Hankel determinants of linear combinations of consecutive Catalan-like numbers
Lili Mu \textsuperscript{a}, Yi Wang \textsuperscript{b,*}, Yeong-Nan Yeh \textsuperscript{b,c}
\textsuperscript{a} School of Mathematics, Liaoning Normal University, Dalian 116024, China
\textsuperscript{b} School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
\textsuperscript{c} Institute of Mathematics, Academia Sinica, Taipei 10617, China

\textbf{A R T I C L E I N F O}

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\textbf{ABSTRACT}

Let \((a_n)_{n \geq 0}\) be a sequence of the Catalan-like numbers. We evaluate Hankel determinants \(\det[\lambda a_{ij} + \mu a_{i+j+1}]_{0 \leq i,j \leq n}\) and \(\det[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{0 \leq i,j \leq n}\) for arbitrary coefficients \(\lambda\) and \(\mu\). Our results unify many known results of Hankel determinant evaluations for classic combinatorial counting coefficients, including the Catalan, Motzkin and Schröder numbers. © 2017 Elsevier B.V. All rights reserved.

1. Introduction

Given a sequence \((a_n)_{n \geq 0}\), define its \textit{Hankel matrix} \([a_{ij}]_{i,j \geq 0}\) and the \textit{nth Hankel determinant} \(\det[a_{ij}]_{0 \leq i,j \leq n}\). Hankel determinants occur naturally in diverse areas of mathematics. In recent years, there has been a considerable amount of interest in the evaluation of Hankel determinants \(\det[a_{i+j+m}]_{0 \leq i,j \leq n}\) and \(\det[a_{i+j+m} + a_{i+j+2}]_{0 \leq i,j \leq n}\) involving various combinatorial sequences [1–3,5–12,14–16,18]. As we will see in Examples 2.4 and 2.5, these combinatorial sequences, including the Catalan numbers, the Motzkin numbers and the Schröder numbers, turn out to be the so-called Catalan-like numbers (or generalized Motzkin numbers [19]). The purpose of this paper is to provide a unified framework for previous results from the viewpoint of Catalan-like numbers.

Let \(s = (s_k)_{k \geq 0}\) and \(t = (t_k)_{k \geq 1}\) be two sequences of nonnegative numbers and define an infinite lower triangular matrix \(A = [a_{n,k}]_{n,k \geq 0}\) by the recurrence

\begin{equation}
\begin{aligned}
a_{0,0} &= 1, \\
a_{n+1,k} &= a_{n,k-1} + s_k a_{n,k} + t_k + 1 a_{n,k+1},
\end{aligned}
\end{equation}

where \(a_{n,k} = 0\) unless \(n \geq k \geq 0\). Clearly, all \(a_{n,0} = 1\). Following Aigner [3], we say that \(A\) is the \textit{recursive matrix} and \(a_{n,0}\) are the \(n\)th Catalan-like numbers corresponding to \((s, t)\).

\begin{example}
The Catalan-like numbers unify many well-known counting coefficients, such as
\begin{itemize}
\item[(i)] the Catalan numbers \(C_n\) when \(s = (1, 2, 2, \ldots)\) and \(t = (1, 1, 1, \ldots)\);
\item[(ii)] the shifted Catalan numbers \(C_{n+1}\) when \(s = (2, 2, 2, \ldots)\) and \(t = (1, 1, 1, \ldots)\);
\item[(iii)] the Motzkin numbers \(M_n\) when \(s = t = (1, 1, 1, \ldots)\);
\item[(iv)] the central binomial coefficients \(\binom{2n}{n}\) when \(s = (2, 2, 2, \ldots)\) and \(t = (2, 1, 1, \ldots)\);
\end{itemize}
\end{example}

\* Corresponding author.

E-mail addresses: lily-mu@hotmail.com (L. Mu), wangyi@dlut.edu.cn (Y. Wang), mayeh@gate.sinica.edu.tw (Y.-N. Yeh).
The Catalan-like numbers have a nice combinatorial interpretation from the viewpoint of weighted lattice paths. A Motzkin path of length $n$ is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $(1, 1)$, down steps $(1, -1)$, and horizontal steps $(0, 1)$. The weight of a step in a Motzkin path is the $y$-coordinate of the starting point. Assign a weight 1 to up steps and horizontal steps, and a weight $i$ to down steps of length $i$. Define the weight of a Motzkin path to be the product of weights of its steps. Then the Catalan-like number $a_n$ counts the total weight of all Motzkin paths of length $n$.

The Catalan-like numbers are closely related to continued fractions and orthogonal polynomials. Let $a_n$ be the Catalan-like numbers corresponding to $(s, t)$. Then

$$
\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - s_0 x - \frac{t_1 x^2}{1 - s_1 x - \frac{t_2 x^2}{1 - s_2 x - \cdots}}}.
$$

Let $(p_n(x))_{n \geq 0}$ be the sequence of orthogonal polynomials with respect to the linear operator $L(x^n) = a_n$. Then $L(p_n(x)p_m(x)) = \delta_{m,n} t_1 \cdots t_n$ and

$$p_{n+1}(x) = (x - s_n) p_n(x) - t_n p_{n-1}(x), \quad p_0(x) = 1.
$$

For an infinite matrix $M = [m_{i,j}]_{i,j \geq 0}$, let $M_n = [m_{i,j}]_{0 \leq i,j \leq n}$ denote its $n$th leading principal submatrix and $\delta_n(M) = \det M_n$. For convenience, denote $\delta_{-1}(M) = 1$. Let

$$J = \begin{bmatrix}
s_0 & 1 \\
t_1 & s_1 & 1 \\
t_2 & s_2 & 1 \\
t_3 & s_3 & \ddots \\
\vdots & \ddots & \ddots
\end{bmatrix}
$$

denote the coefficient matrix of the recursive relation (1.1) and $d_n = \delta_n(J)$. Denote $T_0 = 1$ and $T_n = t_1 t_2 \cdots t_n$ for $n \geq 1$. The following result is folklore (see [3,19] for instance).

**Proposition 1.2.** Let $a_n$ be the Catalan-like numbers corresponding to $(s, t)$. Then

(i) $\det(a_{i+j})_{0 \leq i,j \leq n} = T_1 \cdots T_n$.

(ii) $\det(a_{i+j+1})_{0 \leq i,j \leq n} = T_1 \cdots T_n d_n$.

(iii) $\det(a_{i+j+2})_{0 \leq i,j \leq n} = T_1 \cdots T_n T_{n+1} \sum_{i=1}^{n} \frac{d_i}{T_i} d_{n-i}$.

For $\lambda, \mu \in \mathbb{R}$, let

$$d_n^{(\lambda, \mu)} = \det \begin{bmatrix}
\lambda + \mu s_0 & \mu \\
\mu t_1 & \lambda + \mu s_1 & \mu \\
\mu t_2 & \ddots & \ddots \\
\mu t_n & \lambda + \mu s_{n-1} & \mu \\
\mu t_n & \lambda + \mu s_n & \mu 
\end{bmatrix}.
$$

Then $d_n^{(1,0)} = 1$ and $d_n^{(0,1)} = d_n$. Our main results are the following general formulae.

**Theorem 1.3.** Let $a_n$ be the Catalan-like numbers corresponding to $(s, t)$. Then

(i) $\det(\lambda a_{i+j} + \mu a_{i+j+1})_{0 \leq i,j \leq n} = T_1 \cdots T_n d_n^{(\lambda, \mu)}$.

(ii) $\det(\lambda a_{i+j+1} + \mu a_{i+j+2})_{0 \leq i,j \leq n} = T_1 \cdots T_n T_{n+1} \sum_{i=1}^{n} \frac{d_i}{T_i} d_{n-i}^{(\lambda, \mu)}$.

In the next section, we give the proof of the theorem and then present applications on some interesting Catalan-like numbers. Our results unify many known results of Hankel determinant evaluations for classic combinatorial counting coefficients, including the Catalan, Motzkin and Schröder numbers.
2. Proof and applications of Theorem 1.3

We first present the proof of the theorem.

Proof of Theorem 1.3. Let \( A = [a_{n,k}]_{n,k \geq 0} \) be the recursive matrix corresponding to \((s, t)\). Then the recurrence (1.1) is equivalent to \( \tilde{A} = AJ \), where

\[
\tilde{A} = \begin{bmatrix}
a_{1,0} & a_{1,1} \\
a_{2,0} & a_{2,1} & a_{2,2} \\
a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

On the other hand, let

\[
K = \begin{bmatrix}
0 & 1 \\
0 & 0 & 1 \\
& & \ddots & \ddots
\end{bmatrix}.
\]

Then \( KA = \tilde{A} \). Thus we have \( KA = AJ \).

Clearly,

\[
[\lambda a_{i+j} + \mu a_{i+j+1}]_{i,j \geq 0} = \begin{bmatrix}
\lambda & \mu \\
\lambda & \mu \\
\vdots & \vdots
\end{bmatrix} \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_1 & a_2 & a_3 & \cdots \\
a_2 & a_3 & a_4 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Let \( I \) be the identity matrix and \( H = [a_{i+j}]_{i,j \geq 0} \) the Hankel matrix of the Catalan-like numbers \( a_n \). Then

\[
[\lambda a_{i+j} + \mu a_{i+j+1}]_{i,j \geq 0} = (\lambda I + \mu K)H.
\]

Recall that the Aigner’s Fundamental Theorem [3]:

\[
a_{m+n} = \sum_k a_{m,k} a_{n,k} T_k, \quad m, n \geq 0.
\]

In other words, \( H = ATA^T \), where \( T = \text{diag}(T_0, T_1, T_2, \ldots) \). It follows that

\[
[\lambda a_{i+j} + \mu a_{i+j+1}]_{i,j \geq 0} = (\lambda I + \mu K)ATA^T = A(\lambda I + \mu J)TA^T.
\]

Note that \( A \) is a lower triangular matrix with all diagonal entries equal to 1 and \( T \) is a diagonal matrix. Hence

\[
\det[\lambda a_{i+j} + \mu a_{i+j+1}]_{0 \leq i,j \leq n} = \delta_n (\lambda I + \mu J)T_0 T_1 \cdots T_n.
\]

This proves (i).

Similarly, we have

\[
[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{i,j \geq 0} = \begin{bmatrix}
0 & \lambda & \mu \\
0 & \lambda & \mu \\
\vdots & \vdots & \ddots
\end{bmatrix} \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_1 & a_2 & a_3 & \cdots \\
a_2 & a_3 & a_4 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Thus

\[
[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{i,j \geq 0} = (\lambda K + \mu K^2)ATA^T = A(\lambda J + \mu J^2)TA^T,
\]

and so

\[
\det[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{0 \leq i,j \leq n} = \delta_n (\lambda J + \mu J^2)T_0 T_1 \cdots T_n.
\]

Let

\[
J_i = \begin{bmatrix}
s_0^{(i)} & r_1^{(i)} \\
\tilde{t}_1^{(i)} & s_1^{(i)} & r_2^{(i)} \\
\tilde{t}_2^{(i)} & s_2^{(i)} & r_3^{(i)} \\
\tilde{t}_3^{(i)} & s_3^{(i)} & \ddots
\end{bmatrix}, \quad i = 1, 2
\]

be two tridiagonal matrices. Then by the Cauchy–Binet formula,

$$
\delta_n(J J_2) = \delta_n(J) \delta_n(J_2) + \delta_{n-1}(J_1) r_{n+1}^{(1)} \delta_{n-1}(J_2) t_{n+1}^{(2)} + \delta_{n-2}(J_1) r_{n+1}^{(1)} \delta_{n-2}(J_2) t_{n+1}^{(2)} + \cdots
$$

+ \sum_{j=1}^{n} \delta_j(J_1) \delta_j(J_2) \prod_{k=0}^{j} \frac{t_{n+1}^{(2)}}{\prod_{k=0}^{j} t_{n+1}^{(2)}} \prod_{k=0}^{j} t_{n+1}^{(2)},
$$

where $\delta_{-1}(J_1) = r_{0}^{(1)} = t_{0}^{(2)} = 1$. In particular, setting $J_1 = J$ and $J_2 = \lambda I + \mu J$, we obtain

$$
\delta_n(\lambda I + \mu J^2) = \delta_n(J(\lambda I + \mu J)) = \sum_{j=1}^{n} \delta_j(J) \delta_j(\lambda I + \mu J) \frac{T_{n+1}}{T_{j+1}} \mu^{n-j}.
$$

Thus (ii) follows. This completes the proof. □

In the rest of this section, we apply Theorem 1.3 to a class of particularly interesting Catalan-like numbers. Let $p, q, s, t$ be four nonnegative numbers. Define a recursive matrix $A(p, s; q, t) = [a_{n,k}]_{n,k \geq 0}$ by

$$
a_{0,0} = 1, \quad a_{n+1,0} = pa_{n,0} + qa_{n,1}, \quad a_{n+1,k} = a_{n,k-1} + sa_{n,k} + ta_{n,k+1}.
$$

(2.1)

In this case, $s = (p, s, s, \ldots)$ and $t = (q, t, t, \ldots)$. We denote by $a_n(p, s; q, t)$ the corresponding Catalan-like numbers. Many famous counting coefficients are such numbers. For example, the Catalan numbers $C_n = a_n(1, 2; 1, 1)$, the shifted Catalan numbers $C_n = a_n(2, 2; 1, 1)$, the Motzkin numbers $M_n = a_n(1, 1; 1, 1)$, and the large Schröder numbers $r_n = a_n(2, 3; 2, 2)$. Such Catalan-like numbers can be characterized by the generating function since $A(p, s; q, t)$ is also a Riordan array. See [13] for details.

**Lemma 2.1.** If $a_n$ are the Catalan-like numbers $a_n(p, s; q, t)$, then

$$
\sum_{n \geq 0} a_n x^n = \frac{2t}{2t - q + (qs - 2pt)x + q\sqrt{1 - 2sx + (s^2 - 4t)x^2}}.
$$

The converse is also true.

**Lemma 2.2.** Let

$$
\tilde{d}_0 = 1, \quad \tilde{d}_n = \det \begin{bmatrix}
  s & r \\
  t & s & r \\
  & & \ddots & \ddots \\
  & & & s & r \\
  & & & t & s
\end{bmatrix}_n
$$

for $n \geq 1$ and

$$
D_n = \det \begin{bmatrix}
  p & h \\
  q & s & r \\
  t & s & r \\
  & & \ddots & \ddots \\
  & & & s & r \\
  & & & t & s
\end{bmatrix}_{n+1}
$$

for $n \geq 0$. Then

$$
\sum_{n \geq 0} \tilde{d}_n x^n = \frac{1}{1 - sx + rt x^2}, \quad \sum_{n \geq 0} D_n x^n = \frac{p - h q x}{1 - sx + rt x^2}.
$$
Proof. We have \( \tilde{d}_n = s\tilde{d}_{n-1} - rt\tilde{d}_{n-2} \), with \( \tilde{d}_0 = 1 \) and \( \tilde{d}_1 = s \). Let \( \tilde{d}(x) = \sum_{n \geq 0} \tilde{d}_n x^n \). Then

\[
\tilde{d}(x) = \tilde{d}_0 + \tilde{d}_1 x + \sum_{n \geq 2} (s\tilde{d}_{n-1} - rt\tilde{d}_{n-2}) x^n = 1 + sx\tilde{d}(x) - rt^2\tilde{d}(x).
\]

It follows that \( \tilde{d}(x) = 1/(1 - sx + rt^2) \).

On the other hand, \( D_n = \tilde{p}\tilde{d}_n - hq\tilde{d}_{n-1} \) for \( n \geq 0 \), where \( \tilde{d}_{-1} = 0 \). Thus

\[
\sum_{n \geq 0} D_n x^n = p\tilde{d}(x) - hqx\tilde{d}(x) = \frac{p - hqx}{1 - sx + rt^2}.
\]

This completes the proof. \( \square \)

Corollary 2.3. Let \( a_n = a_n(p, s; q, t) \) be the Catalan-like numbers. Then

(i) \( \det(\lambda a_{ij} + \mu a_{ij+1})_{0 \leq i, j \leq n} = q^n t^\binom{n}{2} d_n^{(\lambda, \mu)} \) and

(ii) \( \det(\lambda a_{ij+1} + \mu a_{ij+2})_{0 \leq i, j \leq n} = q^n t^\binom{n}{2} \sum_{i=1}^n d_i^{(\lambda, \mu)} (\mu t)^{n-i} \),

where

\[
\sum_{n \geq 0} d_n^{(\lambda, \mu)} x^n = \frac{(\lambda + \mu p - \mu^2 q x)}{1 - (\lambda + \mu s) x + \mu^2 tx^2},
\]

and

\[
\sum_{n \geq 0} D_n^{(\lambda, \mu)} x^n = \frac{p - q x}{1 - sx + tx^2}.
\]

Example 2.4. Consider the generalized Catalan numbers \( C_n(t) \) defined in [4], which have the generating function

\[
\sum_{n \geq 0} C_n(t) x^n = \frac{2}{1 - (t - 1)x + \sqrt{1 - 2(t + 1)x + (t - 1)^2x^2}}. \tag{2.2}
\]

Therefore \( C_n(t) \) are precisely the Catalan-like numbers \( a_n(t, t + 1; t, t) \). In particular, \( C_n(1) \) are the Catalan numbers and \( C_n(2) \) are the large Schröder numbers. It follows from (2.2) that

\[
\sum_{n \geq 0} C_{n+1}(t) x^n = \frac{2t}{1 - (t + 1)x + \sqrt{1 - 2(t + 1)x + (t - 1)^2x^2}}.
\]

In other words, \( C_{n+1}(t)/t \) are the Catalan-like numbers \( a_n(t + 1, t + 1; t, t) \). By Corollary 2.3(i), we have

(i) \( \det(\lambda C_{ij}(t) + \mu C_{ij+1}(t))_{0 \leq i, j \leq n} = t^\binom{n}{2} d_n^{(\lambda, \mu)} \) and

(ii) \( \det(\lambda C_{ij+1}(t) + \mu C_{ij+2}(t))_{0 \leq i, j \leq n} = t^\binom{n}{2} D_n^{(\lambda, \mu)} \),

where

\[
\sum_{n \geq 0} d_n^{(\lambda, \mu)} x^n = \frac{\lambda + \mu (t + 1) - \mu^2 tx}{1 - (\lambda + \mu (t + 1)) x + \mu^2 tx^2}
\]

and

\[
\sum_{n \geq 0} D_n^{(\lambda, \mu)} x^n = \frac{\lambda + \mu (t + 1) - \mu^2 tx}{1 - (\lambda + \mu (t + 1)) x + \mu^2 tx^2},
\]

or equivalently,

\[
\sum_{n \geq 0} d_n^{(\lambda, \mu)} x^n = \frac{1 - \mu x}{1 - (\lambda + \mu (t + 1)) x + \mu^2 tx^2}, \quad d_{-1}^{(\lambda, \mu)} = 1
\]

and

\[
\sum_{n \geq 0} D_n^{(\lambda, \mu)} x^n = \frac{1}{1 - (\lambda + \mu (t + 1)) x + \mu^2 tx^2}, \quad D_{-1}^{(\lambda, \mu)} = 1.
\]
The result in the special case $\lambda = \mu = 1$ has been obtained in [15] by an analytic approach. In particular, taking $t = 1$, we have the following result about the Catalan numbers $C_n$:

$$\det(C_{i+j} + C_{i+j+1})_{0 \leq i,j \leq n} = F_{2n+2}$$

and

$$\det(C_{i+j} + C_{i+j+2})_{0 \leq i,j \leq n} = F_{2n+3},$$

where $F_n$ are the Fibonacci numbers: $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. This elegant result first occurred in [8].

A Schröder path of length $n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ using up steps $(1, 1)$, down steps $(1, -1)$, and level steps $(2, 0)$, and never passing below the $x$-axis. Let each up step and each down step have weight $1$ and each level step have weight $t$. The $t$-Schröder number $r_n^{(t)}$ counts the total weight of all Schröder paths of length $n$. Sun et al. [16, Proposition 2.1] gave the generating function

$$\sum_{n \geq 0} r_n^{(t)} x^n = \frac{2}{1 - tx + \sqrt{1 - 2(t + 2)x + t^2x^2}}.$$

Hence the $t$-Schröder number $r_n^{(t)}$ are precisely the generalized Catalan numbers $C_n(t + 1)$. Sun et al. obtained

$$\det(r_n^{(t)})_{0 \leq i,j \leq n-1} = (1 + t)^{n+1}/2$$

and

$$\det(r_n^{(t)})_{0 \leq i,j \leq n-1} = (1 + t)^{n+1}/2.$$  

Raković et al. [15] gave an explicit formula for $\det(r_n^{(t)})_{0 \leq i,j \leq n-1}$. Eu et al. [10] evaluated $\det(\lambda r_n^{(t)} + \mu r_{n+1}^{(t)})_{0 \leq i,j \leq n-1}$ by taking lattice path techniques. These results are immediate from our results.

**Example 2.5.** Sun [17] defined the generalized Motzkin numbers $M_n(s, t)$ by

$$M_n(s, t) = \sum_{k=0}^{[n/2]} \binom{n}{2k} C_k s^{n-2k} t^k,$$

which are common generalizations of the Motzkin numbers and the Catalan numbers:

$$M_n(1, 1) = M_n, \quad M_n(2, 1) = C_{n+1}.$$

Sun gave the generating function

$$\sum_{n \geq 0} M_n(s, t) x^n = \frac{2}{1 - sx + \sqrt{1 - 2sx - (s^2 - 4t)x^2}}.$$

Therefore $M_n(s, t)$ are precisely the Catalan-like numbers $a_n(s, s; t, t)$. Thus we have

$$\det(\lambda M_{i+j}(s, t) + \mu M_{i+j+1}(s, t))_{0 \leq i,j \leq n} = t^{n+1} d_n^{(s, \mu)}$$

and

$$\det(\lambda M_{i+j+1}(s, t) + \mu M_{i+j+2}(s, t))_{0 \leq i,j \leq n} = t^{n+1} d_n^{(s, \mu)} = \sum_{i=-1}^{n} d_i^{(s, \mu)} t^{i+1},$$

where

$$\sum_{n \geq 0} d_n^{(s, \mu)} x^n = \frac{1}{1 - (\lambda + \mu s)x + \mu^2 t x^2}, \quad d_{-1}^{(s, \mu)} = 1$$

and

$$\sum_{n \geq 0} d_n^{(s, \mu)} x^n = \frac{1}{1 - sx + tx}, \quad d_{-1} = 1.$$

An $s$-Motzkin path is a Motzkin path in which the up step, down step, and unit level step have weights $1, 1$ and $s$ respectively. Let $M^{(s)}_n$ be the total weight of all $s$-Motzkin paths of length $n$. Then $M^{(s)}_n$ are precisely the generalized Motzkin numbers $M_n(s, 1)$. Thus the following determinant evaluations, obtained by Cameron and Yip [6] using lattice path techniques, are now immediate:

(i) $\det(M^{(s)}_{i+j+1})_{0 \leq i,j \leq n} = d_n(s)$;
(ii) $\det(M^{(s)}_{i+j+2})_{0 \leq i,j \leq n} = \sum_{i=-1}^{n} d_i^{(s)}$;
(iii) $\det(M^{(s)}_{i+j} + M^{(s)}_{i+j+1})_{0 \leq i,j \leq n} = d_n(s + 1)$;
(iv) $\det(M^{(s)}_{i+j+1} + M^{(s)}_{i+j+2})_{0 \leq i,j \leq n} = \sum_{i=-1}^{n} d_i(s)d_i(s + 1)$,

where

$$\sum_{n \geq 0} d_n^{-1}(s)x^n = \frac{1}{1 - sx + x^2}.$$
In particular, \( d_n(1) = \sin(n + 2)\omega / \sin \omega \), where \( \omega = \pi / 3 \), and \( d_n(2) = n + 2 \). Thus the results reduce to Hankel determinants for the Motzkin numbers:

(i) \( \det[M_{ij+1}]_{0 \leq i,j \leq n} = \begin{cases} 
1, & \text{if } n = 0, 5 \pmod{6}; \\
0, & \text{if } n = 1, 4 \pmod{6}; \\
-1, & \text{if } n = 2, 3 \pmod{6}.
\end{cases} \)

(ii) \( \det[M_{ij+2}]_{0 \leq i,j \leq n} = \begin{cases} 
2m + 2, & \text{if } n = 3m \text{ or } n = 3m + 1; \\
2m + 3, & \text{if } n = 3m + 2.
\end{cases} \)

(iii) \( \det[M_{ij+1} + M_{ij+2}]_{0 \leq i,j \leq n} = \begin{cases} 
(-1)^{m+1} \chi(m + 1 \chi - 1)^m, & \text{if } n = 3m + 2; \\
\chi(m) & \text{if } n = 3m \text{ or } n = 3m + 1.
\end{cases} \)

These results for the Motzkin numbers have occurred in the literature \([1,3,6,9]\).

3. Remarks

Given a sequence \( (x_n)_{n \geq 0} \), let \( h_n^{(k)} = \det[x_{i+j+k}]_{0 \leq i,j \leq n} \). Then

\[
h_n^{(k+2)} = h_n^{(k)} h_n^{(k+2)} - [h_n^{(k+1)}]^2
\]

by the famous Desnanot–Jacobi identity (see, e.g., [20]). It follows that

\[
\frac{h_n^{(k+2)}}{h_n^{(k)}} = \sum_{i=0}^{n} \frac{[h_i^{(k+1)}]^2}{h_i^{(k)} h_i^{(k+1)}}.
\]

In other words, \( h_n^{(k+2)} \) can be determined by all \( h_n^{(k)} \) and \( h_n^{(k+1)} \). For example, in Proposition 1.2, (iii) follows from (i) and (ii).

Thus it is possible to determine \( h_n^{(k+2)} \) by \( h_n^{(0)} \) and \( h_n^{(1)} \) recursively. In particular, we can evaluate \( \det[\lambda a_{i+j+k} + \mu a_{i+j+k+1}]_{0 \leq i,j \leq n} \) for \( k \geq 2 \) by Theorem 1.3.

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