Parking functions on nonsingular M-matrices

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Abstract

In this paper, let Δ be a nonsingular M-matrix. A generalization of G-parking functions, which is called Δ-parking functions, is studied. An explicit characterization for Δ-parking functions is given. It is shown that Δ-parking functions can be obtained by a simple way from recurrent configurations on the nonsingular M-matrix Δ. It is proved that the number of Δ-parking functions is equal to the determinant of Δ.

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1 Introduction

In 1966, Konheim and Weiss [8] introduced the conception of parking functions in the study of the linear probes of random hashing function. Many generalizations of parking functions were studied. Please refer to [4, 9, 12, 14, 15, 16, 17]. In 2004, Postnikov and Shapiro [13] introduced a new generalization, the G-parking functions, in the study of certain quotients of the polynomial ring. Let G be a digraph with vertex set $V(G) = \{0, 1, 2, \ldots, n\}$ and edge set $E(G)$. We allow G to have multiple edges and loops. For any $I \subseteq V(G) \setminus \{0\}$ and $v \in I$, define $\text{outdeg}_{I,G}(v)$ to be the number of edges directed from the vertex v to a vertex outside of the subset I in G. G-parking functions are defined as follows.

- A G-parking function is a function $f : V(G) \setminus \{0\} \to \{0, 1, 2, \ldots\}$, such that for every $I \subseteq V(G) \setminus \{0\}$ there exists a vertex $v \in I$ such that $0 \leq f(v) < outdeg_{I,G}(v)$.

For the complete graph $G = K_{n+1}$ on $n+1$ vertices, K_{n+1}-parking functions are exactly the classical parking functions. G-parking functions are an important tool for the determination of the rank defined by Baker and Norine in [2] on a Riemann Roch theorem for graphs.

In 1990, Dhar [5] introduced the abelian sandpile model, which is also known as the chip-firing game, and showed that the number of recurrent configurations on a toppling matrix equals the determinant of the matrix. In 1993, Gabrielov [6] studied the sandpile model for a class of toppling matrices, which is more general than in [5]. An explicit characterization for recurrent configurations on a toppling-matrix appeared originally in [11] and later in [1], [10] and [7]. Throughout the paper, we always let Δ be an integer matrix. We state the definition of toppling matrices as follows.

Definition 1.1. *A Z-matrix is an $n \times n$ matrix $\Delta = (\Delta_{ij})_{1 \leq i,j \leq n}$ such that $\Delta_{ij} \leq 0$ for all $i \neq j$.*

Let Δ be a Z-matrix. We say that Δ is a toppling matrix if there exists an integer vector $\mathbf{h}$ of length n with $\mathbf{h} \geq \mathbf{0}$ such that $\Delta\mathbf{h} > \mathbf{0}$. Here the notation $\mathbf{0}$ denotes a vector of length n in which

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all coordinates have value 0, and the notation $\mathbf{h} \geq \mathbf{h}'$ ($\mathbf{h} > \mathbf{h}'$ resp.) means that $h_i \geq h_i'$ ($h_i > h_i'$ resp.) for every i.

Toppling matrices are simply *nonsingular M-matrices* which have been extensively studied in the matrix theory literature, see also Berman and Plemmons's book [3]. In this book, the 50 conditions which are equivalent to the statement:"Δ is a nonsingular M-matrix" are given. Following [3], we define nonsingular M-matrices as follows:

Definition 1.2. *Let Δ be a Z-matrix. If any of the following equivalent conditions hold then Δ is called a non-singular M-matrix:*

(1) There exists a vector $\mathbf{h} > \mathbf{0}$ such that $\Delta\mathbf{h} > \mathbf{0}$.

(2) Δ is nonsingular and all principal minors of Δ are positive.

(3) There exists an integer vector $\mathbf{h}$ of length n with $\mathbf{h} \geq \mathbf{0}$ such that $\Delta\mathbf{h} > \mathbf{0}$.

(4) There exists a positive diagonal matrix H such that ΔH has all positive row sums.

Clearly, Condition (2) implies that a Z-matrix Δ is a nonsingular M-matrix if and only if its transposed matrix Δ^T of Δ is a nonsingular M-matrix, moreover, we have a Z-matrix Δ is a nonsingular M-matrix if and only if each principal submatrix of Δ is a nonsingular M-matrix.

The main objective of the present paper is to generalize the G-parking functions associated to a nonsingular M-matrix. Let Δ be a nonsingular M-matrix. In this paper, the generalization for G-parking functions are called Δ-parking functions. We give an explicit characterization for Δ-parking functions and show that Δ-parking functions can be obtained by a simple way from recurrent configurations on the nonsingular M-matrix Δ. It is proved that the number of Δ-parking functions is equal to the determinant of Δ.

The rest of this paper is organized as follows. In Section 2, we consider an equivalent definition for nonsingular M-matrices. In Section 3, we define Δ-parking functions and give their explicit characterization. In Section 4, we define Δ-recurrent configurations, show that Δ-parking functions can be obtained by a simple way from Δ-recurrent configurations, and prove that the number of Δ-parking functions is equal to the determinant of Δ.

2 An equivalent definition for nonsingular M-matrices

In this section, we study an equivalent definition for nonsingular M-matrices. Let Δ be a Z-matrix. In [6], Gabrielov gave a sufficient condition for nonsingular M-matrices.

Proposition 2.1. *(Gabrielov, [6]) Let Δ be a Z-matrix. Suppose that Δ is nonsingular and has all nonnegative column sums. Then Δ is a nonsingular M-matrix.*

The condition, Δ has all nonnegative column sums, is not necessary for nonsingular M-matrices. For example, let us consider the matrix

$$\Delta = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}.$$

It is easy to check that

$$\Delta \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and so Δ is a nonsingular M-matrices. But Δ has a negative column row.

We are interested in the following equivalent definition for Nonsingular M-matrices.

Proposition 2.2. *Let Δ be be a Z-matrix. Δ is a nonsingular M-matrix if and only if Δ is nonsingular and there exists a vector $\mathbf{r} = (r_1, \ldots, r_n) > \mathbf{0}$ such that $\mathbf{r}\Delta \geq \mathbf{0}$.*

Proof. Suppose that Δ is nonsingular and there exists a vector $\mathbf{r} = (r_1, \ldots, r_n) > \mathbf{0}$ such that $\mathbf{r}\Delta \geq \mathbf{0}$. Let

$$\tilde{\Delta} = \tilde{\Delta}(\mathbf{r}) = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_n \end{pmatrix} \Delta = \begin{pmatrix} r_1\Delta_{11} & r_1\Delta_{12} & \cdots & r_1\Delta_{1n} \\ r_2\Delta_{21} & r_2\Delta_{22} & \cdots & r_2\Delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_n\Delta_{n1} & r_n\Delta_{n2} & \cdots & r_n\Delta_{nn} \end{pmatrix}$$

Since $\mathbf{r}\Delta \geq \mathbf{0}$, the matrix $\tilde{\Delta}$ has all nonnegative column sums. $\tilde{\Delta}$ is nonsingular since $\det \tilde{\Delta} = r_1 \cdots r_n \cdot \det \Delta \neq 0$. By Proposition 2.1, we have $\tilde{\Delta}$ is a nonsingular M-matrix. By Definition 1.2, there exists a column vector $\mathbf{h} > \mathbf{0}$ such that $\tilde{\Delta}\mathbf{h} > \mathbf{0}$. Suppose $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T = \Delta\mathbf{h}$. We have

$$\tilde{\Delta}\mathbf{h} = (r_1v_1, r_2v_2, \ldots, r_nv_n)^T > \mathbf{0}.$$

This implies $\Delta\mathbf{h} > \mathbf{0}$ and Δ is a nonsingular M-matrix.

Conversely, suppose that Δ is a nonsingular M-matrix. We have $\det \Delta \neq 0$ and the transposed matrix Δ^T of Δ is a nonsingular M-matrix. By Definition 1.2, there exists a column vector $\mathbf{h} > \mathbf{0}$ such that $\Delta^T\mathbf{h} > \mathbf{0}$. So $\mathbf{h}^T\Delta > \mathbf{0}$. □

3 Δ-parking functions

For any nonsingular M-matrix Δ, let

$$\mathscr{R}(\Delta) = \{\mathbf{r} \in \mathbb{Z}^n \mid \mathbf{r}\Delta \geq \mathbf{0} \text{ and } \mathbf{r} > \mathbf{0}\}$$

where $\mathbb{Z}$ is the set of integers. For any $\mathbf{r} \in \mathscr{R}(\Delta)$, denote by $\Omega(\mathbf{r})$ the set of nonzero integer vectors

$$\chi = (\chi(1), \cdots, \chi(n))$$

such that

$$0 \leq \chi(i) \leq r_i \text{ for every } i.$$

Let $\Delta^j = (\Delta_{1j}, \ldots, \Delta_{nj})^T$ be the j-th column of Δ. For any two vectors X, Y of length n, we consider the standard inner product given by $\langle X, Y\rangle = \sum_{i=1}^{n} X_iY_i$. We define $(\Delta, \mathbf{r})$-parking functions as follows:

Definition 3.1. *Let $\mathbf{r} \in \mathscr{R}(\Delta)$. A $(\Delta, \mathbf{r})$-parking function is a function $f : \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots\}$ such that for any $\chi \in \Omega(\mathbf{r})$ there exists an integer $j \in \{1, 2, \ldots, n\}$ with $\chi(j) \geq 1$ such that*

$$0 \leq f(j) < \langle \chi, \Delta^j \rangle.$$

Denote by $\mathcal{P}(\Delta, \mathbf{r})$ the set of $(\Delta, \mathbf{r})$-parking functions.

Example 3.2. *Let us consider a digraph G with vertex set $\{0, 1, \ldots, n\}$ and at least one 0-sink spanning tree. Let L_G be the Laplace matrix that corresponds to the digraph G and L_0 the truncated Laplace matrix obtained from the matrix L_G by deleting the rows and columns indexed by 0. The transposed matrix L_0^T of L_0 satisfies the conditions in (1) and the vector $\mathbf{1} \in \mathscr{R}(L_0^T)$, where the notation $\mathbf{1}$ denotes a vector of length n in which all coordinates have value 1. $(L_0^T, \mathbf{1})$-parking functions are exactly G-parking functions.*

Example 3.3. *The matrix* Δ *and the vector* $\mathbf{r}$ *are given as follows:*

$$\Delta = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}, \mathbf{r} = (2, 1).$$

Since the sum of entries of the first column of Δ *is less than* 0*, the transposed matrix of* Δ *is not a truncated Laplace matrix of any digraph. We have*

$$\Omega(\mathbf{r}) = \{(1, 0), (2, 0), (0, 1), (1, 1), (2, 1)\}$$

and

$$\mathcal{P}(\Delta, \mathbf{r}) = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1)\}.$$

For any $\mathbf{r} \in \mathscr{R}(\Delta)$, let $m = m(\mathbf{r}) = \sum_{i=1}^{n} r_i$ and denote by $V(\mathbf{r})$ a multiset in which the multiplicity of the integer i is r_i for every $i \in \{1, 2, \dots, n\}$. For any submultiset W of $V(\mathbf{r})$, let $\chi(i)$ be the multiplicity of the integer i in W for every $i \in \{1, 2, \cdots, n\}$. Then $\chi = (\chi(1), \dots, \chi(n)) \in \Omega(\mathbf{r})$. We also call χ the characteristic function of W.

Example 3.4. *Let us consider the matrix* Δ *and the vector* $\mathbf{r}$ *in the example 3.3. Then*

$$V(\mathbf{r}) = \{1, 1, 2\} \text{ and } m(\mathbf{r}) = 3.$$

Take $W = \{1, 1\}$*. Then the characteristic function* χ *of* W *is* $(2, 0) \in \Omega(\mathbf{r})$*.*

Lemma 3.5. *For any* $\mathbf{r} \in \mathscr{R}(\Delta)$*, let* $m = m(\mathbf{r}) = \sum_{i=1}^{n} r_i$*. Then* f *is a* $(\Delta, \mathbf{r})$*-parking function if and only if there is a sequence of integers in the multiset* $V(\mathbf{r})$

$$\pi(1), \dots, \pi(m)$$

such that for every $i \in \{1, 2, \cdots, m\}$*,*

$$0 \le f(\pi(i)) < \langle \chi_i, \Delta^{\pi(i)} \rangle$$

where χ_i *is the characteristic function of the multiset* $\{\pi(i), \pi(i+1), \dots, \pi(m)\}$*.*

Proof. Suppose that f is a $(\Delta, \mathbf{r})$-parking function. We construct a sequence

$$\pi(1), \pi(2), \dots, \pi(m)$$

of integers in $V(\mathbf{r})$ by the following algorithm.
Algorithm A.

- Step 1. Let $W_1 = V(\mathbf{r})$, χ_1 the characteristic function of W_1 and

  $$U_1 = \left\{ j \in W_1 \mid 0 \le f(j) < \langle \chi_1, \Delta^j \rangle \right\}.$$

  Set $\pi(1) \in U_1$ by choose $\pi(1)$ as any integer in U_1.

- Step 2. At time $i \ge 2$, suppose $\pi(1), \dots, \pi(i-1)$ are determined. Let

  $$W_i = V(\mathbf{r}) \setminus \{\pi(1), \dots, \pi(i-1)\},$$

  χ_i the characteristic function of W_i and

  $$U_i = \left\{ j \in W_i \mid 0 \le f(j) < \langle \chi_i, \Delta^j \rangle \right\}.$$

  Set $\pi(i) \in U_i$ by choose $\pi(i)$ as any integer in U_i.

By Algorithm A, iterating Step 2 until $i = m$, we obtain the sequence of integers as desired.

Conversely, suppose that there is a sequence of integers in $V(\mathbf{r})$

$$\pi(1), \dots, \pi(m)$$

such that for every $i \in \{1, 2, \dots, m\}$,

$$0 \leq f(\pi(i)) < \langle \chi_i, \Delta^{\pi(i)} \rangle,$$

where χ_i is the characteristic function of $\{\pi(i), \pi(i+1), \dots, \pi(m)\}$.

For any $\chi \in \Omega(\mathbf{r})$, let k be the largest index $i \in \{1, 2, \dots, m\}$ such that $\chi_i \geq \chi$. Let $j = \pi(k)$. Then $\chi(j) = \chi_k(j)$ and $\chi_{k+1}(j) = \chi_k(j) - 1$. The algorithm A tells us that

$$0 \leq f(j) < \langle \chi_k, \Delta^{\pi(k)} \rangle = \langle \chi_k, \Delta^j \rangle.$$

By the choice of k we have

$$\langle \chi, \Delta^j \rangle = \chi(j)\Delta_{jj} - \sum_{i \neq j} \chi(i)(-\Delta_{ij}) \geq \chi(j)\Delta_{jj} - \sum_{i \neq j} \chi_k(i)(-\Delta_{ij}).$$

Moveover $\chi(j) = \chi_k(j)$ gives:

$$\langle \chi, \Delta^j \rangle \geq \langle \chi_k, \Delta^j \rangle.$$

Proving that for any $\chi \in \Omega(\mathbf{r})$ there exists an integer j such that $f(j) < \langle \chi, \Delta^j \rangle$; hence f is a $(\Delta, \mathbf{r})$-parking function. □

Example 3.6. *Let us consider the matrix Δ and the vector $\mathbf{r}$ in the example 3.3. We have*

$$V(\mathbf{r}) = \{1, 1, 2\} \text{ and } \mathcal{P}(\Delta, \mathbf{r}) = \{(0,0), (0,1), (0,2), (1,0), (1,1)\}.$$

In the following table, we list all $(\Delta, \mathbf{r})$-parking functions f and their corresponding sequences π of integers in $V(\mathbf{r})$ obtained by Algorithm A.

f	π	f	π	f	π
(0,0)	1,2,1	(0,1)	2,1,1	(0,2)	1,2,1
(1,0)	2,1,1	(1,1)	2,1,1		

Table 1. $(\Delta, \mathbf{r})$-parking functions f and their corresponding sequences π.

Note that we can also obtain the sequence $2, 1, 1$ of integers in $V(\mathbf{r})$ by the algorithm A for the Δ-parking function $(0, 0)$. So, in general, for a Δ-parking function f, the sequence of integers in $V(\mathbf{r})$ obtained by the algorithm A is not unique.

Lemma 3.7. *Suppose that $\mathbf{r}, \mathbf{r}' \in \mathscr{R}(\Delta)$ and $\mathbf{r} \leq \mathbf{r}'$. Then $\mathcal{P}(\Delta, \mathbf{r}') \subseteq \mathcal{P}(\Delta, \mathbf{r})$.*

Proof. Note that $\Omega(\mathbf{r}) \subseteq \Omega(\mathbf{r}')$ since $\mathbf{r} \leq \mathbf{r}'$. So f is a $(\Delta, \mathbf{r})$-parking function if it is a $(\Delta, \mathbf{r}')$-parking function. Hence we have $\mathcal{P}(\Delta, \mathbf{r}') \subseteq \mathcal{P}(\Delta, \mathbf{r})$. □

Lemma 3.8. *Suppose that $\mathbf{r}, \mathbf{r}' \in \mathscr{R}(\Delta)$. Then*

$$\mathcal{P}(\Delta, \mathbf{r} + \mathbf{r}') = \mathcal{P}(\Delta, \mathbf{r}) \cap \mathcal{P}(\Delta, \mathbf{r}').$$

Proof. By Lemma 3.7, we have $\mathcal{P}(\Delta, \mathbf{r}+\mathbf{r}') \subseteq \mathcal{P}(\Delta, \mathbf{r})$ and $\mathcal{P}(\Delta, \mathbf{r}+\mathbf{r}') \subseteq \mathcal{P}(\Delta, \mathbf{r}')$. So, $\mathcal{P}(\Delta, \mathbf{r}+\mathbf{r}') \subseteq \mathcal{P}(\Delta, \mathbf{r}) \cap \mathcal{P}(\Delta, \mathbf{r}')$.

Conversely, let $m = \sum_{i=1}^{n} r_i$ and $m' = \sum_{i=1}^{n} r'_i$. For any $f \in \mathcal{P}(\Delta, \mathbf{r}) \cap \mathcal{P}(\Delta, \mathbf{r}')$, by Lemma 3.5, there is a sequence

$$\pi(1), \dots, \pi(m)$$

of integers in $V(\mathbf{r})$ such that

$$0 \le f(\pi(i)) < \langle \chi_i, \Delta^{\pi(i)} \rangle$$

for every $i \in \{1, 2, \cdots, m\}$ and a sequence

$$\pi'(1), \ldots, \pi'(m')$$

of integers in $V(\mathbf{r}')$ such that

$$0 \le f(\pi'(i)) < \langle \chi'_i, \Delta^{\pi'(i)} \rangle$$

for every $i \in \{1, 2, \cdots, m'\}$ where χ_i and χ'_i are the characteristic functions of $\{\pi(i), \pi(i+1), \ldots, \pi(m)\}$ and $\{\pi'(i), \pi'(i+1), \ldots, \pi'(m')$ respectively.

Let us consider the following sequence

$$\sigma(1), \ldots, \sigma(m), \sigma(m+1), \ldots, \sigma(m+m')$$

where

$$\sigma(i) = \begin{cases} \pi(i) & \text{if} \quad 1 \le i \le m \\ \pi'(i-m) & \text{if} \quad 1+m \le i \le m+m' \end{cases}$$

For every $i = 1, 2, \cdots, m+m'$, let $\hat{\chi}_i$ is the characteristic functions of $\{\sigma(i), \cdots, \sigma(m+m')\}$. Then we have

$$f(\sigma(i)) = \begin{cases} f(\pi(i)) & \text{if} \quad 1 \le i \le m \\ f(\pi'(i-m)) & \text{if} \quad 1+m \le i \le m+m' \end{cases}$$

and

$$\langle \hat{\chi}_i, \Delta^{\sigma(i)} \rangle = \begin{cases} \langle \chi_i + \mathbf{r}', \Delta^{\sigma(i)} \rangle = \langle \chi_i, \Delta^{\sigma(i)} \rangle + \langle \mathbf{r}', \Delta^{\sigma(i)} \rangle & \text{if} \quad 1 \le i \le m \\ \langle {\chi'}_i, \Delta^{\sigma(i)} \rangle & \text{if} \quad 1+m \le i \le m+m' \end{cases}$$

Since $\mathbf{r}'\Delta \ge 0$, we have

$$f(\sigma(i)) < \langle \hat{\chi}_i, \Delta^{\sigma(i)} \rangle$$

for every $i = 1, 2, \ldots, m+m'$. By Lemma 3.5, f is a $(\Delta, \mathbf{r}+\mathbf{r}')$-parking function. Hence, $\mathcal{P}(\Delta, \mathbf{r}+\mathbf{r}') = \mathcal{P}(\Delta, \mathbf{r}) \cap \mathcal{P}(\Delta, \mathbf{r}')$. □

Corollary 3.9. *(1) Suppose that $\mathbf{r} \in \mathscr{R}(\Delta)$ and b is a positive integer. Then*

$$\mathcal{P}(\Delta, b\mathbf{r}) = \mathcal{P}(\Delta, \mathbf{r}).$$

(2) Suppose that $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_k \in \mathscr{R}(\Delta)$ and $b_1, b_2, \cdots, b_k$ are k positive integers. Then

$$\mathcal{P}(\Delta, b_1\mathbf{r}_1 + b_2\mathbf{r}_2 + \cdots + b_k\mathbf{r}_k) = \bigcap_{i=1}^{k} \mathcal{P}(\Delta, \mathbf{r}_i).$$

Theorem 3.10. *For any $\mathbf{r}, \mathbf{r}' \in \mathscr{R}(\Delta)$, $\mathcal{P}(\Delta, \mathbf{r}) = \mathcal{P}(\Delta, \mathbf{r}')$.*

Proof. Note that there is a positive b such that $b\mathbf{r} \ge \mathbf{r}'$ since $\mathbf{r} > 0$. By Lemma 3.7 and Corollary 3.9(1), we have

$$\mathcal{P}(\Delta, \mathbf{r}) = \mathcal{P}(\Delta, b\mathbf{r}) \subseteq \mathcal{P}(\Delta, \mathbf{r}').$$

Similarly, we have $\mathcal{P}(\Delta, \mathbf{r}') \subseteq \mathcal{P}(\Delta, \mathbf{r})$. Hence, $\mathcal{P}(\Delta, \mathbf{r}) = \mathcal{P}(\Delta, \mathbf{r}')$. □

Theorem 3.10 tells us that the set of $(\Delta, \mathbf{r})$-parking functions is independent of $\mathbf{r}$ for any $\mathbf{r} \in \mathscr{R}(\Delta)$. So, $(\Delta, \mathbf{r})$-parking functions are simply called Δ-parking functions and let $\mathcal{P}(\Delta)$ be the set of Δ-parking functions.

Let $\langle \Delta \rangle = \mathbb{Z}\Delta_1 \oplus \mathbb{Z}\Delta_2 \oplus \cdots \oplus \mathbb{Z}\Delta_n$ be the sublattice in $\mathbb{Z}^n$ spanned by the vectors Δ_i, where $\Delta_i = (\Delta_{i1}, \ldots, \Delta_{in})$ be the i-th row of Δ. We define an equivalence relation $\sim$ on $\mathbb{Z}^n$ by declaring that $f \sim f'$ if and only if $f - f' \in \langle \Delta \rangle$.

Lemma 3.11. *Let $\mathbf{r} \in \mathscr{R}(\Delta)$. Suppose f and f' are two $(\Delta, \mathbf{r})$-parking functions. If $f' - f \in \langle \Delta \rangle$, then $f' = f$.*

Proof. Assume that $f' \neq f$. Then $f' - f = x\Delta$ and $x \neq 0$. By symmetry, we may suppose that $x_j > 0$ for some $j \in \{1, 2, \ldots, n\}$.

Let b be a positive integer such that

$$\min\{br_i \mid i = 1, 2, \cdots, n\} \geq \max\{x_j \mid x_j > 0 \text{ and } 1 \leq j \leq n\}.$$

Let

$$\chi(j) = \begin{cases} x_j & \text{if} \quad x_j > 0 \\ 0 & \text{if} \quad x_j \leq 0 \end{cases}$$

for each $j = 1, 2, \cdots, n$. Then $\chi \in \Omega(\Delta, b\mathbf{r})$ and for any j with $\chi(j) > 0$

$$\begin{aligned} 0 \leq f(j) &= f'(j) - \sum_{k=1}^{n} x_k \Delta_{k,j} \\ &\leq f'(j) - \sum_{\substack{k=1 \\ x_k > 0}}^{n} x_k \Delta_{k,j} \\ &= f'(j) - \langle \chi, \Delta^j \rangle. \end{aligned}$$

So, for any j with $\chi(j) > 0$, we have $f'(j) \geq \langle \chi, \Delta^j \rangle$. Hence f' is not a $(\Delta, b\mathbf{r})$-parking function since $\chi \in \Omega(b\mathbf{r})$. But Corollary 3.9(1) implies that f' is a $(\Delta, b\mathbf{r})$-parking functions since f' is a $(\Delta, \mathbf{r})$-parking functions, a contradiction. □

Lemma 3.11 implies distinct Δ-parking functions cannot be equivalent and every equivalent class of $\mathbb{Z}^n$ contains at most one Δ-parking function. So we obtain the following corollary.

Corollary 3.12. *The number of Δ-parking functions is less than or equal to* $\det \Delta$.

Proof. Since the order of the quotient of the integer lattice $\mathbb{Z}^n / \langle \Delta \rangle$ is $\det \Delta$, it follows from Lemma 3.11 and Theorem 3.10 that $|\mathcal{P}(\Delta)| \leq \det \Delta$. □

4 Δ-recurrent configurations

For a nonsingular M-matrix Δ, let $\Delta_i = (\Delta_{i1}, \ldots, \Delta_{in})$ be the i-th row of Δ. A row vector $\mathbf{u} = (u_1, \ldots, u_n)$ is called a configuration if $u_i \geq 0$ for all i. In the sandpile model, the number u_i is interpreted as the the number of particles, or grains of sand, at site $i = 1, \ldots, n$. For any site i, if $u_i \geq \Delta_{ii}$, we say that the site i is critical. A configuration u is called stable if no site is critical, i.e., $0 \leq u_i < \Delta_{ii}$ for all sites i. A critical site i is toppled, that is a subtraction the vector Δ_i from the vector u.

$$u \xrightarrow{\text{the vertex } i \text{ is toppled}} u - \Delta_i$$

Furthermore, a sequence of topplings is a sequence of sites $i_1, i_2, \ldots, i_k$ such that i_j is a critical site of $u - \Delta_{i_1} - \cdots - \Delta_{i_{j-1}}$ for any $1 \leq j \leq k$.

$$u \xrightarrow{i_1, i_2, \ldots, i_k \text{ are toppled}} u - \sum_{j=1}^{k} \Delta_{i_j}$$

A representation vector for the sequence of topplings is a vector $\mathbf{r} = (r_1, \ldots, r_n)$ with

$$r_s = |\{j \mid i_j = s, 1 \leq j \leq k\}|.$$

Clearly, $\mathbf{u} - \sum_{j=1}^{k} \Delta_{i_j} = \mathbf{u} - \mathbf{r}\Delta$.

Remark 4.1. *In [5], it is proved that every configuration can be transformed into a stable configuration by a sequence of topplings and the stable configuration does not depend on the order in which topplings are performed.*

For any $\mathbf{r} \in \mathscr{R}(\Delta)$, we define $(\Delta, \mathbf{r})$-recurrent configurations as follows.

Definition 4.2. *Let $\mathbf{u}$ be a configuration and $\mathbf{r} \in \mathscr{R}(\Delta)$. We say that $\mathbf{u}$ is a $(\Delta, \mathbf{r})$-recurrent configuration if $\mathbf{u}$ is stable and the configuration $\mathbf{u} + \mathbf{r}\Delta$ can be transformed into $\mathbf{u}$ by a sequence of topplings. Denote by $\mathcal{R}(\Delta, \mathbf{r})$ the set of $(\Delta, \mathbf{r})$-recurrent configurations.*

Example 4.3. *The matrix Δ and the vector $\mathbf{r}$ are given as those in Example 3.3. Then*

$$\mathcal{R}(\Delta, \mathbf{r}) = \{(1,3), (1,2), (1,1), (0,3), (0,2)\}.$$

Lemma 4.4. *Let*

$$\mathbf{d} = \mathbf{d}(\Delta) = (\Delta_{11} - 1, \Delta_{22} - 1, \ldots, \Delta_{nn} - 1).$$

For any $\mathbf{r} \in \mathscr{R}(\Delta)$, a configuration $\mathbf{u}$ is a $(\Delta, \mathbf{r})$-recurrent configuration if and only if $\mathbf{d} - \mathbf{u}$ is a $(\Delta, \mathbf{r})$-parking function.

Proof. Let $m = m(\mathbf{r}) = \sum_{j=1}^{n} r_j$. Suppose that $\mathbf{u}$ is a $(\Delta, \mathbf{r})$-recurrent configuration. By Definition 4.2, the configuration $\mathbf{u} + \mathbf{r}\boldsymbol{\Delta}$ can be transformed into $\mathbf{u}$ by a sequence $i_1, i_2, \ldots, i_m$ of topplings. Note that $\mathbf{r}$ is the representation vector for the sequence $i_1, i_2, \ldots, i_m$. For every $j \in \{1, 2, \ldots, m\}$, let χ_j be the characteristic function of the multiset $\{i_j, i_{j+1}, \ldots, i_m\}$. Then we have

$$u_{i_j} + \sum_{k=j}^{m} \Delta_{i_k, i_j} \geq \Delta_{i_j, i_j}$$

and

$$(\mathbf{d} - \mathbf{u})_{i_j} = \Delta_{i_j, i_j} - 1 - u_{i_j} \leq \sum_{k=j}^{m} \Delta_{i_k, i_j} - 1 = \langle \chi_j, \Delta^{i_j} \rangle - 1 < \langle \chi_j, \Delta^{i_j} \rangle.$$

It follows from Lemma 3.5 that $\mathbf{d} - \mathbf{u}$ is a $(\Delta, \mathbf{r})$-parking function.

Conversely, suppose $f = \mathbf{d} - \mathbf{u}$ is a $(\Delta, \mathbf{r})$-parking function. By Proposition 3.5, there is a sequence of integers in $V(\mathbf{r})$

$$\pi(1), \ldots, \pi(m)$$

such that for every $i \in \{1, 2, \ldots, m\}$

$$0 \leq f(\pi(i)) < \langle \chi_i, \Delta^{\pi(i)} \rangle,$$

where χ_i is the characteristic function of $\{\pi(i), \pi(i+1), \ldots, \pi(m)\}$. So,

$$\begin{aligned} u_{\pi(i)} &= \Delta_{\pi(i),\pi(i)} - 1 - f(\pi(i)) \\ &> \Delta_{\pi(i),\pi(i)} - 1 - \langle \chi_i, \Delta^{\pi(i)} \rangle \\ &= \Delta_{\pi(i),\pi(i)} - 1 - \sum_{k=i}^{m} \Delta_{\pi(k),\pi(i)} \end{aligned}$$

and

$$u_{\pi(i)} + \sum_{k=i}^{m} \Delta_{\pi(k),\pi(i)} \geq \Delta_{\pi(i),\pi(i)}.$$

This implies that $\mathbf{u}+\mathbf{r}\Delta$ can be transformed into $\mathbf{u}$ by the sequence $\pi(1), \pi(2), \ldots, \pi(m)$ of topplings. □

Example 4.5. *Let us consider the matrix* Δ *and the vector* $\mathbf{r}$ *in the example 3.3. We have*

$$\mathbf{d} = (1,3), \mathbf{r}\Delta = (1,2), V(\mathbf{r}) = \{1,1,2\}$$

and

$$\mathcal{R}(\Delta, \mathbf{r}) = \{(1,3),(1,2),(1,1),(0,3),(0,2)\}.$$

In the following table, we list all $(\Delta, \mathbf{r})$*-recurrent configurations* $\mathbf{u}$*, their corresponding* $(\Delta, \mathbf{r})$*-parking functions* $\mathbf{d}$-$\mathbf{u}$*, the configurations* $\mathbf{u} + \mathbf{r}\Delta$ *and sequences of topplings for* $\mathbf{u} + \mathbf{r}\Delta$*.*

$\mathbf{u}$	$\mathbf{d}$-$\mathbf{u}$	$\mathbf{u} + \mathbf{r}\Delta$	*a sequence of topplings for* $\mathbf{u}$+$\mathbf{r}\Delta$
(1,3)	*(0,0)*	*(2,5)*	*1,2,1*
(1,2)	*(0,1)*	*(2,4)*	*2,1,1*
(1,1)	*(0,2)*	*(2,3)*	*1,2,1*
(0,3)	*(1,0)*	*(1,5)*	*2,1,1*
(0,2)	*(1,1)*	*(1,4)*	*2,1,1*

Table 2. $(\Delta, \mathbf{r})$*-recurrent configurations* $\mathbf{u}$ *and the sequences of topplings for* $\mathbf{u} + \mathbf{r}\Delta$

Theorem 4.6. *For any* $\mathbf{r} \in \mathscr{R}(\Delta)$, $\mathcal{R}(\Delta, \mathbf{r}) = \mathcal{R}(\Delta, \mathbf{r}')$.

Proof. The required results follows from Lemma 4.4 and Theorem 3.10. □

Theorem 4.6 tells us that the set of $(\Delta, \mathbf{r})$-recurrent configurations is independent of $\mathbf{r}$ for any $\mathbf{r} \in \mathscr{R}(\Delta)$. So, $(\Delta, \mathbf{r})$-parking functions are simply called Δ-recurrent configurations and let $\mathcal{R}(\Delta)$ be the set of Δ-recurrent configurations.

Lemma 4.7. *Let* $\mathbf{r} \in \mathscr{R}(\Delta)$*. For any integer vector* $\mathbf{v} = (v_1, \ldots, v_n)$*, there exists a* $(\Delta, \mathbf{r})$*-recurrent configuration* $\mathbf{u}$ *such that* $\mathbf{v} - \mathbf{u} \in \langle\Delta\rangle$*.*

Proof. Note that $\det\Delta > 0$ and $(\det\Delta)\mathbf{1} = (\mathbf{1}adj(\Delta))\Delta \in \langle\Delta\rangle$. For any integer vector $\mathbf{v} = (v_1, \ldots, v_n)$, there exists a positive integer k such that $\mathbf{v} + k(\det\Delta)\mathbf{1} > \mathbf{0}$. It is sufficient to prove for any configuration $\mathbf{v} = (v_1, \ldots, v_n)$, there exists a $(\Delta, \mathbf{r})$-recurrent configuration $\mathbf{u}$ such that $\mathbf{v} - \mathbf{u} \in \langle\Delta\rangle$.

We now suppose $\mathbf{v}$ is a configuration. We start from $\mathbf{v}$, increase v_i by $(\mathbf{r}\Delta)_i$ for all $i \in \{1, 2, \cdots, n\}$ and then transform $\mathbf{v} + \mathbf{r}\Delta$ into a stable configuration by a sequence of topplings. If we repeat the process, we shall reach another stable configuration. This procedure can be repeated as often as we please, whereas the number of stable configurations is finite. So at least one of them must recur. This means that there exists a stable configuration $\mathbf{u}$ for which $\mathbf{u} + b \cdot \mathbf{r}\Delta$ can be transformed into $\mathbf{u}$ by a sequence of topplings. Hence, $\mathbf{u}$ is a $(\Delta, b\mathbf{r})$-recurrent configuration. By Corollary 3.9 and Lemma 4.4, we have $\mathbf{u}$ is a $(\Delta, \mathbf{r})$-recurrent configuration and $\mathbf{u} - \mathbf{v} \in \langle\Delta\rangle$. □

Lemma 4.7 implies that every equivalent class of $\mathbb{Z}^n$ contains at least one Δ-recurrent configuration. So we have the following corollary.

Corollary 4.8. *The number of* Δ*-recurrent configurations is larger than or equal to* $\det\Delta$*.*

Proof. Since the order of the quotient of the integer lattice $\mathbb{Z}^n/\langle\Delta\rangle$ is $\det\Delta$, it follows from Lemma 4.7 and Theorem 4.6 that $|\mathcal{R}(\Delta)| \geq \det\Delta$. □

Theorem 4.9. $|\mathcal{P}(\Delta)| = |\mathcal{R}(\Delta)| = \det\Delta$*.*

Proof. Combining Corollaries 3.12, 4.8, Lemma 4.4 and Theorem 4.6, we have $|\mathcal{P}(\Delta)| = |\mathcal{R}(\Delta)| = \det\Delta$. □

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