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Several variants of the Dumont differential system and permutation statistics

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Abstract The Dumont differential system on the Jacobi elliptic functions was introduced by Dumont (Math Comp, 1979, 33: 1293–1297) and was extensively studied by Dumont, Viennot, Flajolet and so on. In this paper, we first present a labeling scheme for the cycle structure of permutations. We then introduce two types of Jacobi-pairs of differential equations. We present a general method to derive the solutions of these differential equations. As applications, we present some characterizations for several permutation statistics.

Keywords Jacobi elliptic functions; Dumont differential system; permutation statistics; context-free grammars

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1 Introduction

The Jacobi elliptic functions occur naturally in geometry, analysis, number theory, algebra and combinatorics (see [5, 7, 8, 20] for instance). The three basic *Jacobi elliptic functions* $\mathrm{sn}\,(u,k), \mathrm{cn}\,(u,k), \mathrm{dn}\,(u,k)$ are respectively defined by

$$u = \int_0^{\mathrm{sn}\,(u,k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$
$$\mathrm{cn}\,(u,k) = \sqrt{1-\mathrm{sn}^{\,2}(u,k)},$$
$$\mathrm{dn}\,(u,k) = \sqrt{1-k^2\mathrm{sn}^{\,2}(u,k)},$$

where the modulus is often confined to the normal case $0 < k < 1$. These functions are generalizations of the trigonometric functions and hyperbolic functions satisfying

$$\mathrm{sn}\,(u,0) = \sin u, \mathrm{cn}\,(u,0) = \cos u, \mathrm{dn}\,(u,0) = 1,$$

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$$\mathrm{sn}\,(u,1)=\tanh u, \mathrm{cn}\,(u,1)=\mathrm{dn}\,(u,1)=\mathrm{sech} u.$$

The Taylor series expansions of these Jacobian elliptic functions are given as follows:

$$\begin{aligned}
\mathrm{sn}\,(u,k)&=u-(1+k^2)\frac{u^3}{3!}+(1+14k^2+k^4)\frac{u^5}{5!}+\cdots,\\
\mathrm{cn}\,(u,k)&=1-\frac{u^2}{2!}+(1+4k^2)\frac{u^4}{4!}-(1+44k^2+16k^4)\frac{u^6}{6!}+\cdots,\\
\mathrm{dn}\,(u,k)&=1-k^2\frac{u^2}{2!}+k^2(4+k^2)\frac{u^4}{4!}-k^2(16+44k^2+k^4)\frac{u^6}{6!}+\cdots.
\end{aligned}$$

Using formal methods, Abel [1] discovered the following differential system:

$$\begin{cases}
\frac{d}{du}\mathrm{sn}\,(u,k)=\mathrm{cn}\,(u,k)\mathrm{dn}\,(u,k),\\
\frac{d}{du}\mathrm{cn}\,(u,k)=-\mathrm{sn}\,(u,k)\mathrm{dn}\,(u,k),\\
\frac{d}{du}\mathrm{dn}\,(u,k)=-k^2\mathrm{sn}\,(u,k)\mathrm{cn}\,(u,k).
\end{cases} \tag{1.1}$$

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. An *interior peak* in π is an index $i\in\{2,3,\ldots,n-1\}$ such that $\pi(i-1)<\pi(i)>\pi(i+1)$. Given a permutation $\pi\in\mathfrak{S}_n$, a value $i\in[n]$ is called a *cycle peak* if $\pi^{-1}(i)<i>\pi(i)$. Throughout this paper, we always let $X(\pi)$ (resp., $Y(\pi)$) be the number of odd (resp., even) cycle peaks of π. For example, for $\pi=241365$, we have $X(\pi)=0$ and $Y(\pi)=2$.

Let D be the derivative operator on the polynomials in three variables. The *Dumont differential system* on the Jacobi elliptic functions is defined by

$$\begin{cases}
D(x)=yz,\\
D(y)=xz,\\
D(z)=xy.
\end{cases} \tag{1.2}$$

For $n\geqslant 0$, we define the numbers $s_{n,i,j}$ by

$$\begin{aligned}
D^{2n}(x)&=\sum_{i,j\geqslant 0}s_{2n,i,j}x^{2i+1}y^{2j}z^{2n-2i-2j},\\
D^{2n+1}(x)&=\sum_{i,j\geqslant 0}s_{2n+1,i,j}x^{2i}y^{2j+1}z^{2n-2i-2j+1}.
\end{aligned} \tag{1.3}$$

The study of (1.2) was initiated by Schett [17] (in a slightly different form) and he found that

$$\sum_{i,j\geqslant 0}s_{n,i,j}=n!,\quad \sum_{j\geqslant 0}s_{n,i,j}=P_{n,\lfloor (n-1)/2\rfloor -i},$$

where $P_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ with k interior peaks. Dumont [4] deduced the recurrence relation

$$\begin{aligned}
s_{2n,i,j}&=(2j+1)s_{2n-1,i,j}+(2i+2)s_{2n-1,i+1,j-1}+(2n-2i-2j+1)s_{2n-1,i,j-1},\\
s_{2n+1,i,j}&=(2i+1))s_{2n,i,j}+(2j+2)s_{2n,i-1,j+1}+(2n-2i-2j+2)s_{2n,i-1,j},
\end{aligned} \tag{1.4}$$

and established that

$$s_{n,i,j}=|\{\pi\in\mathfrak{S}_n : X(\pi)=i, Y(\pi)=j\}|. \tag{1.5}$$

Moreover, Dumont [4, Corollary 1] obtained the following result:

(i) the coefficient of $(-1)^n k^{2j}u^{2n+1}/(2n+1)!$ in the Taylor expansion of $\mathrm{sn}\,(u,k)$ is equal to the number of permutations in $\mathfrak{S}_{2n}$ (or in $\mathfrak{S}_{2n+1}$) having j even cycle peaks and with no odd cycle peaks;

(ii) the coefficient of $(-1)^n k^{2i}u^{2n}/(2n)!$ (resp. $(-1)^n k^{2n-2i}u^{2n}/(2n)!$) in the Taylor expansion of $\mathrm{cn}\,(u,k)$ (resp. $\mathrm{dn}\,(u,k)$) is equal to the number of permutations in $\mathfrak{S}_{2n-1}$ (or in $\mathfrak{S}_{2n}$) having i odd cycle peaks and with no even cycle peaks.

Subsequently, Dumont [5] studied the symmetric variant of (1.1):

$$\frac{d}{du}\mathrm{sn}\,(u;a,b)=\mathrm{cn}\,(u;a,b)\mathrm{dn}\,(u;a,b),$$
$$\frac{d}{du}\mathrm{cn}\,(u;a,b)=a^2\mathrm{sn}\,(u;a,b)\mathrm{dn}\,(u;a,b),$$
$$\frac{d}{du}\mathrm{dn}\,(u;a,b)=b^2\mathrm{sn}\,(u;a,b)\mathrm{cn}\,(u;a,b),$$

with the initial conditions $\mathrm{sn}\,(0;a,b)=0,\mathrm{cn}\,(0;a,b)=1$ and $\mathrm{dn}\,(0;a,b)=1$. In particular, for the Dumont differential system (1.2), Dumont [5, Proposition 2.1] showed that

$$\sum_{n\geqslant 0}D^n(x)\frac{u^n}{n!}=\frac{yz\mathrm{sn}\,(u;v,w)+x\mathrm{cn}\,(u;v,w)\mathrm{dn}\,(u;v,w)}{1-x^2\mathrm{sn}^2(u;v,w)}, \tag{1.6}$$

where $v=\sqrt{y^2-x^2}$ and $w=\sqrt{z^2-x^2}$.

The grammatical method was systematically introduced by Chen [2] in the study of exponential structures in combinatorics. Many combinatorial structures can be generated by using context-free grammars. We refer the reader to [3, 14, 15] for recent progress on this topic. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. A *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. It is clear that (1.2) is equivalent to the context-free grammar

$$G=\{x\to yz, y\to xz, z\to xy\}. \tag{1.7}$$

This paper is organized as follows. In Section 2, we present a constructive proof of (1.5) by using the grammatical labeling introduced by Chen and Fu [3]. In Section 3, we introduce and study two types of Jacobi-pairs of differential equations. In Section 4, we present some characterizations for several permutation statistics.

2 A constructive proof of (1.5)

In this section, we always write $\pi\in\mathfrak{S}_n$ using the *standard cycle decomposition*, where each cycle is written with its smallest entry first and the cycles are written in increasing order of their smallest entry. In what follows, we present a labeling scheme for the cycle structure of permutations.

Let

$$\mathfrak{S}_{n,i,j}=\{\pi\in\mathfrak{S}_n: X(\pi)=i, Y(\pi)=j\}.$$

Definition 2.1. Let $\pi\in\mathfrak{S}_{n,i,j}$. Then we put the superscript label x immediately before and right after each odd cycle peak of π, and we put the superscript label y immediately before and right after each even cycle peak. In each of the remaining positions except the first position of each cycle, we put the superscript label z. Moreover, we put the superscript label x (resp. y) at the end of π if n is even (resp. odd).

For example, for $\pi=(132)(45)(68)(7)\in\mathfrak{S}_{8,2,1}$ and $\pi'=(132)(45)\in\mathfrak{S}_{5,2,0}$, the labeled π and π' are respectively given by

$$(1^x3^x2^z)(4^x5^x)(6^y8^y)(7^z)^x,\ (1^x3^x2^z)(4^x5^x)^y.$$

When $n=1$, we have $\mathfrak{S}_{1,0,0}=\{(1^z)^y\}$. When $n=2$, we have $\mathfrak{S}_{2,0,0}=\{(1^z)(2^z)^x\}$ and $\mathfrak{S}_{2,0,1}=\{(1^y2^y)^x\}$. Let $n=m$. Suppose we get all labeled permutations in $\mathfrak{S}_{m,i,j}$ for all i,j, where $m\geqslant 2$. We now consider the case $n=m+1$. Let $\widehat{\pi}\in\mathfrak{S}_{m+1}$ be obtained from $\pi\in\mathfrak{S}_{m,i,j}$ by inserting the entry $m+1$ into π. In the following, we construct a correspondence, denoted by Φ, between π and $\widehat{\pi}$.

If m is odd and the entry $m+1$ is inserted at the end of π as a new cycle $(m+1)$, then we leave all labels of π unchanged except the last label y. We define Φ by

$$\pi=\cdots(\cdots)^y\leftrightarrow\widehat{\pi}=\cdots(\cdots)((m+1)^z)^x,$$

which corresponds to the operation $y \to xz$. Note that $X(\widehat{\pi}) = X(\pi)$ and $Y(\widehat{\pi}) = Y(\pi)$. Hence $\widehat{\pi} \in \mathfrak{S}_{m+1,i,j}$. If m is odd and the entry $m+1$ occurs in a cycle with at least two elements, there are three cases to consider:

(i) Suppose c_r is the rth odd cycle peak of π and we put the entry $m+1$ immediately before or right after c_r. Then we have

$$\pi = \cdots(\ldots^x c_r\, ^x \ldots)\cdots(\cdots)^y \leftrightarrow \widehat{\pi} = \cdots(\ldots^y (m+1)^y c_r\, ^z \ldots)\cdots(\cdots)^x,$$

or

$$\pi = \cdots(\ldots^x c_r\, ^x \ldots)\cdots(\cdots)^y \leftrightarrow \widehat{\pi} = \cdots(\ldots^z c_r\, ^y (m+1)^y \ldots)\cdots(\cdots)^x.$$

In this case, the corresponding operation of Φ is $x \to yz$ and we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i-1,j+1}$.

(ii) Suppose d_ℓ is the ℓth even cycle peak of π and we put the entry $m+1$ immediately before or right after d_ℓ. Then we have

$$\pi = \cdots(\ldots^y d_\ell\, ^y \ldots)\cdots(\cdots)^y \leftrightarrow \widehat{\pi} = \cdots(\ldots^y (m+1)^y d_\ell\, ^z \ldots)\cdots(\cdots)^x,$$

or

$$\pi = \cdots(\ldots^y d_\ell\, ^y \ldots)\cdots(\cdots)^y \leftrightarrow \widehat{\pi} = \cdots(\ldots^z d_\ell\, ^y (m+1)^y \ldots)\cdots(\cdots)^x.$$

In this case, the corresponding operation of Φ is $y \to xz$ and we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i,j}$.

(iii) If we insert $m+1$ into a position of π with label z, then we have

$$\pi = \cdots(\ldots w\, ^z \ldots)\cdots(\cdots)^y \leftrightarrow \widehat{\pi} = \cdots(\ldots w^y (m+1)^y \ldots)\cdots(\cdots)^x.$$

In this case, the corresponding operation of Φ is $z \to xy$ and we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i,j+1}$.

If m is even and the entry $m+1$ is inserted at the end of π as a new cycle $(m+1)$, then we leave all labels of π unchanged except the last label x. We define Φ by

$$\pi = \cdots(\cdots)^x \leftrightarrow \widehat{\pi} = \cdots(\cdots)((m+1)^z)^y,$$

which corresponds to the operation $x \to yz$. In this case, we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i,j}$. If m is even and the entry $m+1$ occurs in a cycle with at least two elements, there are also three cases to consider:

(i) Suppose c_r is the rth odd cycle peak of π and we put the entry $m+1$ immediately before or right after c_r. Then we have

$$\pi = \cdots(\ldots^x c_r\, ^x \ldots)\cdots(\cdots)^x \leftrightarrow \widehat{\pi} = \cdots(\ldots^x (m+1)^x c_r\, ^z \ldots)\cdots(\cdots)^y,$$

or

$$\pi = \cdots(\ldots^x c_r\, ^x \ldots)\cdots(\cdots)^x \leftrightarrow \widehat{\pi} = \cdots(\ldots^z c_r\, ^x (m+1)^x \ldots)\cdots(\cdots)^y.$$

In this case, the corresponding operation of Φ is $x \to yz$ and we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i,j}$.

(ii) Suppose d_ℓ is the ℓth even cycle peak of π and we put the entry $m+1$ immediately before or right after d_ℓ. Then we have

$$\pi = \cdots(\ldots^y d_\ell\, ^y \ldots)\cdots(\cdots)^x \leftrightarrow \widehat{\pi} = \cdots(\ldots^x (m+1)^x d_\ell\, ^z \ldots)\cdots(\cdots)^y,$$

or

$$\pi = \cdots(\ldots^y d_\ell\, ^y \ldots)\cdots(\cdots)^x \leftrightarrow \widehat{\pi} = \cdots(\ldots^z d_\ell\, ^x (m+1)^x \ldots)\cdots(\cdots)^y.$$

In this case, the corresponding operation of Φ is $y \to xz$ and we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i+1,j-1}$.

(*iii*) If we insert $m+1$ into a position of π with label z, then we have

$$\pi = \cdots(\ldots w\ ^{z} \ldots)\cdots(\cdots)^{x} \leftrightarrow \widehat{\pi} = \cdots(\ldots w^{x}(m+1)^{x}\ldots)\cdots(\cdots)^{y}.$$

In this case, the corresponding operation of Φ is $z \to xy$ and we have $\widehat{\pi} \in \mathfrak{S}_{m+1,i+1,j}$.

By induction and (1.4), we see that Φ is the desired correspondence between permutations in $\mathfrak{S}_m$ and $\mathfrak{S}_{m+1}$, which also gives a constructive proof of (1.5).

Example 2.2. Given $\pi = (14)(23) \in \mathfrak{S}_{4,1,1}$. The correspondence between π and x^3y^2 is built up as follows:

$$(1^{z})^{\mathbf{y}} \leftrightarrow y \to xz(1^{z})(2^{\mathbf{z}})^{x} \leftrightarrow z \to xy(1^{\mathbf{z}})(2^{x}3^{x})^{y} \leftrightarrow z \to xy(1^{y}4^{y})(2^{x}3^{x})^{x}.$$

3 Solutions of two types of Jacobi-pairs

3.1 Basic definitions and notation

Let

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{3.1}$$

which is the incomplete elliptic integral of the first kind in Jacobi's form. Define

$$\begin{aligned}
h_{p,q} &= F\left(\sqrt{\frac{q(1-p)}{q-p}}, \sqrt{\frac{q-p}{1-p}}\right),\\
\ell_{p,q} &= F\left(q\sqrt{\frac{1-p}{q^2-p}}, \sqrt{\frac{q^2-p}{1-p}}\right),\\
k_{p,q} &= \sqrt{\frac{p-1}{q-p}}\arctan\left(\sqrt{\frac{q(p-1)}{q-p}}\right),\\
x_{\pm} &= (p-1)x \pm k_{p,q}.
\end{aligned}$$

For any sequence $a_{n,i,j}$, we define the following generating functions

$$\begin{aligned}
A = A(x,p,q) &= \sum_{n,i,j\geqslant 0} a_{n,i,j}\frac{x^n}{n!}p^iq^j,\\
AE = AE(x,p,q) &= \sum_{n,i,j\geqslant 0} a_{2n,i,j}\frac{x^{2n}}{(2n)!}p^iq^j = \frac{1}{2}(A(x,p,q)+A(-x,p,q)),\\
AO = AO(x,p,q) &= \sum_{n,i,j\geqslant 0} a_{2n+1,i,j}\frac{x^{2n+1}}{(2n+1)!}p^iq^j = \frac{1}{2}(A(x,p,q)-A(-x,p,q)),
\end{aligned}$$

where we use the small letters $a, b, c, \ldots$ for sequences, capital letters $A, B, C, \ldots$ for generating functions, and $AE, BE, CE, \ldots, AO, BO, CO, \ldots$ for the even and odd parts of the generating functions, respectively. Also, we denote by H_y the partial derivative of the function H with respect to y.

Recall that the numbers $s_{n,i,j}$ are defined by (1.3). Then

$$\begin{aligned}
S = S(x,p,q) &= \sum_{n,i,j\geqslant 0} s_{n,i,j}\frac{x^n}{n!}p^iq^j,\\
SE = SE(x,p,q) &= \sum_{n,i,j\geqslant 0} s_{2n,i,j}\frac{x^{2n}}{(2n)!}p^iq^j \text{ and}\\
SO = SO(x,p,q) &= \sum_{n,i,j\geqslant 0} s_{2n+1,i,j}\frac{x^{2n+1}}{(2n+1)!}p^iq^j.
\end{aligned}$$

Using (1.4), we get the following comparable result of (1.6).

Theorem 3.1. *We have*

$$\begin{cases} SO(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{q}}\left(K\left(\frac{1-q}{1-p},\sqrt{p-1}x-h_{p,q}\right)-K\left(\frac{1-q}{1-p},\sqrt{p-1}x+h_{p,q}\right)\right),\\ SE(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{p}}\left(K\left(\frac{1-q}{1-p},\sqrt{p-1}x-h_{p,q}\right)+K\left(\frac{1-q}{1-p},\sqrt{p-1}x+h_{p,q}\right)\right),\end{cases} \tag{3.2}$$

where $K(p,x)=\sqrt{1-p}\,\mathrm{cn}\,(\sqrt{p}x,\sqrt{1-1/p}),\quad -1<p<1$ *and* $0<q<1$.

Proof. By (1.4), we have

$$\begin{cases} SO_x &= SE+2p(1-p)SE_p+2p(1-q)SE_q+pxSE_x,\\ SE_x &= SO+2q(1-p)SO_p+2q(1-q)SO_q+qxSO_x.\end{cases}$$

Set

$$(\widetilde{SO},\widetilde{SE})=\frac{1}{\sqrt{p-1}}(\sqrt{q}SO,\sqrt{p}SE).$$

Then

$$\begin{cases} \widetilde{SO}_x &= 2\sqrt{pq}(1-p)\widetilde{SE}_p+2\sqrt{pq}(1-q)\widetilde{SE}_q+\sqrt{pq}x\widetilde{SE}_x,\\ \widetilde{SE}_x &= 2\sqrt{pq}(1-p)\widetilde{SO}_p+2\sqrt{pq}(1-q)\widetilde{SO}_q+\sqrt{pq}x\widetilde{SO}_x.\end{cases} \tag{3.3}$$

Solving (3.3) for $\widetilde{SO}_x-\widetilde{SE}_x$ and $\widetilde{SO}_x+\widetilde{SE}_x$ (with the help of maple), we obtain that there exist two (analytical) functions K_1 and K_2 such that

$$\begin{cases} \widetilde{SO}-\widetilde{SE} &= K_1\left(\frac{1-q}{1-p},\sqrt{p-1}x+h_{p,q}\right),\\ \widetilde{SO}+\widetilde{SE} &= K_2\left(\frac{1-q}{1-p},\sqrt{p-1}x-h_{p,q}\right).\end{cases} \tag{3.4}$$

In order to provide explicit formulas for the generating functions $\widetilde{SO}_x$ and $\widetilde{SE}_x$, we solve (3.3) for $q=0$. In this case, we obtain

$$\begin{cases} SO_x(x,p,0) &= SE(x,p,0)+2p(1-p)SE_p(x,p,0)+2pSE_q(x,p,0)+pxSE_x(x,p,0),\\ SE_x(x,p,0) &= SO(x,p,0).\end{cases}$$

Note that our initial conditions are $SO(0,p,q)=0$, $SE(0,p,q)=1$,

$$SO(x,0,0)=\frac{e^x-e^{-x}}{2},\ SE(x,0,0)=\frac{e^x+e^{-x}}{2}.$$

Thus, it is obvious to see that the solution of this system of partial differential equations is given by

$$SO(x,p,0)=-I\mathrm{dn}\,(Ix,\sqrt{p})\mathrm{sn}\,(Ix,\sqrt{p})\text{ and }SE(x,p,0)=\mathrm{cn}\,(Ix,\sqrt{p}),$$

with $I^2=-1$. Therefore, solving (3.4) for $q=0$ gives

$$-\frac{\sqrt{p}}{\sqrt{p-1}}\mathrm{cn}\,(Ix,\sqrt{p})=K_1\left(\frac{1}{1-p},\sqrt{p-1}x\right),$$
$$\frac{\sqrt{p}}{\sqrt{p-1}}\mathrm{cn}\,(Ix,\sqrt{p})=K_2\left(\frac{1}{1-p},\sqrt{p-1}x\right),$$

which leads to $K_2(p,x)=-K_1(p,x)=K(p,x)$. Hence, by (3.4) we get (3.2), as claimed. □

In order to provide a unified approach to the sequences discussed in this paper, we introduce the following definitions.

Definition 3.2. A pair $(F,G)=(F(x,p,q),G(x,p,q))$ of functions is called *the Jacobi-pair of the first type* if they satisfy the following system of PDEs:

$$\begin{cases} F_x &= 2p\sqrt{q}(1-p)G_p+2p\sqrt{q}(1-q)G_q+2p\sqrt{q}xG_x,\\ G_x &= 2p\sqrt{q}(1-p)F_p+2p\sqrt{q}(1-q)F_q+2p\sqrt{q}xF_x.\end{cases} \tag{3.5}$$

Remark 3.3. Concerning the solution to (3.5), note that by defining

$$P(x,p,q)=F(x,p,q)-G(x,p,q),\ Q(x,p,q)=F(x,p,q)+G(x,p,q),$$

we have

$$\begin{cases} P_x(x,p,q)+2p\sqrt{q}((1-p)P_p(x,p,q)+(1-q)P_q(x,p,q)+xP_x(x,p,q))=0,\\ Q_x(x,p,q)-2p\sqrt{q}((1-p)Q_p(x,p,q)+(1-q)Q_q(x,p,q)+xQ_x(x,p,q))=0. \end{cases}$$

Using the Maple package, it is not hard to check that the solution (with $p,q\neq 1$ and $q\neq 0$) of these PDEs is given by

$$P(x,p,q)=V\left(\frac{1-q}{1-p},x_+\right),\ Q(x,p,q)=\widetilde{V}\left(\frac{1-q}{1-p},x_-\right),$$

for any two functions V and $\widetilde{V}$.

Definition 3.4. A pair $(M,N)=(M(x,p,q),N(x,p,q))$ of functions is called *the Jacobi-pair of the second type* if they satisfy the following system of PDEs:

$$\begin{cases} M_x &= 2q\sqrt{p}(1-p)N_p+\sqrt{p}(1-q^2)N_q+xq\sqrt{p}N_x,\\ N_x &= 2q\sqrt{p}(1-p)M_p+\sqrt{p}(1-q^2)M_q+xq\sqrt{p}M_x. \end{cases} \tag{3.6}$$

Remark 3.5. Concerning the solution to (3.6), note that by defining

$$\widetilde{P}(x,p,q)=M(x,p,q)-N(x,p,q),\ \widetilde{Q}(x,p,q)=M(x,p,q)+N(x,p,q),$$

we have

$$\begin{cases} \widetilde{P}_x(x,p,q)+\sqrt{q}(2q(1-p)\widetilde{P}_p(x,p,q)+(1-q^2)\widetilde{P}_q(x,p,q)+xq\widetilde{P}_x(x,p,q))=0,\\ \widetilde{Q}_x(x,p,q)-\sqrt{q}(2q(1-p)\widetilde{Q}_p(x,p,q)+(1-q^2)\widetilde{Q}_q(x,p,q)+xq\widetilde{Q}_x(x,p,q))=0. \end{cases}$$

Using the Maple package, it is not hard to check that the solution (with $p,q\neq 1$ and $q\neq 0$) of these PDEs is given by

$$\widetilde{P}(x,p,q)=W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right),\ \widetilde{Q}(x,p,q)=\widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right),$$

for any two functions W and $\widetilde{W}$.

3.2 Jacobi-pairs of the first type

There are countless combinatorial structures related to the differential operators xD and Dx (e.g., [8, 10, 13]). It is natural to further study (1.2) via these differential operators.

Write

$$(xD)^{n+1}(x)=(xD)(xD)^n(x)=xD((xD)^n(x)),$$
$$(Dx)^{n+1}(x)=(Dx)(Dx)^n(x)=D(x(Dx)^n(x)),$$
$$(Dx)^{n+1}(y)=(Dx)(Dx)^n(y)=D(x(Dx)^n(y)).$$

In particular, from (1.2), we have

$$(xD)(x)=xyz,\quad (xD)^2(x)=xy^2z^2+x^3y^2+x^3z^2,$$
$$(Dx)(x)=2xyz,\quad (Dx)^2(x)=4xy^2z^2+2x^3y^2+2x^3z^2,$$
$$(Dx)(y)=y^2z+x^2z,\quad (Dx)^2(y)=y^3z^2+5x^2yz^2+x^2y^3+x^4y.$$

For $n \geqslant 0$, we define the numbers $a_{n,i,j}, c_{n,i,j}$ and $d_{n,i,j}$ by

$$\begin{aligned}
(xD)^{2n}(x) &= \sum_{i,j\geqslant 0} a_{2n,i,j} x^{2i+1} y^{2j} z^{4n-2i-2j}, \\
(xD)^{2n+1}(x) &= \sum_{i,j\geqslant 0} a_{2n+1,i,j} x^{2i+1} y^{2j+1} z^{4n-2i-2j+1}, \\
(Dx)^{2n}(x) &= \sum_{i,j\geqslant 0} c_{2n,i,j} x^{2i+1} y^{2j} z^{4n-2i-2j}, \\
(Dx)^{2n+1}(x) &= \sum_{i,j\geqslant 0} c_{2n+1,i,j} x^{2i+1} y^{2j+1} z^{4n-2i-2j+1}, \\
(Dx)^{2n}(y) &= \sum_{i,j\geqslant 0} d_{2n,i,j} x^{2i} y^{2j+1} z^{4n-2i-2j}, \\
(Dx)^{2n+1}(y) &= \sum_{i,j\geqslant 0} d_{2n+1,i,j} x^{2i} y^{2j} z^{4n-2i-2j+3}.
\end{aligned}$$

For convenience, we list the first terms of the corresponding generating functions:

$$\begin{aligned}
A(x,p,q) &= 1 + x + (p(1+q)+q)\frac{x^2}{2!} + (4p^2 + 5p(1+q)+q)\frac{x^3}{3!} \\
&\quad + (p^3(4+4q) + p^2(5+50q+5q^2) + p(18q^2+18q) + q^2)\frac{x^4}{4!} \\
&\quad + (16p^4 + p^3(148+148q) + p^2(61+394q+61q^2) + p(58q+58q^2) + q^2)\frac{x^5}{5!} + \cdots,
\end{aligned}$$

$$\begin{aligned}
C(x,p,q) &= 1 + 2x + 2(p(1+q)+2q)\frac{x^2}{2!} + 8(p^2 + 2p(1+q)+q)\frac{x^3}{3!} \\
&\quad + 8(p^3(1+q) + 2p^2(1+9q+q^2) + 11pq(1+q) + 2q^2)\frac{x^4}{4!} \\
&\quad + 16(2p^4 + 26p^3(1+q) + p^2(17+98q+17q^2) + 26pq(1+q) + 2q^2)\frac{x^5}{5!} + \cdots,
\end{aligned}$$

$$\begin{aligned}
D(x,p,q) &= 1 + (p+q)x + (p^2 + p(5+q)+q)\frac{x^2}{2!} + (p^3 + p^2(5+18q) + pq(18+5q) + q^2)\frac{x^3}{3!} \\
&\quad + (p^4 + p^3(58+18q) + p^2(61+164q+5q^2) + pq(58+18q) + q^2)\frac{x^4}{4!} + \cdots.
\end{aligned}$$

Note that

$$\begin{aligned}
(xD)^{2n+1}(x) &= (xD)(xD)^{2n}(x) \\
&= xD\left(\sum_{i,j\geqslant 0} a_{2n,i,j} x^{2i+1} y^{2j} z^{4n-2i-2j}\right) \\
&= \sum_{i,j\geqslant 0} (2i+1) a_{2n,i,j} x^{2i+1} y^{2j+1} z^{4n-2i-2j+1} + \sum_{i,j\geqslant 0} 2j a_{2n,i,j} x^{2i+3} y^{2j-1} z^{4n-2i-2j+1} + \\
&\quad \sum_{i,j\geqslant 0} (4n-2i-2j) a_{2n,i,j} x^{2i+3} y^{2j+1} z^{4n-2i-2j-1}.
\end{aligned}$$

Hence

$$a_{2n+1,i,j} = (2i+1)a_{2n,i,j} + (2j+2)a_{2n,i-1,j+1} + (4n-2i-2j+2)a_{2n,i-1,j}. \tag{3.7}$$

Similarly,

$$a_{2n,i,j} = (2i+1)a_{2n-1,i,j-1} + (2j+1)a_{2n-1,i-1,j} + (4n-2i-2j+1)a_{2n-1,i-1,j-1}. \tag{3.8}$$

Equivalently, recurrences (3.7) and (3.8) can be written as the following lemma.

Lemma 3.6. *We have*

$$\begin{cases} AO_x &= AE + 2p(1-p)AE_p + 2p(1-q)AE_q + 2xpAE_x, \\ AE_x &= (q+p-pq)AO + 2pq(1-p)AO_p + 2pq(1-q)AO_q + 2xpqAO_x. \end{cases}$$

Equivalently, $(\widetilde{AO}, \widetilde{AE})$ *is a Jacobi-pair of the first type, where* $\widetilde{AO} = \sqrt{\frac{pq}{p-1}}AO$ *and* $\widetilde{AE} = \sqrt{\frac{p}{p-1}}AE$*.*

Theorem 3.7. *Let* $y = \frac{1-q}{1-p}$*. Define*

$$G(x,p) = \sqrt{\frac{1-p}{\cos^2(x\sqrt{p(1-p)}) - p}} \quad \text{and } H(x,p) = \frac{(1-p)\sin(2x\sqrt{p(1-p)})}{2\sqrt{p}(\cos^2(x\sqrt{p(1-p)}) - p)^{3/2}}.$$

Then

$$\begin{aligned} AO(x,p,q) &= \frac{1}{2}\sqrt{\frac{p-q}{pq}}(H(yx_-, 1-1/y) - G(yx_+, 1-1/y)), \\ AE(x,p,q) &= \frac{1}{2}\sqrt{\frac{p-q}{p}}(H(yx_-, 1-1/y) + G(yx_+, 1-1/y)). \end{aligned}$$

Proof. By Remark 3.3 and Lemma 3.6, we obtain that

$$\sqrt{\frac{pq}{p-1}}AO(x,p,q) - \sqrt{\frac{p}{p-1}}AE(x,p,q) = V(y, x_+)$$

and

$$\sqrt{\frac{pq}{p-1}}AO(x,p,q) + \sqrt{\frac{p}{p-1}}AE(x,p,q) = \widetilde{V}(y, x_-),$$

for some functions V and $\widetilde{V}$. Moreover, at $q = 0$, the above equations reduce to

$$-V(p,x) = \widetilde{V}(p,x) = \sqrt{1-p}AE(-px, 1-1/p, 0).$$

Hence, if we guess that $AE(x,p,0) = G(x,p)$ and $AO(x,p,0) = H(x,p)$, then we get

$$\begin{aligned} \sqrt{\frac{pq}{p-1}}AO(x,p,q) - \sqrt{\frac{p}{p-1}}AE(x,p,q) &= -\sqrt{1-y}G(yx_+, 1-1/y), \\ \sqrt{\frac{pq}{p-1}}AO(x,p,q) + \sqrt{\frac{p}{p-1}}AE(x,p,q) &= \sqrt{1-y}H(yx_-, 1-1/y), \end{aligned}$$

which implies

$$\begin{aligned} AO(x,p,q) &= \frac{1}{2}\sqrt{\frac{(1-y)(p-1)}{pq}}(H(yx_-, 1-1/y) - G(yx_+, 1-1/y)), \\ AE(x,p,q) &= \frac{1}{2}\sqrt{\frac{(p-1)(1-y)}{p}}(H(yx_-, 1-1/y) + G(yx_+, 1-1/y)). \end{aligned}$$

To complete the proof, we have to check that the functions AO and AE are satisfying Lemma 3.6, which is a routine procedure. □

Along the same lines, we get

$$\begin{aligned} c_{2n,i,j} &= (2i+2)c_{2n-1,i,j-1} + (2j+1)c_{2n-1,i-1,j} + (4n-2i-2j+1)c_{2n-1,i-1,j-1}, \\ c_{2n+1,i,j} &= (2i+2)c_{2n,i,j} + (2j+2)c_{2n,i-1,j+1} + (4n-2i-2j+2)c_{2n,i-1,j}, \end{aligned} \tag{3.9}$$

which leads to the following result.

Lemma 3.8. *We have*

$$\begin{cases} CO_x &= 2CE + 2p(1-p)CE_p + 2p(1-q)CE_q + 2xpCE_x, \\ CE_x &= (p+2q-pq)CO + 2pq(1-p)CO_p + 2pq(1-q)CO_q + 2xpqCO_x. \end{cases} \tag{3.10}$$

Equivalently, $(\widetilde{CO}, \widetilde{CE})$ *is a Jacobi-pair of the first type, where* $\widetilde{CO} = \frac{p\sqrt{q}}{p-1}CO$ *and* $\widetilde{CE} = \frac{p}{p-1}CE$.

Theorem 3.9. *Define* $y = \frac{1-q}{1-p}$ *and* $G(x,p) = \frac{1-p}{p\cos^2(x\sqrt{p-1})+1-p}$. *Then*

$$\begin{aligned} CO(x,p,q) &= \tfrac{p-1}{2p\sqrt{q}}(G(x_-,y) - G(x_+,y)), \\ CE(x,p,q) &= \tfrac{p-1}{2p}(G(x_-,y) + G(x_+,y)), \\ C(x,p,q) &= \tfrac{p-1}{2p\sqrt{q}}(G(x_-,y) - G(x_+,y)) + \tfrac{p-1}{2p}(G(x_-,y) + G(x_+,y)). \end{aligned} \tag{3.11}$$

Proof. By Remark 3.3 and Lemma 3.8, we obtain that

$$\frac{p\sqrt{q}}{p-1}CO(x,p,q) - \frac{p}{p-1}CE(x,p,q) = \widetilde{V}(y,x_+)$$

and

$$\frac{p\sqrt{q}}{p-1}CO(x,p,q) + \frac{p}{p-1}CE(x,p,q) = V(y,x_-)$$

for some functions V and $\widetilde{V}$. Moreover, at $q=0$, then above equations reduce to

$$V(1/(1-p),(p-1)x) = -\widetilde{V}(1/(1-p),(p-1)x).$$

Hence, if we take $(1-p)CE(-px, 1-1/p, 0) = G(x,p)$, $CO(x,p,q) = \frac{p-1}{2p\sqrt{q}}(G(x_-,y) - G(x_+,y))$ and $CE(x,p,q) = \frac{p-1}{2p}(G(x_-,y)+G(x_+,y))$, then (3.11) is a solution for (3.10), where

$$V(1/(1-p),(p-1)x) = -\widetilde{V}(1/(1-p),(p-1)x) = (1-p)CE(-px,1-1/p,0) = G(x,p).$$

To complete the proof, we have to check that the functions CO and CE are satisfying Lemma 3.8, which is a routine procedure. □

Corollary 3.10. *We have*

$$\begin{aligned} C(x,0,q) &= \cosh(2\sqrt{q}x) + \frac{1}{\sqrt{q}}\sinh(2\sqrt{q}x), \\ C(x,1,q) &= \frac{(x^2(q-1)+2x+1)}{(x^2(1-q)-2x+1)(x^2(1-q)+2x+1)}, \\ C(x,p,0) &= \frac{(1-p)\sqrt{1-p}\sin(2x\sqrt{p(1-p)})}{\sqrt{p}(\cos^2(x\sqrt{p(1-p)})-p)^2} + \frac{1-p}{\cos^2(x\sqrt{p(1-p)})-p}, \\ C(x,p,1) &= \frac{p-1}{p-e^{2x(p-1)}}. \end{aligned}$$

Proof. By applying Theorem 3.9 for $q=0$ or $p=1$, we obtain the formulas of $C(x,p,0)$ and $C(x,1,q)$. Solving (3.10) for $p=0$, we obtain

$$\begin{aligned} CE(x,0,q) &= \alpha_q e^{2\sqrt{q}x} + \beta_q e^{-2\sqrt{q}x}, \\ CO(x,0,q) &= \frac{1}{\sqrt{q}}(\alpha_q e^{2\sqrt{q}x} - \beta_q e^{-2\sqrt{q}x}). \end{aligned}$$

By using the initial conditions $CE(0,p,q)=1$ and $CO(0,p,q)=0$, we obtain $CE(x,0,p) = \cosh(2\sqrt{q}x)$ and $CO(x,0,q) = \frac{1}{\sqrt{q}}\sinh(2\sqrt{q}x)$, which completes the first part of the proof.

Again, solving (3.10) with $q=1$ for $CO(x,p,1) - CE(x,p,1)$ and $CO(x,p,1)+CE(x,p,1)$, we obtain

$$CO(x,p,1) - CE(x,p,1) = \frac{p-1}{p}V(x(p-1) + \frac{1}{2}\ln p),$$

$$CO(x,p,1)+CE(x,p,1)=\frac{p-1}{p}\widetilde{V}(x(p-1)-\frac{1}{2}\ln p),$$

where $V,\widetilde{V}$ are two fixed functions. By the initial values $CE(0,p,q)=1$ and $CO(0,p,q)=0$, we get

$$V(y)=\frac{e^{2y}}{1-e^{2y}} \text{ and } \widetilde{V}(y)=\frac{1}{1-e^{2y}}.$$

Hence,

$$\begin{aligned} CO(x,p,1)-CE(x,p,1)&=\frac{(p-1)e^{2x(p-1)}}{1-pe^{2x(p-1)}},\\ CO(x,p,1)+CE(x,p,1)&=\frac{p-1}{p-e^{2x(p-1)}}, \end{aligned}$$

which completes the proof. □

Along the same lines, we get

$$\begin{aligned} d_{2n,i,j}&=(2i+1)d_{2n-1,i,j}+(2j+2)d_{2n-1,i-1,j+1}+(4n-2i-2j+1)d_{2n-1,i-1,j},\\ d_{2n+1,i,j}&=(2i+1)d_{2n,i,j-1}+(2j+1)d_{2n,i-1,j}+(4n-2i-2j+4)d_{2n,i-1,j-1}, \end{aligned} \tag{3.12}$$

which leads to the following result.

Lemma 3.11. *We have*

$$\begin{cases} DO_x=(p+q)DE+2pq(1-p)DE_p+2pq(1-q)DE_q+2pqxDE_x,\\ DE_x=(1+p)DO+2p(1-p)DO_p+2p(1-q)DO_q+2pxDO_x. \end{cases} \tag{3.13}$$

Equivalently, $(\widetilde{DO},\widetilde{DE})$ *is a Jacobi-pair of the first type, where* $\widetilde{DO}=\sqrt{\frac{p}{p-1}}DO$ *and* $\widetilde{DE}=\sqrt{\frac{pq}{p-1}}DE$.

By similar arguments as in the proof of Theorem 3.9 with help from Remark 3.3 and Lemma 3.11, we obtain the following result.

Theorem 3.12. *Define* $y=\frac{1-q}{1-p}$ *and* $G(x,p)=\frac{\sinh(x\sqrt{p-1})}{1-\frac{p}{p-1}\cosh^2(x\sqrt{p-1})}$. *Then*

$$\begin{aligned} DO(x,p,q)&=\frac{\sqrt{p-1}}{2\sqrt{p}}(G(x_-,y)+G(x_+,y)),\\ DE(x,p,q)&=\frac{\sqrt{p-1}}{2\sqrt{pq}}(G(x_-,y)-G(x_+,y)),\\ D(x,p,q)&=\frac{\sqrt{p-1}}{2\sqrt{pq}}(G(x_-,y)-G(x_+,y))+\frac{\sqrt{p-1}}{2\sqrt{p}}(G(x_-,y)+G(x_+,y)). \end{aligned}$$

Corollary 3.13. *Let* $\widetilde{p}=\sqrt{p(p-1)}$. *Then we have*

$$\begin{aligned} D(x,p,0)&=\frac{(p-1)\cosh(x\widetilde{p})(\cosh^2(x\widetilde{p})-2+p)}{((p-1)\cosh^2(x\widetilde{p})-p\sinh^2(x\widetilde{p}))^2}+\frac{\widetilde{p}\sinh(x\widetilde{p})}{p-\cosh^2(x\widetilde{p})},\\ D(x,1,q)&=\frac{(x^2(q-1)+2x-1)(x^2(1-q)+2x+1)(x^3(q-1)^2+x^2(q-1)-x(q+1)-1)}{(x^2(q-1)-2x\sqrt{q}+1)^2(x^2(q-1)+2x\sqrt{q}+1)^2},\\ D(x,p,1)&=\frac{(1-p)e^{(1-p)x}}{1-pe^{2(1-p)x}}. \end{aligned}$$

From Corollary 3.10 and Corollary 3.13, it is easy to verify that

$$\begin{aligned} C(x,1,q)&=\sum_{n\geqslant0}\sum_{k\geqslant0}\binom{2n+1}{2k}q^kx^{2n}+\sum_{n\geqslant1}\sum_{k\geqslant0}\binom{2n}{2k+1}q^kx^{2n-1},\\ D(x,1,q)&=\sum_{n\geqslant0}\sum_{k\geqslant0}\binom{2n+1}{2k+1}q^kx^{2n}+\sum_{n\geqslant1}\sum_{k\geqslant0}\binom{2n}{2k}q^kx^{2n-1}. \end{aligned}$$

3.3 Jacobi-pairs of the second type

In [6], Dumont considered chains of general substitution rules on words. In particular, Dumont discovered the following.

Proposition 3.14. *If*

$$G = \{w \to wx, x \to wx\}, \tag{3.14}$$

then

$$D^n(w) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle w^{k+1} x^{n-k},$$

where $\left\langle {n \atop k} \right\rangle$ *is the Eulerian number, i.e., the number of permutations in* $\mathfrak{S}_n$ *with* k *descents.*

As a conjunction of (1.7) and (3.14), it is natural to consider the context-free grammar

$$G = \{w \to wx, x \to yz, y \to xz, z \to xy\}. \tag{3.15}$$

From (3.15), we have

$$\begin{aligned} D(w) &= wx,\ D^2(w) = w(x^2+yz),\ D^3(x) = w(x^3+xz^2+3xyz+xy^2),\\ D^4(w) &= w(x^4+10x^2yz+4x^2z^2+4x^2y^2+3y^2z^2+y^3z+yz^3),\\ D(w^2) &= 2w^2x,\ D^2(w^2) = w^2(4x^2+2yz),\ D^3(w^2) = w^2(8x^3+12xyz+2xz^2+2xy^2). \end{aligned}$$

For $n \geqslant 0$, we define the numbers $t_{n,i,j}$ and $r_{n,i,j}$ by

$$\begin{aligned} D^{2n}(w) &= w \sum_{i,j\geqslant 0} t_{2n,i,j} x^{2i} y^j z^{2n-2i-j},\\ D^{2n+1}(w) &= w \sum_{i,j\geqslant 0} t_{2n+1,i,j} x^{2i+1} y^j z^{2n-2i-j},\\ D^{2n}(w^2) &= w^2 \sum_{i,j\geqslant 0} r_{2n,i,j} x^{2i} y^j z^{2n-2i-j},\\ D^{2n+1}(w^2) &= w^2 \sum_{i,j\geqslant 0} r_{2n+1,i,j} x^{2i+1} y^j z^{2n-2i-j}. \end{aligned}$$

The first terms of the corresponding generating functions are given as follows:

$$\begin{aligned} T(x,p,q) &= 1 + x + (p+q)\frac{x^2}{2!} + (1+p+3q+q^2)\frac{x^3}{3!}\\ &\quad + (p^2+4p+(10p+1)q+(4p+3)q^2+q^3)\frac{x^4}{4!}\\ &\quad + (p^2+14p+1+(30p+15)q+(14p+29)q^2+15q^3+q^4)\frac{x^5}{5!} + \cdots, \end{aligned}$$

$$\begin{aligned} R(x,p,q) &= 1 + 2x + (4p+2q)\frac{x^2}{2!} + (2+8p+12q+2q^2)\frac{x^3}{3!}\\ &\quad + (16p+16p^2+(2+56p)q+(12+16p)q^2+2q^3)\frac{x^4}{4!}\\ &\quad + (2+88p+32p^2+(60+240p)q+(148+88p)q^2+60q^3+2q^4)\frac{x^5}{5!} + \cdots. \end{aligned}$$

Note that

$$D^{2n+1}(w) = D(D^{2n}(w))$$

$$
\begin{aligned}
&= D\left(w\sum_{i,j\geqslant 0} t_{2n,i,j}x^{2i}y^jz^{2n-2i-j}\right)\\
&= w\sum_{i,j\geqslant 0} t_{2n,i,j}x^{2i+1}y^jz^{2n-2i-j} + w\sum_{i,j\geqslant 0} 2it_{2n,i,j}x^{2i-1}y^{j+1}z^{2n-2i-j+1}+\\
&w\sum_{i,j\geqslant 0} jt_{2n,i,j}x^{2i+1}y^{j-1}z^{2n-2i-j+1} + w\sum_{i,j\geqslant 0}(2n-2i-j)t_{2n,i,j}x^{2i+1}y^{j+1}z^{2n-2i-j-1}.
\end{aligned}
$$

Hence

$$t_{2n+1,i,j} = t_{2n,i,j} + (2i+2)t_{2n,i+1,j-1} + (j+1)t_{2n,i,j+1} + (2n-2i-j+1)t_{2n,i,j-1}. \tag{3.16}$$

Similarly,

$$t_{2n,i,j} = t_{2n-1,i-1,j} + (2i+1)t_{2n-1,i,j-1} + (j+1)t_{2n-1,i-1,j+1} + (2n-2i-j+1)t_{2n-1,i-1,j-1}. \tag{3.17}$$

By rewriting these recurrence relations in terms of generating functions TE and TO, we obtain the following result.

Lemma 3.15. *We have*

$$
\begin{cases}
TO_x &= TE + 2q(1-p)TE_p + (1-q^2)TE_q + xqTE_x,\\
TE_x &= (p+q-qp)TO + 2pq(1-p)TO_p + p(1-q^2)TO_q + xqpTO_x.
\end{cases} \tag{3.18}
$$

Equivalently, $(\widetilde{TO},\widetilde{TE})$ *is a Jacobi-pair of the second type, where* $\widetilde{TO} = \sqrt{\frac{p(1+q)}{1-q}}TO$ *and* $\widetilde{TE} = \sqrt{\frac{1+q}{1-q}}TE$.

Theorem 3.16. *Let* $\ell'_{p,q} = \sqrt{\frac{1-q^2}{1-p}}\ell_{p,q}$. *Then we have*

$$
\begin{cases}
TO(x,p,q) &= \frac{q-1}{\sqrt{p(p-1)}}\mathrm{sn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}),\\
TE(x,p,q) &= \sqrt{\frac{1-q}{1+q}}\mathrm{dn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}).
\end{cases}
$$

Proof. By Remark 3.5, we see that Lemma 3.15 leads to

$$
\begin{cases}
\sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) - \sqrt{\frac{1+q}{1-q}}TE(x,p,q) &= W\left(\frac{1-q^2}{1-p}, \sqrt{p-1}x - \ell_{p,q}\right),\\
\sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) + \sqrt{\frac{1+q}{1-q}}TE(x,p,q) &= \widetilde{W}\left(\frac{1-q^2}{1-p}, \sqrt{p-1}x + \ell_{p,q}\right),
\end{cases} \tag{3.19}
$$

for some functions W and $\widetilde{W}$. Thus, at $q=0$, we have

$$
\begin{cases}
\sqrt{p}TO(I\sqrt{p}x, 1-1/p, 0) - TE(I\sqrt{p}x, 1-1/p, 0) &= W(p,x),\\
\sqrt{p}TO(I\sqrt{p}x, 1-1/p, 0) + TE(I\sqrt{p}x, 1-1/p, 0) &= \widetilde{W}(p,x),
\end{cases}
$$

where $I^2=-1$. Therefore, if we set

$$TE(x,p,0) = \mathrm{dn}\,(Ix,\sqrt{p}) \text{ and } TO(x,p,0) = -I\mathrm{sn}\,(Ix,\sqrt{p}),$$

then

$$
\begin{cases}
-I\sqrt{p}\mathrm{sn}\,(-\sqrt{p}x, \sqrt{1-1/p}) - \mathrm{dn}\,(-\sqrt{p}x, \sqrt{1-1/p}) &= W(p,x),\\
-I\sqrt{p}\mathrm{sn}\,(-\sqrt{p}x, \sqrt{1-1/p}) + \mathrm{dn}\,(-\sqrt{p}x, \sqrt{1-1/p}) &= \widetilde{W}(p,x).
\end{cases}
$$

By (3.19), we obtain

$$
\begin{cases}
\sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) - \sqrt{\frac{1+q}{1-q}}TE(x,p,q)\\
\qquad = -\sqrt{\frac{q^2-1}{1-p}}\mathrm{sn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}) - \mathrm{dn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}),\\
\sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) + \sqrt{\frac{1+q}{1-q}}TE(x,p,q)\\
\qquad = -\sqrt{\frac{q^2-1}{1-p}}\mathrm{sn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}) + \mathrm{dn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}),
\end{cases}
$$

which implies

$$\begin{cases} TO(x,p,q) &= \frac{q-1}{\sqrt{p(p-1)}}\mathrm{sn}\,(-\sqrt{q^2-1}x+\ell'_{p,q},\sqrt{\frac{p-q^2}{1-q^2}}),\\ TE(x,p,q) &= \sqrt{\frac{1-q}{1+q}}\mathrm{dn}\,(-\sqrt{q^2-1}x-\ell'_{p,q},\sqrt{\frac{p-q^2}{1-q^2}}),\end{cases}$$

which agrees with the case $q=0$. To complete the proof, we have to check that the functions TO and TE satisfy Lemma 3.15, which is a routine procedure. □

By the above theorem (or by a direct check using Lemma 3.15), we obtain the following result.

Corollary 3.17. *Let* $h(x,p)=\frac{\sqrt{p-1}}{\sqrt{p-1}\cosh(x\sqrt{p-1})-\sqrt{p}\sinh(x\sqrt{p-1})}$. *Then, we have*

$$T(x,p,1)=\frac{1}{2}(h(x,p)+h(-x,p))+\frac{1}{2\sqrt{p}}(h(x,p)-h(-x,p)),$$

$$T(x,1,q)=\frac{q^2-1+\sqrt{q^2-1}\sinh(x\sqrt{q^2-1})}{(1+q)(q-\cosh(x\sqrt{q^2-1}))}.$$

Along the same lines, we have

$$\begin{aligned} r_{2n+1,i,j}&=2r_{2n,i,j}+(2i+2)r_{2n,i+1,j-1}+(j+1)r_{2n,i,j+1}+(2n-2i-j+1)r_{2n,i,j-1},\\ r_{2n,i,j}&=2r_{2n-1,i-1,j}+(2i+1)r_{2n-1,i,j-1}+(j+1)r_{2n-1,i-1,j+1}+\\ &\quad(2n-2i-j+1)r_{2n-1,i-1,j-1}, \end{aligned} \tag{3.20}$$

which implies the following result.

Lemma 3.18. *We have*

$$\begin{cases} RO_x &= 2RE+2q(1-p)RE_p+(1-q^2)RE_q+xqRE_x,\\ RE_x &= (2p+q-pq)RO+2pq(1-p)RO_p+p(1-q^2)RO_q+xpqRO_x. \end{cases} \tag{3.21}$$

Equivalently, $(\widetilde{RO},\widetilde{RE})$ *is a Jacobi-pair of the second type, where* $\widetilde{RO}=\frac{\sqrt{p}(1+q)}{1-q}RO$ *and* $\widetilde{RE}=\frac{1+q}{1-q}RE$.

Along the line of the proof of Theorem 3.16, we state the following result.

Theorem 3.19. *Let*

$$\begin{cases} U\,(p,x) &= -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p')-2p\mathrm{cn}^2(-\sqrt{p}x,p')+1-2/p,\\ \widetilde{U}\,(p,x) &= -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p')+2p\mathrm{cn}^2(-\sqrt{p}x,p')-1+2/p, \end{cases}$$

where $p'=\sqrt{1-1/p}$. *Then*

$$\begin{cases} RO(x,p,q) &= \frac{\sqrt{p}(1-q)}{2(1+q)}\left(U\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)+\widetilde{U}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)\right),\\ RE(x,p,q) &= \frac{1-q}{2(1+q)}\left(\widetilde{U}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)-U\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)\right). \end{cases}$$

Proof. By Remark 3.5, we obtain

$$\begin{cases} \frac{\sqrt{p}(1+q)}{1-q}RO(x,p,q)-\frac{1+q}{1-q}RE(x,p,q) &= W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right),\\ \frac{\sqrt{p}(1+q)}{1-q}RO(x,p,q)+\frac{1+q}{1-q}RE(x,p,q) &= \widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right), \end{cases} \tag{3.22}$$

for some functions W and $\widetilde{W}$. Thus, at $q=0$, we have

$$\begin{cases} \sqrt{p}RO(I\sqrt{p}x,1-1/p,0)-RE(I\sqrt{p}x,1-1/p,0) &= W\,(p,x)\,,\\ \sqrt{p}RO(I\sqrt{p}x,1-1/p,0)+RE(I\sqrt{p}x,1-1/p,0) &= \widetilde{W}\,(p,x)\,, \end{cases}$$

where $I^2=-1$. Therefore, if we set

$$RE(x,p,0)=2p\mathrm{cn}^2(Ix,\sqrt{p})-2p+1 \text{ and } RO(x,p,0)=-2I\mathrm{dn}\,(Ix,\sqrt{p})\mathrm{sn}\,(Ix,\sqrt{p}),$$

then

$$\begin{cases} -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p') - 2p\mathrm{cn}^{\,2}(-\sqrt{p}x,p') + 1 - 2/p = W\,(p,x)\,, \\ -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p') + 2p\mathrm{cn}^{\,2}(-\sqrt{p}x,p') - 1 + 2/p = \widetilde{W}\,(p,x)\,, \end{cases}$$

where $p' = \sqrt{1-1/p}$. By (3.22), we have

$$\begin{cases} RO(x,p,q) = \frac{\sqrt{p}(1-q)}{2(1+q)}\left(W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)+\widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)\right), \\ RE(x,p,q) = \frac{1-q}{2(1+q)}\left(\widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)-W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)\right), \end{cases}$$

which agrees with the case $q=0$. To complete the proof, we have to check that the functions RO and RE satisfy Lemma 3.18, which is a routine procedure. □

4 Applications

In this section, we apply the results obtained in the previous section to present new characterizations for several combinatorial sequences.

4.1 Peaks, descents and perfect matchings

Perhaps one of the most important permutation statistics is the peaks statistic (see, e.g., [11,12,15,16] and the references contained therein). A *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0)=0$. Denote by $\widetilde{P}_{n,k}$ the number of permutations in $\mathfrak{S}_n$ with k left peaks. Recall that $P_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ with k interior peaks. Define polynomials

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} P_{n,k}x^k, \quad \widetilde{P}_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \widetilde{P}_{n,k}x^k.$$

The polynomial $P_n(x)$ satisfies recurrence relation

$$P_{n+1}(x) = (nx - x + 2)P_n(x) + 2x(1-x)\frac{d}{dx}P_n(x),$$

with the initial values $P_1(x)=1, P_2(x)=2, P_3(x)=4+2x$, and the polynomial $\widetilde{P}_n(x)$ satisfies recurrence relation

$$\widetilde{P}_{n+1}(x) = (nx+1)\widetilde{P}_n(x) + 2x(1-x)\frac{d}{dx}\widetilde{P}_n(x), \tag{4.1}$$

with the initial values $\widetilde{P}_1(x)=1, \widetilde{P}_2(x)=1+x, \widetilde{P}_3(x)=1+5x$ (see [18, A008303,A008971]).

A *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > \pi(i+1)$. Denote by $\mathrm{des}\,(\pi)$ the number of descents of π. Let

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}\,(\pi)} = \sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle x^k.$$

The polynomial $A_n(x)$ is called an *Eulerian polynomial*. Let B_n denote the set of signed permutations of $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm1,\pm2,\ldots,\pm n\}$. Let

$$B_n(x) = \sum_{k=0}^{n} B(n,k)x^k = \sum_{\pi\in B_n} x^{\mathrm{des}\,_B(\pi)},$$

where $\mathrm{des}\,_B(\pi) = |\{i \in [n] : \pi(i-1) > \pi(i)\}|$ with $\pi(0)=0$. The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n,k)$ is called an *Eulerian number of type B*.

Recall that a *perfect matching* of $[2n]$ is a partition of $[2n]$ into n blocks of size 2. Denote by $N(n,k)$ the number of perfect matchings of $[2n]$ with the restriction that only k matching pairs have odd smaller entries (see [18, A185411]). It is easy to verify that

$$N(n+1,k) = 2kN(n,k) + (2n-2k+3)N(n,k-1). \tag{4.2}$$

We can now conclude the following result from the discussion above.

Theorem 4.1. *For $n \geqslant 1$, we have*

(i) $\sum_{i,j\geqslant 0} a_{n,i,j} = (2n-1)!!$.

(ii) $\sum_{j\geqslant 0} a_{n,i,j} = N(n,n-i)$.

(iii) $\sum_{j\geqslant 0} a_{n,i,\lfloor \frac{n}{2}\rfloor} x^i = \sum_{j\geqslant 0} a_{n,i,\lfloor \frac{n}{2}\rfloor - i} x^i = \widetilde{P}_n(x)$.

(iv) $\sum_{j\geqslant 0} c_{n,i,j} = 2^n \left\langle {n \atop i} \right\rangle$.

(v) $\sum_{j\geqslant 0} d_{n,i,j} = B(n,i)$.

(vi) $\sum_{i\geqslant 0} c_{n,i,\lfloor \frac{n}{2}\rfloor} x^i = \sum_{i\geqslant 0} c_{n,i,\lfloor \frac{n}{2}\rfloor - i} x^i = P_{n+1}(x)$.

(vii) $\sum_{i\geqslant 0} c_{2n-1,i,0} x^{2n-2-i} = \sum_{i\geqslant 0} c_{2n,i,0} x^{2n-1-i} = P_{2n}(x)$.

(viii) $\sum_{i\geqslant 0} d_{n,i,\lceil \frac{n}{2}\rceil} x^i = \widetilde{P}_n(x)$ *and* $\sum_{i\geqslant 0} d_{n,i,\lceil \frac{n}{2}\rceil - i} x^i = \widetilde{P}_{n+1}(x)$.

(ix) $\sum_{i\geqslant 0} d_{2n,i,0} x^{2n-i} = \sum_{i\geqslant 0} d_{2n+1,i,0} x^{2n+1-i} = \widetilde{P}_{2n+1}(x)$.

Proof. We only prove the assertion for the sequence $a_{n,i,j}$ and the corresponding assertion for the other sequences follows from similar consideration.

(A) Setting $p, q = 1$ in Lemma 3.6 gives

$$\begin{cases} AO_x(x,1,1) = AE(x,1,1) + 2xAE_x(x,1,1), \\ AE_x(x,1,1) = AO(x,1,1) + 2xAO_x(x,1,1), \end{cases}$$

which implies $A_x(x,1,1) = A(x,1,1) + 2xA_x(x,1,1)$. Therefore,

$$A(x,1,1) = \frac{A(0,1,1)}{\sqrt{1-2x}} = \frac{1}{\sqrt{1-2x}} = \sum_{n\geqslant 0} \frac{n!}{2^n}\binom{2n}{n}\frac{x^n}{n!}.$$

Hence, $\sum_{i,j\geqslant 0} a_{n,i,j} = \frac{n!}{2^n}\binom{2n}{n} = (2n-1)!!$, as required.

(B) Setting $q = 1$ in Lemma 3.6 gives

$$A_x(x,p,1) = A(x,p,1) + 2p(1-p)A_p(x,p,1) + 2xpA_x(x,1,1).$$

By $A(0,1,p) = 1$, it is a routine to check that $A(x,p,1) = \frac{\sqrt{1-p}e^{x(1-p)}}{\sqrt{1-pe^{2x(1-p)}}}$. Therefore, by [13, eq. (25)] we have

$$A(px,1/p,1) = \frac{\sqrt{1-p}}{\sqrt{1-pe^{2x(1-p)}}} = \sum_{n,k\geqslant 0} N(n,k)x^n p^k,$$

which implies that $A(x,p,1) = \sum_{n,k\geqslant 0} N(n,n-k)x^n p^k$. Hence $\sum_{j\geqslant 0} a_{n,k,j} = N(n,n-k)$, as claimed.

(C) Let $f_{n,i} = a_{n,i,\lfloor n/2\rfloor}$. By (3.7) and (3.8), we have

$$f_{n,i} = (2i+1)f_{n-1,i} + (n-2i+1)f_{n-1,i-1}, \quad 0 \leqslant i \leqslant \lfloor n/2\rfloor,$$

with $f_{0,0} = 1$. Define $f_n(x) = \sum_{i\geqslant 0} f_{n,i}x^i$. Then

$$f_{n+1}(x) = (nx+1)f_n(x) + 2x(1-x)\frac{d}{dx}f_n(x), \tag{4.3}$$

with the initial condition $f_0(x) = 1$. By comparing (4.3) with (4.1), we see that the polynomials $f_n(x)$ satisfy the same recurrence relation and initial conditions as $\widetilde{P}_n(x)$, so they agree. Similarly, it is easy to verify that

$$\sum_{j\geqslant 0} a_{n,i,\lfloor\frac{n}{2}\rfloor-i}x^i = \widetilde{P}_n(x),$$

which completes the proof. □

4.2 Alternating runs and up-down runs

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. We say that π changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$, or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in \{2,3,\ldots,n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. The *up-down runs* of a permutation π are the alternating runs of π endowed with a 0 in the front. Let $R(n,k)$ (resp. $a_k(n)$) be the number of permutations of $\mathfrak{S}_n$ with k alternating runs (resp. up-down runs). For $n,k \geqslant 1$, the numbers $R(n,k)$ and $a_k(n)$ respectively satisfy the recurrence relations

$$R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2),$$

$$a_k(n) = ka_k(n-1) + a_{k-1}(n-1) + (n-k+1)a_{k-2}(n-1),$$

where $R(1,0) = a_0(0) = a_1(1) = 1$ and $R(1,k) = a_0(n) = a_k(0) = 0$ for $n,k \geqslant 1$ (see [15,19]).

As in the proof of Theorem 4.1, it is a routine exercise to show the following result.

Theorem 4.2. *For $n \geqslant 1$, we have*

(i) $\sum_{i,j\geqslant 0} t_{n,i,j} = \sum_{i,j\geqslant 0} r_{n-1,i,j} = n!$.

(ii) $\sum_{i\geqslant 0} t_{n,i,j} = a_{n-j}(n)$.

(iii) $\sum_{j\geqslant 0} t_{n,i,j} = \widetilde{P}_{n,\lfloor n/2\rfloor-i}$.

(iv) $\sum_{i\geqslant 0} r_{n,i,j} = R(n+1,n-j)$.

(v) $\sum_{j\geqslant 0} r_{n,i,j} = P_{n+1,\lfloor n/2\rfloor-i}$.

For convenience, we list the tables of the values of $t_{n,i,j}$ and $r_{n,i,j}$ for $1 \leqslant n \leqslant 4$.

$t_{1,i,j}$	$j=0$
$i=0$	1

$t_{2,i,j}$	$j=0$	$j=1$
$i=0$	0	1
$i=1$	1	0

$t_{3,i,j}$	$j=0$	$j=1$	$j=2$
$i=0$	1	3	1
$i=1$	1	0	0

$t_{4,i,j}$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$	0	1	3	1
$i=1$	4	10	4	0
$i=2$	1	0	0	0

$r_{1,i,j}$	$j=0$
$i=0$	2

$r_{2,i,j}$	$j=0$	$j=1$
$i=0$	0	2
$i=1$	4	0

$r_{3,i,j}$	$j=0$	$j=1$	$j=2$
$i=0$	2	12	2
$i=1$	8	0	0

$r_{4,i,j}$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$	0	2	12	2
$i=1$	16	56	16	0
$i=2$	16	0	0	0

Define

$$\operatorname{sn}(x,k) = \sum_{n\geqslant 0}(-1)^n J_{2n+1}(k^2)\frac{x^{2n+1}}{(2n+1)!},$$

$$\operatorname{cn}(x,k) = 1+\sum_{n\geqslant 0}(-1)^n J_{2n}(k^2)\frac{x^{2n}}{(2n)!}.$$

Note that

$$J_n(k^2)=\sum_{0\leqslant 2i\leqslant n-1} J_{n,2i}k^{2i}.$$

Dumont [4, Corollary 1] found that $s_{2n,i,0}=J_{2n,2i}$ and $s_{2n+1,i,0}=J_{2n+2,2i}$. By comparing (1.4) with (3.16) and (3.17), we immediately get the following result.

Theorem 4.3. *For $n\geqslant 1$, we have $J_{n,2i}=t_{n,\lfloor n/2\rfloor-i,0}$.*

It follows from *Leibniz's formula* that

$$\begin{aligned} D^{2n+1}(w) &= D^{2n}(wx)\\ &= \sum_{k\geqslant 0}\binom{2n}{2k}D^{2k}(w)D^{2n-2k}(x)+\sum_{k\geqslant 0}\binom{2n}{2k+1}D^{2k+1}(w)D^{2n-2k-1}(x), \end{aligned}$$

and similarly,

$$\begin{aligned} D^{2n+2}(w) &= D^{2n+1}(wx)\\ &= \sum_{k\geqslant 0}\binom{2n+1}{2k}D^{2k}(w)D^{2n+1-2k}(x)+\sum_{k\geqslant 0}\binom{2n+1}{2k+1}D^{2k+1}(w)D^{2n-2k}(x). \end{aligned}$$

Therefore, combining (1.3), we get

$$t_{2n+1,i,0}=\sum_{k\geqslant 0}\binom{2n}{2k}\sum_{j=0}^{i} t_{2k,j,0}s_{2n-2k,i-j,0},$$

$$t_{2n+2,i+1,0}=\sum_{k\geqslant 0}\binom{2n+1}{2k+1}\sum_{j=0}^{i} t_{2k+1,j,0}s_{2n-2k,i-j,0}.$$

Thus, as a corollary of Theorem 4.3, we get the following.

Corollary 4.4 ([20, eq. (20)])**.** *For $n\geqslant 0$, we have*

$$J_{2n+1,2n-2i}=\sum_{k\geqslant 0}\binom{2n}{2k}\sum_{j=0}^{i} J_{2k,2k-2j}J_{2n-2k,2i-2j},$$

$$J_{2n+2,2n-2i}=\sum_{k\geqslant 0}\binom{2n+1}{2k+1}\sum_{j=0}^{i} J_{2k+1,2k-2j}J_{2n-2k,2i-2j}.$$

Let $s_{n,i,j}$ be the numbers defined by (1.3). Set $\widetilde{s}_{n,i,j} = s_{n,j,i}$, i.e.,

$$\widetilde{s}_{n,i,j} = |\{\pi \in \mathfrak{S}_n : X(\pi) = j, Y(\pi) = i\}|,$$

where $X(\pi)$ (resp., $Y(\pi)$) is the number of odd (resp., even) cycle peaks of π. Based on empirical evidence, we conjecture that

$$\begin{aligned} \widetilde{s}_{2n+1,i,0} &= t_{2n+1,i,0}, \\ \widetilde{s}_{2n+1,i,j} &= t_{2n+1,i,2j-1} + t_{2n+1,i,2j} \quad \text{for } j \geqslant 1, \\ \widetilde{s}_{2n,i,j} &= t_{2n,i,2j} + t_{2n,i,2j+1} \quad \text{for } j \geqslant 0. \end{aligned}$$

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