}} > \frac{1}{1 + \frac{dx}{1+x}}$$
$$= \tau(S(C_2), x) = \tau(S(R_1^2), x).$$

So (i) holds. By Lemma 3.2, for any vertex $w \in V(L_{d+1}^2)$, there are d pendent vertices or no pendent vertices in $N(w)$. Thus $R_1^2 = C_2$, which implies that $\tau(S(L_1^2), x) > \tau(S(R_1^2), x)$. Hence Corollary 2.6, we have $\tau(S(L_j^2), x) \ge \tau(S(C_2), x)$ for $j = 2, \ldots, d$. Therefore by Lemmas 2.4 and 3.2, we have $L_j^2 = C_1$ or C_2 for $j = 2, \ldots, d$. So (ii) holds. Since P is a longest path with ends v_1 and w_0, the distance between any vertex in R_j^2 and w_2 is no more than 2. By $\tau(S(R_j^2), x) \le \tau(S(L_1^2), x)$ and Lemma 3.2, we have $R_j^2 = C_2$ for $j = 2, \ldots, d$. So (iii) holds.

Suppose that the claim holds for less than t. By the induction prothesis, $\tau(S(C_{t-1}), x) < \tau(S(L_1^{t-1}), x) < \tau(S(C_{t-2}), x)$ and $\tau(S(C_{t-1}), x) \le \tau(S(L_j^{t-1}), x) \le \tau(S(C_{t-2}), x)$ for $j = 2, \ldots, d$. It follows from (11) and (21) that

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$$\tau(S(C_t), x) = \frac{1}{1+\sum_{i=1}^{d}\frac{x}{1+x\tau(S(C_{t-1}),x)}} < \frac{1}{1+\sum_{i=1}^{d}\frac{x}{1+x\tau(S(L_i^{t-1}),x)}}$$

$$= \tau(S(L_1^t), x) < \frac{1}{1+\sum_{i=1}^{d}\frac{x}{1+x\tau(S(C_{t-2}),x)}} = \tau(S(C_{t-1}), x)$$

Hence (i) holds for t. In order to prove (ii) holding for t, we first prove the following several claims.

Claim 3.1 $\tau(S(C_t), x) \leq \tau(S(L_i^t), x) \leq \tau(S(C_{t-1}, x)$ *for* $i = 2, \ldots, d$.

In fact, there are d branches $L_1^t, \ldots, L_d^t$ containing no w_t in $\widetilde{T} - v_t$ and there are d branches $R_1^t = C_t, R_2^t, \ldots, R_d^t$ containing no v_t in $\widetilde{T} - w_t$. By Claim (i), we have $\tau(S(C_t), x) < \tau(S(L_1^t), x)$. On the other hand, by the induction hypothesis, $R_1^{t-1} = \ldots = R_d^{t-1} = C_{t-1}$ which implies $R_1^t = C_t$. Hence by Corollary 2.6, $\min\{\tau(S(L_1^t), x), \ldots, \tau(S(L_d^t), x)\} \geq \tau(S(C_t), x)$. Further there are d branches $L_1^t, \ldots, L_d^t$ containing no w_{t-1} in $\widetilde{T} - v_t$ and there are d branches $R_1^{t-1} = C_{t-1}, \ldots, R_d^{t-1} = C_{t-1}$ containing no v_t in $\widetilde{T} - w_{t-1}$. By Claim (i), $\tau(S(L_1^t), x) < \tau(S(C_{t-1}), x) = \tau(S(R_1^{t-1}), x)$. Hence by Corollary 2.6, $\max\{S(L_2^t), x), \ldots, S(L_d^t), x)\} \leq \tau(S(C_{t-1}), x)$. So Claim 3.1 holds.

Let $l+1$ be the maximum distance between v_t and any vertex in $\bigcup_{i=1}^{t} V(L_i^t)$. Then $l \geq t-1$. Denote by

$$V_j = \left\{u \mid dist(u, v_t) = l - j + 1, u \in \bigcup_{i=1}^{d} V(L_i^t)\right\}, \ j = 0, \ldots, l.$$

Claim 3.2 *For any* $u \in V_{l-j}$, *there are* d *branches* $L_1^u, \ldots, L_d^u$ *containing no* v_t *in* $\widetilde{T} - u$ *such that*

$$\tau(S(C_{t-j-1}), x) \leq \tau(S(L_i^u), x) \leq \tau(S(C_{t-j-2}), x), \ \ i = 1, \ldots, d, \tag{22}$$

where $j = 0, \ldots, \min\{t, l\} - 2$.

We prove Claim 3.2 by the induction on j. Let $L_1^u, \ldots, L_d^u$ be d the branches containing no v_t in $\widetilde{T} - u$ and T^u be the subtree of T consisting of u and $L_1^u, \ldots, L_d^u$. For $j = 0$, there clearly exists an $1 \leq p \leq d$ such that $T^u = L_p^t$. If there exists an $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) < \tau(S(C_{t-1}), x)$, let $R_1^{t-1} = C_{t-1}, \ldots, R_d^{t-1}$ be d the branches containing no v_t in $\widetilde{T} - w_{t-1}$. Hence by Corollary 2.6, $\max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-1}), x)$. Then

$$\tau(S(L_p^t), x) = \tau(S(T^u), x) = \frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(L_q^u),x)}} < \frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(C_{t-1}),x)}}$$

$$= \tau(S(C_t), x),$$

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which contradicts to Claim 3.1. Therefore,

$$\tau(S(C_{t-1}),x) \le \tau(S(L_i^u),x), i = 1,\dots,d.$$

On the other hand, if there exists $1 \le i \le d$ such that $\tau(S(L_i^u),x) > \tau(S(C_{t-2}),x)$, let $R_1^{t-2} = C_{t-2},\dots,R_d^{t-2}$ be d the branches containing no v_t in $\widetilde{T} - w_{t-2}$. By Corollary 2.6,

$$\min\{\tau(S(L_1^u),x),\dots,\tau(S(L_d^u),x)\} \le \tau(S(C_{t-2}),x).$$

Then

$$\tau(S(L_p^t),x) = \tau(S(T^u),x) = \frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(L_q^u),x)}} > \frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(C_{t-1}),x)}}$$
$$= \tau(S(C_{t-1}),x),$$

which contradicts to Claim 3.1. Hence Claim 3.2 holds for $j = 0$. Now assume that Claim 3.2 holds for j and consider the claim for $j+1$. For any $u \in V_{l-(j+1)}$, let $L_1^u,\dots,L_d^u$ be d the branches containing no v_t in $\widetilde{T} - u$ and T^u be the subtree consisting of u and $L_1^u,\dots,L_d^u$. Clearly there exists a $u' \in V_{l-j}$ such that there exists a branch $L_1^{u'}$ in $T_{d+1}^* - u'$ such that $T^u = L_1^{u'}$. If there exists an $1 \le i \le d$ such that $\tau(S(L_i^u),x) < \tau(S(C_{t-j-2}),x)$, let $R_1^{t-j-2} = C_{t-j-2},\dots,R_d^{t-j-2}$ be d the branches containing no u in $\widetilde{T} - w_{t-j-1}$. By Corollary 2.6,

$$\max\{\tau(S(L_1^u),x),\dots,\tau(S(L_d^u),x)\} \le \tau(S(C_{t-j-2}),x).$$

Then

$$\tau(S(L_1^{u'}),x) = \tau(S(T^u),x) = \frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(L_q^u),x)}} < \frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(C_{t-j-2}),x)}}$$
$$= \tau(S(C_{t-j-1}),x),$$

which contradicts to the induction hypothesis. Therefore, for any $u \in V_{l-j-1}$,

$$\tau(S(C_{t-j-2}),x) \le \tau(S(L_i^u),x), i = 1,\dots,d.$$

On the other hand, if there exists $1 \le i \le d$ such that $\tau(S(L_i^u),x) > \tau(S(C_{t-j-3}),x)$. Let $R_1^{t-j-3} = C_{t-j-3},\dots,R_d^{t-j-3}$ be d the branches containing no u in $\widetilde{T} - w_{t-j-2}$. By Corollary 2.6,

$$\min\{\tau(S(L_1^u),x),\dots,\tau(S(L_d^u),x)\} \ge \tau(S(C_{t-j-3}),x).$$

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Then

$$\tau(S(L_1^{u'}),x)=\tau(S(T^u),x)=\frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(L_q^u),x)}}>\frac{1}{1+\sum_{q=1}^{d}\frac{x}{1+x\tau(S(C_{t-j-3}),x)}}$$
$$=\tau(S(C_{t-j-2}),x),$$

which contradicts to the induction hypothesis. Hence Claim 3.2 holds for $j+1$. Therefore Claim 3.2 holds.

Claim 3.3 $l=t-1$. *If* $l>t-1$, *by Claim* 3.2, *for any* $u\in V_{l-t+2}$,

$$\tau(S(C_1),x)\le\tau(S(L_i^u),x),\ i=1,\ldots,d.$$

On the other hand, there exists a $u'\in V_{l-t+2}$ such that the largest distance between u' and the pendent vertex is at least 2, then C_2 is a proper subgraph $L_1^{u'}$, which implies $\tau(S(L_1^{u'}),x)\le\tau(S(C_2),x)$. It is a contradiction. Hence $l\le t-1$. In addition $l\ge t-1$, we have $l=t-1$.

Claim 3.4 *For any* $u\in V_{t-j-1}$, $j=0,\ldots,t-3$. *Let* $L_1^u,\ldots,\ldots,L_d^u$ *be the* d *branches containing no* v_t *in* $\widetilde{T}-u$ *and* T^u *consist of* u *and* d *branches* $L_1^u,\ldots,\ldots,L_d^u$. *Then* $L_1^u=\cdots=L_d^u=C_{t-j-1}$ *or* $L_1^u=\cdots=L_d^u=C_{t-j-2}$, *i.e.,* $T^u=C_{t-j}$ *or* $T^u=C_{t-j-1}$.

We prove Claim 3.4 by the induction for $t-j-1$. In fact, for $j=t-3$ and $u\in V_2$, by Claim 3.2, $\tau(S(C_2),x)\le\tau(S(L_i^u),x)\le\tau(S(C_1),x)$ for $i=1,\ldots,d$. Hence $L_i^u=C_2$ or $L_i^u=C_1$ for $i=1,\ldots,d$. If, say $L_1^u=C_2$ and $L_2^u=C_1$, then by $\tau(S(L_1^2),x)>\tau(S(L_1^u),x)$ and Corollary 2.6, $\tau(S(L_1^2),x)\ge\max\{\tau(S(L_1^u),x),\ldots,\tau(S(L_d^u),x)\}\ge\tau(S(C_1),x)$, which is a contradiction. Hence $L_1^u=\cdots=L_d^u=C_2$ or $L_1^u=\cdots=L_d^u=C_1$, i.e., $T^u=C_3$ or $T^u=C_2$ for $u\in V_2$. Assume that Claim 3.4 ~~hold~~ for any vertex in V_{t-j-2}. Now for $u\in V_{t-j-1}$. Let $z_1,\ldots,z_d\in V_{t-j-1}$ be the roots of $L_1^u,\ldots,L_d^u$, respectively. By the induction hypothesis, $L_1^u,\ldots,L_d^u\in\{C_{t-j-1},C_{t-j-2}\}$. Further $L_1^u=\cdots=L_d^u=C_{t-j-1}$ or $L_1^u=\cdots=L_d^u=C_{t-j-2}$. In fact, if, say $L_1^u=C_{t-j-1}$ and $L_2^u=C_{t-j-2}$, By $\tau(S(L_1^{t-j-1}),x)>\tau(S(C_{t-j-1}),x)$ and Corollary 2.6,

$$\tau(S(L_1^{t-j-1}),x)\ge\max\{\tau(S(C_{t-j-1}),x),\tau(S(C_{t-j-2}),x)\ge\tau(S(C_{t-j-2}),x),$$

which ~~contradiction~~ to Claim 3.1. Hence $L_1^u=\cdots=L_d^u=C_{t-j-1}$ or $L_1^u=\cdots=L_d^u=C_{t-j-2}$, i.e., $T^u=C_{t-j}$ or $T^u=C_{t-j-1}$. So Claim 3.4 holds.

Hence $L_i^t=C_t$ or $L_i^t=C_{t-1}$ for $i=2,\ldots,d$. In other words, Claim (ii) holds.

Similarly, we can prove Claim (iii) and omit the detail. It is easy from ~~Claim~~ that $\widetilde{T}$ is greedy tree. If the length of P is even, we can prove this assertion by similar method. So we finish our proof. □

Lemma 3.4 *Let* $S(\widetilde{T})$ *be an optimal tree in* $\mathcal{S}(\mathcal{T})_{n,d+1}$. *If the degree of any vertex is* 1 *or* $d+1$, *then there exists at most one vertex* u *such that there are* $1\le k\le d-1$

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pendent vertices in $N(u)$. Further suppose that there exists a vertex u such that there are $1 \leq k \leq d-1$ pendent vertices in $N(u)$. If the branches $T_1, \ldots, T_{d+1}$ of $\widetilde{T}-u$ contains no C_3, then $\widetilde{T}$ is greedy tree. If one of the branches $T_1, \ldots, T_{d+1}$ of $T-u$ contains C_3, say $T_{d+1} \supseteq C_3$, then $T_i = C_1$ or C_2 for $i = 1, \ldots, d$.

Proof Suppose that there exist two vertices u, v such that there are $1 \leq p \leq d-1$ and $1 \leq q \leq d-1$ in $N(u)$ and $N(v)$, respectively. Then $\widetilde{T}$ can be decomposed as three subtrees T_0, T_1 and T_2, where T_0 contains vertices u, v; T_1 has root u and the branches $L_1 = C_1, L_2 \neq C_1, L_3, \ldots, L_d$; T_2 has root v and the branches $R_1 = C_1, R_2 \neq C_1, R_3, \ldots, R_d$. Clearly $\tau(S(L_1), x) > \tau(S(R_2), x)$. Hence by Corollary 2.6,

$$1 > \min\{\tau(S(L_i), x),\ 1 \leq i \leq d\} \geq \max\{\tau(S(R_i), x),\ 1 \leq i \leq d\} = 1,$$

which is a contradiction. Hence there exists at most one vertex u with $N(u)$ ~~has~~ $1 \leq k \leq d-1$ pendent vertices. Further suppose that there exists a vertex u such that there are $1 \leq k \leq d-1$ pendent vertices in $N(u)$.

If the branches $T_1, \ldots, T_{d+1}$ of $T-u$ contains no C_3, it is easy to see that $T_i = C_1$ or C_2 for $i = 1, \ldots, d+1$, which implies $\widetilde{T}$ is a greedy tree T^*_{d+1}. If one of the branches $T_1, \ldots, T_{d+1}$ contains C_3, say $\widetilde{T}$ contains C_3, then there exists a vertex w such that $\widetilde{T}$ can be decomposed as trees T_0, T_1 and T_2, where T_0 contains vertices u, w, T_1 has root u and the branches $L_1 = C_1, L_2 \neq C_1, L_3, \ldots, L_d$, T_2 has root w and the branches $R_1 = C_2, R_2, R_3, \ldots, R_d$. Clearly $\tau(S(L_1), x) > \tau(S(R_1), x)$. By Corollary 2.6,

$$\min\{\tau(S(L_i), x),\ 1 \leq i \leq d\} \geq \max\{\tau(S(R_i), x),\ 1 \leq i \leq d\} \geq \tau(S(C_2), x).$$

Hence $L_i = C_1$ or C_2 for $i = 1, \ldots, d$. □

The proofs of the following two lemmas are similar to the proof of Lemma 3.3, thus we omit the proof. The readers can refer to appendix.

Lemma 3.5 *Let $S(T)$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If ~~degree~~ of each vertex is 1 or $d+1$ in $V(T)$, and there is a vertex v such that there are $1 \leq h \leq d-1$ pendent vertices in $N(v)$, then T is greedy tree T^*_{d+1}.*

Lemma 3.6 *Let $S(T)$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If ~~degree~~ of each vertex is 1 or $d+1$ in $V(T)$, and there are d pendente vertices or no pendent vertices in $N(u)$ for any $u \in V(T)$. Then T is greedy tree T^*_{d+1}.*

Theorem 3.7 *$S(T^*_{d+1})$ is only optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. In other words, If T is any tree of order n with maximum degree $d+1$, then*

$$M(S(T^*_{d+1}), x) \leq M(S(T), x),\ \textit{for}\ \ x > 0,$$

with equality if and only if $T = T^*_{d+1}$.

Proof Let T be any tree of order n with maximum degree $d+1$. If there exists a vertex with degree less than $d+1$, then by Corollary 2.7, Lemmas 3.2 and 3.3, $M(S(T^*_{d+1}, x) \leq M(S(T), x)$, for $x > 0$ with equality if and only if $T = T^*_{d+1}$. If the degree of each vertex in $V(T)$ is 1 or $d+1$, and there exists a vertex u such that there $1 \leq k \leq d-1$ pendent vertices in $N(u)$, then by Lemmas 3.4 and 3.5, $M(S(T^*_{d+1}, x) \leq M(S(T), x)$, for $x > 0$ with equality if and only if $T = T^*_{d+1}$. Hence we assume that degree of each vertex v in $V(T)$ is 1 or $d+1$, and there are d pendent vertices or no pendent vertices in $N(v)$, then by Lemma 3.6, $M(S(T^*_{d+1}, x) \leq M(S(T), x)$, for $x > 0$ with equality if and only if $T = T^*_{d+1}$. Therefore the assertion holds. □

Lemma 3.8 (Zhou and Gutman 2008) *For every tree T of order n,*

$$c_k(T) = m_k(S(T)), k = 0, \ldots, n.$$

Moreover, $\varphi(T, x) = M(S(T), x)$.

Now we area ready to prove Theorem 1.3.

Proof of Theorem 1.3. It follows from Theorem 3.7 and Lemma 3.8 that the assertion holds. □

Hosoya (1971) introduced a molecular graph structure descriptor $Z(T)$, which is now called the *Hosoya index*, $Z(T) = \sum_{k \geq 0} m(T, k)$. Wagner and Gutman (2010) surveyed properties and techniques for the Hosoya index. We present a result for the Hosoya index.

Corollary 3.9 (Wagner and Gutman 2010) *Let T be any tree of order n with the maximum degree $d+1$. Then $Z(S(T)) \geq Z(S(T^*_{d+1}))$ with equality if and only if $T = T^*_{d+1}$.*

Proof It follows from Theorem 3.7 with $x = 1$. □

4 Proof of Theorem 1.4

In order to prove Theorem 1.4, we need more notions and Lemmas. *The Laplacian-like energy* Liu and Liu (2008) of a tree T, LEL for short, is defined as $LEL(T) = \sum_{k=1}^{n-1} \sqrt{\mu_k}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the eigenvalues of $L(G)$. The *characteristic polynomial* of a tree T is $\mathcal{A}(T, \lambda) = \det(\lambda I - A(T)) = \sum_{k=0}^{n} (-1)^k a_k \lambda^{n-k}$. If $\lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq \lambda_n(T)$ are the eigenvalues of $A(T)$, then the *energy* of T is $E(T) = \sum_{k=1}^{n} |\lambda_k(T)|$. Moreover, the *matching polynomial* of T is defined to be $\mathcal{M}(T, \lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m(T, k) \lambda^{n-2k}$.

Lemma 4.1 *Let T be any tree of order n. Then*

$$IE(T) = LEL(T) = \frac{1}{2} E(S(T)). \tag{23}$$

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Proof By the definition of $IE(T)$, $IE(T)$ is the sum of the square roots of all eigenvalues of $I(G)I(G)^T$. Note that $I(G)I(G)^T = Q(T)$, which is signless Laplacian matrix. Since T is bipartite, $L(T)$ and $Q(T)$ are similar, which implies they have the same eigenvalues. Hence $IE(T) = LEL(T)$. On the other hand, it follows from Yan and Yeh (2006) that

$$\mathcal{L}(T, \lambda^2) = \lambda \mathcal{A}(S(T), \lambda).$$

Then the adjacency eigenvalues of $S(T)$ are $\pm\sqrt{\mu_1(T)}, \ldots, \pm\sqrt{\mu_{n-1}(T)}, 0$, where $\mu_1(T), \ldots, \mu_{n-1}(T), 0$ are all eigenvalues of $L(T)$. Hence

$$E(S(T)) = 2\sum_{i=1}^{n-1} \sqrt{\mu_i(T)} = 2LEL(T).$$

So the assertion holds. □

Now we are ready to prove Theorem 1.4.

Proof Let T be any tree of order n with the maximum degree $d+1$. By the Coulson integral formula for energy (for example, see Heuberger and Wagner 2009),

$$E(S(T)) = \frac{2}{\pi}\int_0^{\infty} x^{-2} \log\left(\sum_{k\geq 0} m(S(T), k)x^{2k}\right) dx.$$

On the other hand, by Theorem 3.7, $M(S(T), x) \geq M(S(T^*_{d+1}), x)$ for $x > 0$ with equality if and only if $T = T^*_{d+1}$. Hence $\sum_{k\geq 0} m(S(T), k)x^{2k} \geq \sum_{k\geq 0} m(S(T^*_{d+1}), k)x^{2k}$, which implies $E(S(T)) \geq E(S(T^*_{d+1}))$ with equality if and only if $T = T^*_{d+1}$. Therefore, the assertion follows from Lemma 4.1. □

Denote by $B_{n,d+1}$ the tree obtained by identifying the center vertex of the star S_{d+1} and one of the pendent vertices of the path P_{n-d}.

Lemma 4.2 (Ilić 2009) *Let $T \in \mathcal{T}_{n,d+1}$, then $c_k(T) \leq c_k(B_{n,d+1})$, $k = 0, 1, 2, \ldots, n$.*

Combining with Theorems 1.3 and 3.7 and Lemma 4.2, we are able to get the following results.

Theorem 4.3 *Let T be any tree of order n with the maximum degree $d+1$. Then for $x > 0$,*

$$M(S(T^*_{d+1}), x) \leq M(S(T), x) \leq M(B_{n,d+1}, x),$$
$$\varphi(T^*_{d+1}, x) \leq \varphi(T, x) \leq \varphi(S(B_{n,d+1}), x)$$

with left (right) equality if and only if $T = T^*_{d+1}$ ($T = B_{n,d+1}$).

Lemma 4.4 *Let* $S(T_d^*)$, $S(T_{d+1}^*)$ *be the optimal trees in* $\mathcal{S}(\mathcal{T})_{n,d}$ *and* $\mathcal{S}(\mathcal{T})_{n,d+1}$ *respectively. Then* $M(S(T_d^*), x) > M(S(T_{d+1}^*, x)$ *for* $x > 0$ *and* $n > d+1$.

Proof Since $n > d+1$, then we can find two non-pendent vertices u, v in $S(T_d^*)$. By Corollary 2.6, there is at most one vertex w such that $2 \leq deg(w) \leq d$. Use Lemma 2.5 for vertices u, v in $S(T_d^*)$, then we can get a new graph $T' \in \mathcal{S}(\mathcal{T})_{n,d+1}$ such that $M(S(T_d^*), x) > M(S(T', x) \geq M(S(T_{d+1}^*, x)$. This completes the proof. □

Hence it is easy to see the following assertion holds.

Corollary 4.5 (Zhou and Gutman 2008; Mohar 2007) *Let* T *be any tree of order* n. *Then*

$$\varphi(K_{1,n-1}, x) \leq \varphi(T, x) \leq \varphi(P_n, x),$$
$$M(S(K_{1,n}), x) \leq M(S(T), x) \leq M(S(P_n, x)$$

for $x > 0$ *with left (right) equality if and only if* $T = K_{1,n-1}(T = P_n)$.

Based on the above results, we conclude this paper with the following conjecture.

Conjecture 4.6 *Let* T *be any tree of order* n *with the maximum degree* $d+1$. *Then* $T_{d+1}^* \preceq T$ *with equality if and only if* $T = T_{d+1}^*$, *i.e.,* $c_k(T_{d+1}^*) \leq c_k(T)$ *for* $k = 0, \ldots, n$ *with all equalities if and only if* $T = T_{d+1}^*$.

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Appendix

In here, we present detail proof of Lemmas 3.5 and 3.6

Lemma 3.5 Let $S(T)$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If degree of each vertex is 1 or $d+1$ in $V(T)$, and there is a vertex v such that there are $1 \leq h \leq d-1$ pendent vertices in $N(v)$, then T is greedy tree T_{d+1}^*.

Proof If the branches $T_1, \ldots, T_{d+1}$ of $T-u$ contains no C_3, then by Lemma 3.4, T is a greedy tree. Now assume that the branches $T_1, \ldots, T_{d+1}$ of $T-u$ contains C_3, say $T_{d+1} \supseteq C_3$, then by Lemma 3.4, $T_1 = C_2$ and $T_i = C_1$ or C_2 for $i = 2, \ldots, d$. Let P (see Fig. 1) be the longest path which goes through v and terminates at non-pendent vertices with $v_2 = v$. Assume the length of P is $2k$. Let $L_1^i, \ldots, L_d^i$ be the branches of $T - v_i$ containing no w_1 and L_1^i contains v_2 for $i = 3, \ldots, k$. Let $R_1^i, \ldots, R_d^i$ be the branches of $T - w_i$ containing no v_1, and R_1^i contains w_1 for $2 \leq i \leq k$. Clearly $R_1^2 = R_2^2 = \cdots = R_d^2 = C_2$.

Claim

- (1) $\tau(S(C_t), x) < \tau(S(L_1^t), x) < \tau(S(C_{t-1}), x)$, $t = 3, \ldots, k+1$.
- (2) $L_i^t = C_{t-1}$ or C_t, $t = 3, \ldots, k+1$, $i = 2, \ldots, d$.
- (3) $R_1^t = R_2^t = \cdots = R_d^t = C_t$, $t = 3, \ldots, k$.

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We prove Claim by the induction on t. For $t = 3$, by Lemma 3.4,

$$\begin{aligned}\tau(S(C_t), x) &= \frac{1}{1+\sum_{j=1}^{d}\frac{x}{1+x\tau(S(C_2),x)}}\\ &< \frac{1}{1+\sum_{j=1}^{d}\frac{x}{1+x\tau(S(L_j^2),x)}} = \tau(S(L_1^3), x)\\ &< \frac{1}{1+\sum_{j=1}^{d}\frac{x}{1+x\tau(S(C_1),x)}} = \tau(S(C_2), x).\end{aligned}$$

So Claim (1) holds for $t = 3$. By Lemma 3.4, Claim (2) holds for $t = 3$. Moreover, by $\tau(S(L_1^3), x) > \tau(S(C_3), x) = \tau(S(R_1^3), x)$ and Corollary 2.6, $\tau(S(L_1^3), x) \geq \max\{\tau(S(R_1^3), x), \ldots, \tau(S(R_d^3), x)$. Combining with Lemma 3.4 and $\tau(S(C_2), x) > \tau(S(L_1^3), x)$, we have $R_i^3 = C_3$ for $i = 2, \ldots, d$. Therefore Claim (3) holds for $t = 3$.

Assume that Claim holds for less than t and we consider Claim for t. By the induction prothesis,

$$\tau(S(C_{t-1}), x) \leq \tau(S(L_1^{t-1}), x) \leq \tau(S(C_{t-2}), x)$$

and $\tau(S(C_{t-1}), x) \leq \tau(S(L_j^{t-1}), x) \leq \tau(S(C_{t-2}), x)$ for $j = 2, \ldots, d$. It follows from (11) and (21) that

$$\begin{aligned}\tau(S(C_t), x) &= \frac{1}{1+\sum_{i=1}^{d}\frac{x}{1+x\tau(S(C_{t-1}),x)}}\\ &< \frac{1}{1+\sum_{i=1}^{d}\frac{x}{1+x\tau(S(L_i^{t-1}),x)}}\\ &= \tau(S(L_1^t), x)\\ &< \frac{1}{1+\sum_{i=1}^{d}\frac{x}{1+x\tau(S(C_{t-2}),x)}}.\\ &= \tau(S(C_{t-1}), x)\end{aligned}$$

Hence (1) holds for t. In order to prove (2) holds for t, we first prove the following several Claims

Claim 3.1 $\tau(S(C_t), x) \leq \tau(S(L_i^t), x) \leq \tau(S(C_{t-1}, x)$, for $i = 2, \ldots, d$.

In fact, there are d branches $L_1^t, \ldots, L_d^t$ containing no w_t in $T_{d+1}^* - v_t$ and there are d branches $C_t, R_2^t, \ldots, R_d^t$ containing no v_t in $T_{d+1}^* - w_t$. By Claim (1), we have $\tau(S(C_t), x) < \tau(S(L_1^t), x)$. Hence by Corollary 2.6, $\min\{\tau(S(L_1^t), x), \ldots, \tau(S(L_d^t), x)\} \geq \tau(S(C_t), x)$. On the other hand, there are d branches $L_1^t, \ldots, L_d^t$ containing no w_{t-1} in $T_{d+1}^* - v_t$ and there are d branches $C_{t-1}, \ldots, C_{t-1}$ containing no v_t in $T_{d+1}^* - w_{t-1}$. By Claim (1), $\tau(S(L_1^t), x) < \tau(S(C_{t-1}), x)$. Hence by Corollary 2.6, $\max\{S(L_2^t), x), \ldots, S(L_d^t), x)\} \leq \tau(S(C_{t-1}), x)$. So Claim 3.1 holds.

Let the maximum distance between v_t and any vertex in $L_1^t, \ldots, L_d^t$ is $l+1$. Denote by

$$V_j = \left\{ u \mid dist(u, v_t) = l - j + 1, u \in \bigcup_{i=1}^{d} V(L_i^t) \right\}, j = 0, \ldots, l.$$

Claim 3.2 For any $u \in V_{l-j}$, there are d the branches $L_1^u, \ldots, L_d^u$ containing no v_t in $T_{d+1}^* - u$ such that

$$\tau(S(C_{t-j-1}), x) \leq \tau(S(L_i^u), x) \leq \tau(S(C_{t-j-2}), x), \quad i = 1, \ldots, d, \tag{24}$$

where $j = 0, \ldots, \min\{t, l\} - 2$.

We prove Claim 3.2 by the induction on j. Let $L_1^u, \ldots, L_d^u$ be d the branches containing no v_t in $T_{d+1}^* - u$ and T^u be the subtree consisting of u and $L_1^u, \ldots, L_d^u$. For $j = 0$, there exists a $1 \leq p \leq d$ such that $T^u = L_p^t$. If there exists an $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) < \tau(S(C_{t-1}), x)$, let $R_1^{t-1} = C_{t-1}, \ldots, R_d^{t-1}$ be d the branches containing no v_t in $T_{d+1}^* - w_{t-1}$. Hence by Corollary 2.6, $\max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-1}), x)$. Then

$$\begin{aligned}\tau(S(L_p^t), x) = \tau(S(T^u), x) &= \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(L_q^u),x)}} < \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(C_{t-1}),x)}} \\ &= \tau(S(C_t), x),\end{aligned}$$

which contradicts Claim 3.1. Therefore,

$$\tau(S(C_{t-1}), x) \leq \tau(S(L_i^u), x), i = 1, \ldots, d.$$

On the other hand, if there exists $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) > \tau(S(C_{t-2}), x)$. let $R_1^{t-2} = C_{t-2}, \ldots, R_d^{t-2}$ be d the branches containing no v_t in $T_{d+1}^* - w_{t-2}$. By Corollary 2.6,

$$\min\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-2}), x).$$

Then

$$\begin{aligned}\tau(S(L_p^t), x) = \tau(S(T^u), x) &= \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(L_q^u),x)}} > \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(C_{t-1}),x)}} \\ &= \tau(S(C_{t-1}), x),\end{aligned}$$

which contradicts Claim 3.1. Hence Claim 3.2 holds for $j = 0$. Now assume that Claim 3.2 holds for j and consider the claim for $j + 1$. For any $u \in V_{l-(j+1)}$, let $L_1^u, \ldots, L_d^u$ be d the branches containing no v_t in $T_{d+1}^* - u$ and T^u be the subtree consisting of u and $L_1^u, \ldots, L_d^u$. Clearly there exists a $u' \in V_{l-j}$ such that there exists a branch $L_1^{u'}$ in $T_{d+1}^* - u'$ such that $T^u = L_1^{u'}$.

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If there exists an $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) < \tau(S(C_{t-j-2}), x)$, let $R_1^{t-j-2} = C_{t-j-2}, \ldots, R_d^{t-j-2}$ be d the branches containing no u in $T_{d+1}^* - w_{t-j-1}$. By Corollary 2.6,

$$\max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-j-2}), x).$$

Then

$$\tau(S(L_1^{u'}), x) = \tau(S(T^u), x) = \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(L_q^u),x)}} < \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(C_{t-j-2}),x)}}$$
$$= \tau(S(C_{t-j-1}), x),$$

which contradicts the induction hypothesis. Therefore, for any $u \in V_{l-j-1}$,

$$\tau(S(C_{t-j-2}), x) \leq \tau(S(L_i^u), x), i = 1, \ldots, d.$$

On the other hand, if there exists $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) > \tau(S(C_{t-j-3}), x)$. Let $R_1^{t-j-3} = C_{t-j-3}, \ldots, R_d^{t-j-3}$ be d the branches containing no u in $T_{d+1}^* - w_{t-j-2}$. By Corollary 2.6,

$$\min\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \geq \tau(S(C_{t-j-3}), x).$$

Then

$$\tau(S(L_1^{u'}), x) = \tau(S(T^u), x) = \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(L_q^u),x)}} > \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(C_{t-j-3}),x)}}$$
$$= \tau(S(C_{t-j-2}), x),$$

which contradicts the induction hypothesis. Hence Claim 3.2 holds for $j+1$. Therefore Claim 3.2 holds.

Claim 3.3 $l = t - 1$.

If $l > t - 1$, by Claim 3.2, for any $u \in V_{l-t+2}$,

$$\tau(S(C_1), x) \leq \tau(S(L_i^u), x), \ i = 1, \ldots, d.$$

On the other hand, there exists a $u' \in V_{l-t+2}$ such that the largest distance between u' and the pendent vertex is at least 2, then C_2 is a proper subgraph $L_1^{u'}$, which implies $\tau(S(L_1^{u'}), x) \leq \tau(S(C_2), x)$. it is a contradiction. Hence $l \leq t - 1$. Since $l \geq t - 1$, then $l = t - 1$.

Claim 3.4 For any $u \in V_{t-j-1}$, $j = 0, \ldots, t - 3$. Let $L_1^u, \ldots, \ldots, L_d^u$ be the d branches containing no v_t in $T_{d+1}^* - u$ and T^u consist of u and d branches $L_1^u, \ldots, \ldots, L_d^u$. Then $L_1^u = \cdots = L_d^u = C_{t-j-1}$ or $L_1^u = \cdots = L_d^u = C_{t-j-2}$, i.e., $T^u = C_{t-j}$ or $T^u = C_{t-j-1}$.

We use induction for $t - j - 1$. In fact, for $j = t - 3$ and $u \in V_2$, by Claim 3.2, $\tau(S(C_2), x) \leq \tau(S(L_i^u), x) \leq \tau(S(C_1), x)$ for $i = 1, \ldots, d$.

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Hence $L_i^u = C_2$ or $L_i^u = C_1$ for $i = 1, \ldots, d$. If, say $L_1^u = C_2$ and $L_2^u = C_1$, then by $\tau(S(L_1^2), x) > \tau(S(L_1^u), x)$ and Corollary 2.6, $\tau(S(L_1^2), x) \geq \max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \geq \tau(S(C_1), x)$, which is a contradiction. Hence $L_1^u = \cdots = L_d^u = C_2$ or $L_1^u = \cdots = L_d^u = C_1$, i.e., $T^u = C_3$ or $T^u = C_2$ for $u \in V_2$. Assume that Claim 3.4 hold for any vertex in V_{t-j-2}. Now for $u \in V_{t-j-1}$. Let $z_1, \ldots, z_d \in V_{t-j-1}$ be the roots of $L_1^u, \ldots, L_d^u$, respectively. By the induction hypothesis, $L_1^u, \ldots, L_d^u \in \{C_{t-j-1}, C_{t-j-2}\}$. Further $L_1^u = \cdots = L_d^u = C_{t-j-1}$ or $L_1^u = \cdots = L_d^u = C_{t-j-2}$. In fact, if, say $L_1^u = C_{t-j-1}$ and $L_2^u = C_{t-j-2}$, By $\tau(S(L_1^{t-j-1}), x) > \tau(S(C_{t-j-1}), x)$ and Corollary 2.6,

$$\tau(S(L_1^{t-j-1}), x) \geq \max\{\tau(S(C_{t-j-1}), x), \tau(S(C_{t-j-2}), x) \geq \tau(S(C_{t-j-2}), x),$$

which contradiction to Claim 3.1. Hence $L_1^u = \cdots = L_d^u = C_{t-j-1}$ or $L_1^u = \cdots = L_d^u = C_{t-j-2}$, i.e., $T^u = C_{t-j}$ or $T^u = C_{t-j-1}$. So Claim 3.4 holds. Hence $L_i^t = C_t$ or $L_i = C_{t-1}$ for $i = 2, \ldots, d$. In other words, Claim (2) holds.

Similarly, we can prove Claim (3) and omit the detail. It is easy from Claim that T_{d+1}^* is greedy tree. If the length of P is odd, using similar way to prove this assertion. So we finish our proof. □

Lemma 3.6 Let $S(T)$ be an optimal tree in $\mathcal{S}(\mathcal{T})_{n,d+1}$. If degree of each vertex is 1 or $d+1$ in $V(T)$, and there are d pendente vertices or no pendent vertices in $N(u)$ for any $u \in V(T)$. Then T is greedy tree T_{d+1}^*.

Proof Let $U = \{u \mid deg(u) = 1, u \in V(T)\}$ and $dist(v, U) = \min\{dist(v, u), u \in U\}$.

$$U_i = \{v \mid dist(v, U) = i\},\ i = 1, 2, \ldots.$$

Clearly, for any $v \in U_1, T - v$ has d branches $C_1, \ldots, C_1$. If for any vertex $v \in U_i, T - v$ has d branches $C_i, \ldots, C_i$ for $i = 1, 2, \ldots$, then T is the greedy tree T_{d+1}^*, and we complete the proof. Next assume that there exists a vertex $v \in U_i$ such that $T - v$ has at least two branches different from C_i. Let t_0 be the smallest integer such that for any vertex $v \in U_i, T - v$ has d branches $C_i, \ldots, C_i$ for $i = 1, \ldots, t_0 - 1$ and for some vertex $v \in U_{t_0}, T - v$ has at least two branches different from C_{t_0}. Let P be the longest path through $v = v_s \in U_{t_0}$ (see Fig 2). Then by $v \in V_{t_0}$, we have $s \geq t_0 + 1$. Assume that the length of P is even.

Let $L_1^i, \ldots, L_d^i$ be the branches of $T - v_i$ containing no w_1 and L_1^i contains v_1 for $i = 2, \ldots, k$. Let $R_1^i, \ldots, R_d^i$ be the branches of $T - w_i$ containing no v_1, and R_1^i contains w_1 for $2 \leq i \leq k$. By the definition of $v = v_s, C_{t_0}$ is a branch of $T - v_s$ and C_{t_0+1} is a subtree of L_j^s if $L_i^s \neq C_{t_0},\ i = 1, 2, \ldots, d$. Then $\tau(S(L_j^s), x) \leq \tau(S(C_{t_0+1}), x)$ for $L_i^s \neq C_{t_0}, i = 1, \ldots .d$. Similarly, C_{t_0} is a branch of $T - w_{t_0}$ and C_{t_0+1} is a subtree of $R_j^{t_0}$ if $R_i^{t_0} \neq C_{t_0},\ i = 1, 2, \ldots, d$. Then $\tau(S(R_j^{t_0}), x) \leq \tau(S(C_{t_0+1}), x)$ for $R_i^{t_0} \neq C_{t_0}, i = 1, \ldots .d$. If there is a branch of $R_1^{t_0}, R_2^{t_0}, \ldots, R_d^{t_0}$ is not C_{t_0}, say $R_1^{t_0} \neq C_{t_0}$. By Corollary 2.6, we have

$$\tau(S(R_1^{t_0}), x) \geq \min\{\tau(S(R_1^{t_0}), x), \ldots, \tau(S(R_d^{t_0}), x)\}$$

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$$\geq \max\{\tau(S(L_1^s), x), \ldots, \tau(S(L_d^s), x)\} \geq \tau(S(C_{t_0}), x).$$

It contradicts $\tau(S(R_1^{t_0}), x) \leq \tau(S(C_{t_0+1}), x)$. Thus $R_1^{t_0} = \cdots = R_d^{t_0} = C_{t_0}$. Which will imply that $R_1^{t_0+1} = C_{t_0+1}$. Since $T - v_i$ has branches $L_1^s, \ldots, L_d^s$ containing no w_1 and $T - w_{t_0+1}$ has branches $R_1^{t_0+1}, \ldots, R_d^{t_0+1}$ containing no v_1, by $\tau(S(C_{t_0+1}), x) < \tau(S(C_{t_0}), x)$ and Corollary 2.6, we have

$$\tau(S(C_{t_0+1}), x) \leq \max\{\tau(S(R_1^{t_0+1}), x), \ldots, \tau(S(R_d^{t_0+1}), x)\}$$
$$\leq \min\{\tau(S(L_1^s), x), \ldots, \tau(S(L_d^s), x)\} \leq \tau(S(C_{t_0+1}), x).$$

Since C_{t_0+1} is a subtree of L_i^s if $L_i^s \neq C_{t_0},\ i = 1, 2, \ldots, d$, then $L_i^s = C_{t_0+1}$ if $L_i^s \neq C_{t_0},\ i = 1, 2, \ldots, d$. This implies $s = t_0 + 1$. Without loss of generality, we can assume $L_1^{t_0+1} = C_{t_0+1}$ and $L_2^{t_0+1} = C_{t_0}$.

Claim

- (1) $\tau(S(C_t), x) < \tau(S(L_1^t), x) < \tau(S(C_{t-1}), x),\ t = t_0 + 2, \ldots, k + 1,\ \tau(S(L_i^{t_0+1}), x) = \tau(S(C_{t_0+1}), x)$ or $\tau(S(C_{t_0}), x),\ i = 1, \ldots, d$.
- (2) $L_i^t = C_{t-1}$ or $C_t,\ t = t_0 + 1, \ldots, k + 1,\ i = 2, \ldots, d$.
- (3) $R_1^t = R_2^t = \cdots = R_d^t = C_t,\ t = t_0 + 1, \ldots, k$.

We prove Claim by the induction on t. For $t = t_0 + 1$, by the above argument, we can find Claim holds. Assume that Claim holds for the number less than $t > t_0 + 1$ and we consider Claim for t. By the induction prothesis,

$$\tau(S(C_{t-1}), x) \leq \tau(S(L_1^{t-1}), x) \leq \tau(S(C_{t-2}), x)$$

and $\tau(S(C_{t-1}), x) \leq \tau(S(L_j^{t-1}), x) \leq \tau(S(C_{t-2}), x)$ for $j = 2, \ldots, d$. It follows from (11) and (21) that

$$\tau(S(C_t), x) = \frac{1}{1 + \sum_{i=1}^d \frac{x}{1+x\tau(S(C_{t-1}),x)}}$$
$$< \frac{1}{1 + \sum_{i=1}^d \frac{x}{1+x\tau(S(L_i^{t-1}),x)}}$$
$$= \tau(S(L_1^t), x)$$
$$< \frac{1}{1 + \sum_{i=1}^d \frac{x}{1+x\tau(S(C_{t-2}),x)}}.$$
$$= \tau(S(C_{t-1}), x)$$

Hence (1) holds for t. In order to prove (2) holds for t, we first prove the following several Claims.

Claim 3.1 $\tau(S(C_t), x) \leq \tau(S(L_i^t), x) \leq \tau(S(C_{t-1}, x)$ for $i = 2, \ldots, d$.

In fact, there are d branches $L_1^t, \ldots, L_d^t$ containing no w_t in $T_{d+1}^* - v_t$ and there are d branches $R_1^t = C_t, R_2^t, \ldots, R_d^t$ containing no v_t in $T_{d+1}^* - w_t$. By

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(1) of the Claim, we have $\tau(S(C_t), x) < \tau(S(L_1^t), x)$. Hence by Corollary 2.6, $\min\{\tau(S(L_1^t), x), \ldots, \tau(S(L_d^t), x)\} \geq \tau(S(C_t), x)$. On the other hand, there are d branches $L_1^t, \ldots, L_d^t$ containing no w_{t-1} in $T_{d+1}^* - v_t$ and there are d branches $C_{t-1}, \ldots, C_{t-1}$ containing no v_t in $T_{d+1}^* - w_{t-1}$. By (1) of the Claim, $\tau(S(L_1^t), x) < \tau(S(C_{t-1}), x)$. Hence by Corollary 2.6, $\max\{S(L_2^t), x), \ldots, S(L_d^t), x)\} \leq \tau(S(C_{t-1}), x)$. So Claim 3.1 holds.

Let the maximum distance between v_t and any vertex in $L_1^t, \ldots, L_d^t$ is $l + 1$. By the definition of P, we can find that $l \geq t - 1$. Denote by

$$V_j = \left\{ u \mid dist(u, v_t) = l - j + 1, u \in \bigcup_{i=1}^{d} V(L_i^t) \right\}, j = 0, \ldots, l.$$

Claim 3.2 For any $u \in V_{l-j}$, there are d branches $L_1^u, \ldots, L_d^u$ containing no v_t in $T_{d+1}^* - u$ such that

$$\tau(S(C_{t-j-1}), x) \leq \tau(S(L_i^u), x) \leq \tau(S(C_{t-j-2}), x), \quad i = 1, \ldots, d, \tag{25}$$

where $j = 0, \ldots, \min\{t, l\} - 2$.

We prove Claim 3.2 by the induction on j. Let $L_1^u, \ldots, L_d^u$ be d the branches containing no v_t in $T_{d+1}^* - u$ and T^u be the subtree consisting of u and $L_1^u, \ldots, L_d^u$. For $j = 0$, there exists a $1 \leq p \leq d$ such that $T^u = L_p^t$. If there exists an $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) < \tau(S(C_{t-1}), x)$, let $R_1^{t-1} = C_{t-1}, \ldots, R_d^{t-1}$ be d the branches containing no v_t in $T_{d+1}^* - w_{t-1}$. Hence by Corollary 2.6, $\max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-1}), x)$. Then

$$\begin{aligned}\tau(S(L_p^t), x) = \tau(S(T^u), x) &= \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1 + x\tau(S(L_q^u), x)}} < \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1 + x\tau(S(C_{t-1}), x)}} \\ &= \tau(S(C_t), x),\end{aligned}$$

which contradicts Claim 3.1. Therefore,

$$\tau(S(C_{t-1}), x) \leq \tau(S(L_i^u), x), i = 1, \ldots, d.$$

On the other hand, if there exists $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) > \tau(S(C_{t-2}), x)$. let $R_1^{t-2} = C_{t-2}, \ldots, R_d^{t-2}$ be d the branches containing no v_t in $T_{d+1}^* - w_{t-2}$. By Corollary 2.6,

$$\min\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-2}), x).$$

Then

$$\begin{aligned}\tau(S(L_p^t), x) = \tau(S(T^u), x) &= \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1 + x\tau(S(L_q^u), x)}} > \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1 + x\tau(S(C_{t-1}), x)}} \\ &= \tau(S(C_{t-1}), x),\end{aligned}$$

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which contradicts Claim 3.1. Hence Claim 3.2 holds for $j = 0$. Now assume that Claim 3.2 holds for j and consider the claim for $j+1$. For any $u \in V_{l-(j+1)}$, let $L_1^u, \ldots, L_d^u$ be d the branches containing no v_t in $T_{d+1}^* - u$ and T^u be the subtree consisting of u and $L_1^u, \ldots, L_d^u$. Clearly there exists a $u' \in V_{l-j}$ such that there exists a branch $L_1^{u'}$ in $T_{d+1}^* - u'$ such that $T^u = L_1^{u'}$.

If there exists an $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) < \tau(S(C_{t-j-2}), x)$, let $R_1^{t-j-2} = C_{t-j-2}, \ldots, R_d^{t-j-2}$ be d the branches containing no u in $T_{d+1}^* - w_{t-j-1}$. By Corollary 2.6,

$$\max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \leq \tau(S(C_{t-j-2}), x).$$

Then

$$\begin{aligned}\tau(S(L_1^{u'}), x) = \tau(S(T^u), x) &= \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(L_q^u),x)}} < \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(C_{t-j-2}),x)}} \\ &= \tau(S(C_{t-j-1}), x),\end{aligned}$$

which contradicts the induction hypothesis. Therefore, for any $u \in V_{l-j-1}$,

$$\tau(S(C_{t-j-2}), x) \leq \tau(S(L_i^u), x), i = 1, \ldots, d.$$

On the other hand, if there exists $1 \leq i \leq d$ such that $\tau(S(L_i^u), x) > \tau(S(C_{t-j-3}), x)$. Let $R_1^{t-j-3} = C_{t-j-3}, \ldots, R_d^{t-j-3}$ be d the branches containing no u in $T_{d+1}^* - w_{t-j-2}$. By Corollary 2.6,

$$\min\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \geq \tau(S(C_{t-j-3}), x).$$

Then

$$\begin{aligned}\tau(S(L_1^{u'}), x) = \tau(S(T^u), x) &= \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(L_q^u),x)}} > \frac{1}{1 + \sum_{q=1}^{d} \frac{x}{1+x\tau(S(C_{t-j-3}),x)}} \\ &= \tau(S(C_{t-j-2}), x),\end{aligned}$$

which contradicts the induction hypothesis. Hence Claim 3.2 holds for $j+1$. Therefore Claim 3.2 holds.

Claim 3.3 $l = t - 1$.

If $l > t - 1$, by Claim 3.2, for any $u \in V_{l-t+2}$,

$$\tau(S(C_1), x) \leq \tau(S(L_i^u), x), \ i = 1, \ldots, d.$$

On the other hand, there exists a $u' \in V_{l-t+2}$ such that the largest distance between u' and the pendent vertex is at least 2, then C_2 is a proper subgraph $L_1^{u'}$, which implies $\tau(S(L_1^{u'}), x) \leq \tau(S(C_2), x)$. it is a contradiction. Hence $l \leq t - 1$. Since $l \geq t - 1$, then $l = t - 1$.

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Claim 3.4 For any $u \in V_{t-j-1}$, $j = 0, \ldots, t-3$. Let $L_1^u, \ldots\ldots, L_d^u$ be the d branches containing no v_t in $T_{d+1}^* - u$ and T^u consist of u and d branches $L_1^u, \ldots\ldots, L_d^u$. Then $L_1^u = \cdots = L_d^u = C_{t-j-1}$ or $L_1^u = \cdots = L_d^u = C_{t-j-2}$, i.e., $T^u = C_{t-j}$ or $T^u = C_{t-j-1}$.

We use induction for $t-j-1$. In fact, for $j = t-3$ and $u \in V_2$, by Claim 3.2, $\tau(S(C_2), x) \leq \tau(S(L_i^u), x) \leq \tau(S(C_1), x)$ for $i = 1, \ldots, d$. Hence $L_i^u = C_2$ or $L_i^u = C_1$ for $i = 1, \ldots, d$. If, say $L_1^u = C_2$ and $L_2^u = C_1$, then by $\tau(S(L_1^2), x) > \tau(S(L_1^u), x)$ and Corollary 2.6, $\tau(S(L_1^2), x) \geq \max\{\tau(S(L_1^u), x), \ldots, \tau(S(L_d^u), x)\} \geq \tau(S(C_1), x)$, which is a contradiction. Hence $L_1^u = \cdots = L_d^u = C_2$ or $L_1^u = \cdots = L_d^u = C_1$, i.e., $T^u = C_3$ or $T^u = C_2$ for $u \in V_2$. Assume that Claim 3.4 hold for any vertex in V_{t-j-2}. Now for $u \in V_{t-j-1}$. Let $z_1, \ldots, z_d \in V_{t-j-1}$ be the roots of $L_1^u, \ldots, L_d^u$, respectively. By the induction hypothesis, $L_1^u, \ldots, L_d^u \in \{C_{t-j-1}, C_{t-j-2}\}$. Further $L_1^u = \cdots = L_d^u = C_{t-j-1}$ or $L_1^u = \cdots = L_d^u = C_{t-j-2}$. In fact, if, say $L_1^u = C_{t-j-1}$ and $L_2^u = C_{t-j-2}$, By $\tau(S(L_1^{t-j-1}), x) > \tau(S(C_{t-j-1}), x)$ and Corollary 2.6,

$$\tau(S(L_1^{t-j-1}), x) \geq \max\{\tau(S(C_{t-j-1}), x), \tau(S(C_{t-j-2}), x) \geq \tau(S(C_{t-j-2}), x),$$

which contradiction to Claim 3.1. Hence $L_1^u = \cdots = L_d^u = C_{t-j-1}$ or $L_1^u = \cdots = L_d^u = C_{t-j-2}$, i.e., $T^u = C_{t-j}$ or $T^u = C_{t-j-1}$. So Claim 3.4 holds.

Hence $L_i^t = C_t$ or $L_i^t = C_{t-1}$ for $i = 2, \ldots, d$. In other words, (2) of Claim holds.

Similarly, we can prove (3) of Claim, here we omit the detail. It is easy from Claim that T_{d+1}^* is greedy tree. If the length of P is odd, using similar way to prove this assertion. So we finish our proof. □

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