Catalan-like number sequences and Hausdorff moment sequences

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ABSTRACT

Many well-known Catalan-like sequences turn out to be Stieltjes moment sequences (Liang et al. (2016)). However, a Stieltjes moment sequence is in general not determinate; Liang et al. suggested a further analysis about whether these moment sequences are determinate and how to obtain the associated measures. In this paper we find necessary conditions for a Catalan-like sequence to be a Hausdorff moment sequence. As a consequence, we will see that many well-known counting coefficients, including the Catalan numbers, the Motzkin numbers, the central binomial coefficients, the central Delannoy numbers, are Hausdorff moment sequences. We can also identify the smallest interval including the support of the unique representing measure. Since Hausdorff moment sequences are determinate and are representing measure for above mentioned sequences are already known, we could almost complete the analysis raised by Liang et al. In addition, subsequences of Catalan-like number sequences are also considered; we will see a necessary and sufficient condition for subsequences of Stieltjes Catalan-like number sequences to be Stieltjes Catalan-like number sequences. We will also study representing measure for a linear combination of consecutive terms in Catalan-like number sequences.

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1. Introduction

Let \( \mathbb{N}_0 \) (resp., \( \mathbb{R} \)) be the set of nonnegative integers (resp., real numbers). Let \( K \) be a closed subset of \( \mathbb{R} \). A nonnegative Borel measure \( \mu \) on \( \mathbb{R} \) is called a \( K \)-measure if its support, denoted by \( \text{supp}(\mu) \), is contained in \( K \). The symbol \( \mathbb{R}[x] \) denotes the ring of polynomials in \( x \) with real coefficients. The integral \( \int_K x^n d\mu \), if it exists, is called the \( n \)-th moment of the measure \( \mu \). A sequence \( y = (y_n)_{n \geq 0} \) is said to admit a \( K \)-measure \( \mu \) if

\[
y_n = \int_K x^n d\mu \quad \text{for all } n \in \mathbb{N}_0.
\]

Such a \( \mu \) is called a \( K \)-representing measure for \( y \) and \( y \) is called a \( K \)-moment sequence. When \( K = \mathbb{R} \) (resp. \( K = [0, \infty) \), \( K = [a, b] \)), the sequence \( y \) is also called a Hamburger (resp. Stieltjes, Hausdorff) moment sequence. A \( K \)-moment sequence is said to be determinate, if there is a unique representing measure such that (1.1) holds; otherwise it is said to be indeterminate. For more information, see references [3,11,22,23] and references therein.

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Given a sequence \( y = (y_n)_{n \geq 0} \), we define

\[
H_m(y) := \begin{bmatrix}
y_0 & y_1 & \cdots & y_m \\
y_1 & y_2 & \cdots & y_{m+1} \\
\vdots & \vdots & \ddots & \vdots \\
y_m & y_{m+1} & \cdots & y_{2m}
\end{bmatrix}, \quad \tilde{H}_m(y) := \begin{bmatrix}
y_1 & y_2 & \cdots & y_{m+1} \\
y_2 & y_3 & \cdots & y_{m+2} \\
\vdots & \vdots & \ddots & \vdots \\
y_{m+1} & y_{m+2} & \cdots & y_{2m+1}
\end{bmatrix}.
\]

Denote \( \Delta_m(y) := \det(H_m(y)) \) for all \( m \in \mathbb{N}_0 \). It is well known that \( y \) is a Hamburger moment sequence if and only if \( H_m(y) \) is positive semidefinite for all \( m \in \mathbb{N}_0 \). Also, \( y \) is a Stieltjes moment sequence if and only if both \( H_m(y) \) and \( \tilde{H}_m(y) \) are positive semidefinite for all \( m \in \mathbb{N}_0 \); equivalently, the matrix \( H_m(y) \) is totally positive for all \( m \in \mathbb{N}_0 \), that is, all the minors of \( H_m(y) \) are positive.

Aigner introduced the Catalan-like numbers (which will be called Catalan-like number sequences as below) and many well-known combinatorial sequences turn out to be in the class of the Catalan-like numbers [1, 2]. The class includes the Riordan numbers, the Fine numbers, the Motzkin numbers, the Schröder numbers, and so on. For a formal definition, let \( \sigma = (s_k)_{k \geq 0} \) and \( \tau = (t_k)_{k \geq 1} \) be two sequences of real numbers with \( t_{k+1} \neq 0 \) for all \( k \in \mathbb{N}_0 \) and define an infinite lower triangle matrix \( R := R^{\sigma, \tau} = [r_{n,k}]_{n,k \geq 0} \) by the recurrence relations

\[
r_{0,0} = 1, \quad r_{n+1,k} = r_{n,k-1} + s_k r_{n,k} + t_{k+1} r_{n,k+1},
\]

where \( r_{n,k} = 0 \) unless \( n \geq k \geq 0 \). Clearly, \( r_{n,n} = 1 \) for all \( n \in \mathbb{N}_0 \). The matrix \( R \) is called a recursive matrix and \( r_n \) forms a Catalan-like number sequence corresponding to \( (\sigma, \tau) \).

**Example 1.1.** The following well-known counting coefficients are Catalan-like number sequences.

1. The Catalan numbers \( C_n \) with \( \sigma = (1, 2, 2, \ldots) \) and \( \tau = (1, 1, 1, \ldots) \).
2. The shifted Catalan numbers \( C_{n+1} \) with \( \sigma = (2, 2, 2, \ldots) \) and \( \tau = (1, 1, 1, \ldots) \).
3. The central binomial coefficients \( \binom{2n}{n} \) with \( \sigma = (2, 2, 2, \ldots) \) and \( \tau = (2, 1, 1, \ldots) \).
4. The Motzkin numbers \( M_n \) with \( \sigma = \tau = (1, 1, 1, \ldots) \).
5. The central trinomial coefficients \( T_n \) with \( \sigma = (1, 1, 1, \ldots) \) and \( \tau = (2, 1, 1, \ldots) \).
6. The Riordan numbers \( R_n \) with \( \sigma = (0, 1, 1, 1, \ldots) \) and \( \tau = (1, 1, 1, \ldots) \).
7. The central Delannoy numbers \( D_n \) with \( \sigma = (3, 3, 3, \ldots) \) and \( \tau = (4, 2, 2, \ldots) \).
8. The large Schröder numbers \( r_n \) with \( \sigma = (2, 3, 3, \ldots) \) and \( \tau = (2, 2, 2, \ldots) \).
9. The (restricted) hexagonal numbers \( h_n \) with \( \sigma = (3, 3, 3, \ldots) \) and \( \tau = (1, 1, 1, \ldots) \).

Liang et al. showed that these types of sequences are Stieltjes moment sequences if \( s_0 \geq 1 \) and \( s_k \geq t_k + 1 \) for \( k \geq 1 \) [13]. Wang and Zhu showed that Stieltjes moment sequences are infinitely log-convex, which is an important notion in combinatorics [25]. Chen et al. presented some sufficient conditions such that the recursive matrix is totally positive; equivalently, the corresponding sequence becomes a Stieltjes moment sequence [8]. They also proved that many well-known sequences are log-convex [9]. For more recent work, see [14, 24].

In [13], Liang et al. remarked that one could ask if many well-known sequences are determinate (Recall that Hausdorff moment sequences are determinate.) Checking whether a given moment sequence is determinate is an interesting problem, but it is usually quite difficult; hence, you need to understand some useful criteria in advance. The reader is referred to [22, Chapter 4] for a comprehensive study and [15] for recent results. Considering this question, our main result identifies a class of Catalan-like number sequences that become Hausdorff moment sequences:

**Theorem 1.1.** If a Catalan-like number sequence corresponding to \( \sigma = (p, s, s, \ldots) \) and \( \tau = (q, t, t, \ldots) \) satisfies \( q > 0 \), \( t > 0 \), \( q \leq \sqrt{t}(p - s + 2 \sqrt{t}) \), \( q \leq -\sqrt{t}(p - s - 2 \sqrt{t}) \), and \( \max(q, t) < s + 2 \sqrt{t} \), then it is a \([s - 2 \sqrt{t}, s + 2 \sqrt{t}]\)-moment sequence.

Many well-known Catalan-like number sequences correspond to the sequences in the form of \( \sigma = (p, s, s, \ldots) \) and \( \tau = (q, t, t, \ldots) \). In order for a sequence to have a representing measure, a numerical condition such as in the hypothesis of Theorem 1.1 appears often in the theory of the moment problem. This result shows that all sequences in Example 1.1 are indeed Hausdorff moment sequences, implying that each has a unique representing measure. Specifically, the following can be shown:

1. The Catalan numbers \( C_n \), the shifted Catalan numbers \( C_{n+1} \), and the central binomial coefficients \( \binom{2n}{n} \) are \([0, 4]\)-moment sequences.
2. The Motzkin numbers \( M_n \), the central trinomial coefficients \( T_n \), and the Riordan numbers \( R_n \) are \([-1, 3]\)-moment sequences.
3. The central Delannoy numbers \( D_n \) and the large Schröder numbers \( r_n \) are \([3 - 2 \sqrt{2}, 3 + 2 \sqrt{2}]\)-moment sequences.
4. The (restricted) hexagonal numbers \( h_n \) are \([1, 5]\)-moment sequences.
Using Theorem 1.1, we can find a necessary condition for an extended class derived from sequences in (1)-(4) to be a Hausdorff moment sequence in Corollary 3.1. For such a necessary condition it is required to find the closed interval including the support.

Finding integral representations of well-known counting numbers has been studied in [19] and reference therein. If an interval containing the support of a representing measure for a sequence is discovered, one get help in search for an integral representation.

Next new sequences derived from Catalan-like number sequences are to be investigated. We mainly consider two different types of new sequences: (a) Subsequences of Catalan-like number sequences; (b) Linear combinations of terms in Catalan-like number sequences.

Explicitly, it will be shown that a necessary and sufficient condition on \( n_k \) such that the subsequence \( (y_{n_k})_{k \geq 0} \) is a Stieltjes moment sequence for a Stieltjes moment sequence \( (y_n)_{n \geq 0} \) is

\[
n_k = dk + \ell \quad \text{for all } k \in \mathbb{N}_0,
\]

where \( d, \ell \in \mathbb{N}_0 \).

On the other hand, Bouras considered a linear combination of three successive shifted Catalan numbers defined as

\[
z_n = \alpha_0 C_{n+k} + \alpha_1 C_{n+k+1} + \alpha_2 C_{n+k+2} \quad \text{for all } n \in \mathbb{N}_0,
\]

where \( \alpha_0, \alpha_1, \alpha_2, \in \mathbb{R} \) and \( k \) is an arbitrary positive integer. It was shown that such a linear combination can be expressed in terms of the moments of a linear functional related to a Jacobi linear functional; this result is based on the well-known relation between orthogonal polynomials and Hankel determinants [7].

Mu et al. analyzed sequences of this type in a more general setting [17]. They unified many known results of Hankel determinant evaluations for well-known counting numbers and presented an explicit form of the determinant of a linear combination of consecutive terms in Catalan-like number sequences. We focus on the existence of representing measures for the new sequences instead of their Hankel determinants. For a Catalan-like number sequence \( r = (r_n)_{n \geq 0} \), we consider a new sequence \( \tilde{r} = (\tilde{r}_n)_{n \geq 0} \) defined by

\[
\tilde{r}_n = \alpha_0 r_{dn+i} + \alpha_1 r_{dn+i+1} + \alpha_2 r_{dn+i+2} + \cdots + \alpha_m r_{dn+i+m},
\]

where \( \alpha_i \in \mathbb{R} \) is given for all \( 0 \leq i \leq m \) and \( d, \ell \in \mathbb{N}_0 \). To study this type of a new sequence, as before, we will be in search for the smallest closed interval including the support of a representing measure for each Catalan-like number sequence.

2. Preliminaries

In this section we introduce powerful tools for the study of one-dimensional moment problems: the Riesz functional and orthogonal polynomials. Readers are referred to the references [10,22] for a deeper treatment of the results.

For a sequence, \( y = (y_n)_{n \geq 0} \), define a Riesz functional \( \mathcal{L}_y \) acting on \( \mathbb{R}[x] \) as

\[
\mathcal{L}_y \left[ \sum c_n x^n \right] = \sum c_n y_n.
\]

We write simply \( \mathcal{L} \) instead of \( \mathcal{L}_y \) when it is understood well.

We say that \( \mathcal{L} \) is \( K \)-positive if

\[
\mathcal{L}(p) \geq 0 \quad \forall p \in \mathbb{R}[x] : p|_K \geq 0.
\]

(2.1)

If the conditions, \( p|_K \geq 0 \) and \( p|_K \not\equiv 0 \), imply \( \mathcal{L}(p) > 0 \), then \( \mathcal{L} \) is said to be strictly \( K \)-positive. When \( K = \mathbb{R} \), we use the term positive instead of \( K \)-positive. The \( K \)-positivity of \( \mathcal{L}_y \) is a necessary condition for \( y \) to admit a \( K \)-representing measure. Conversely, the classical theorem of M. Riesz says the \( K \)-positivity is also sufficient for the existence of \( K \)-measures and Haviland generalized the result in \( \mathbb{R}^n \) [12,21].

Theorem 2.1 (Riesz–Haviland’s Theorem). A sequence \( y = (y_n)_{n \geq 0} \) admits a representing measure supported in the closed set \( K \subset \mathbb{R} \) if and only if \( \mathcal{L}_y \) is \( K \)-positive.

From now on, we will collect well-known results about the Riesz functional and orthogonal polynomials, and will see the role of positivity of a sequence. A sequence \( (P_n(x))_{n \geq 0} \) is called an orthogonal polynomial sequence (in short, OPS) with respect to \( \mathcal{L} \) if it satisfies that

\[
\deg(P_n) = n \quad \text{and} \quad \mathcal{L}[P_m P_n] = K_m \delta_{mn} \quad (K_n \not\equiv 0) \quad \text{for all } m, n \in \mathbb{N}_0.
\]

When \( K_n = 1 \) for all \( n \in \mathbb{N}_0 \), such an OPS is called an orthonormal polynomial sequence. There exists an explicit formula for the orthogonal polynomial sequences (see [22, Proposition 5.3]).
Theorem 2.2. For a sequence \( y = (y_n)_{n \geq 0} \), let \( L_y \) be the Riesz functional of \( y \). Then the monic OPS for \( L_y \) is expressed as

\[
P_n(x) = \frac{1}{\Delta_{n-1}(y)} \det \begin{bmatrix} y_0 & y_1 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_{n+1} \\ \vdots & \vdots & & \vdots \\ y_{n-1} & y_n & \cdots & y_{2n-1} \\ 1 & x & \cdots & x^n \end{bmatrix},
\]

provided that \( \Delta_n(y) \neq 0 \) for all \( n \in \mathbb{N}_0 \).

The condition, \( \Delta_n(y) \neq 0 \) for all \( n \in \mathbb{N}_0 \), is a necessary and sufficient condition for the existence of an OPS for \( L_y \); such an \( L_y \) is called quasi-definite and its OPS has a 3-term recurrence relation as follows:

\[
L \phi_{n+1}(x) = (x - s_k) \phi_n(x) - t_k \phi_{n-1}(x) \quad \text{for all} \quad k \in \mathbb{N}_0,
\]

where we assume \( P_{-1}(x) = 0 \) and \( t_0 \) is arbitrary. Furthermore, for each \( k \in \mathbb{N}_0 \)

\[
s_k = \frac{\mathcal{L}_y[xP_k^2(x)]}{\mathcal{L}[xP_k(x)]} \quad \text{and} \quad t_{k+1} = \frac{\mathcal{L}_y[P_{k+1}^2(x)]}{\mathcal{L}_y[P_k^2(x)]} = \frac{\Delta_{k-1}(y) \Delta_{k+1}(y)}{(\Delta_k(y))^2},
\]

where \( \Delta_k(y) \equiv 1 \). Moreover, if \( L_y \) is strictly positive, then \( s_k \in \mathbb{R} \) and \( t_{k+1} > 0 \) for all \( k \in \mathbb{N}_0 \).

The converse of the proceeding result is referred to as the Favard’s Theorem:

Theorem 2.4 ([10], Theorem 4.4, Chapter 1 [Favard’s Theorem]). Let \( \sigma = (s_k)_{k \geq 0} \) and \( \tau = (t_k)_{k \geq 0} \) be arbitrary sequences of complex numbers and let \( (P_n(x))_{n \geq 0} \) be defined by the recurrence formula

\[
P_{-1}(x) = 0, \quad P_0 = 1, \quad P_{k+1}(x) = (x - s_k)P_k(x) - t_k P_{k-1}(x) \quad \text{for all} \quad k \in \mathbb{N}_0,
\]

Then, there exists a unique Riesz functional \( \mathcal{L} \) such that

\[
\mathcal{L}[1] = t_0, \quad \mathcal{L}[P_m(x)P_n(x)] = 0 \quad \text{for all} \quad m, n \in \mathbb{N}_0 \quad \text{with} \quad m \neq n.
\]

Moreover, \( \mathcal{L} \) is quasi-definite and \( (P_n(x))_{n \geq 0} \) is the corresponding monic OPS if and only if \( t_{k+1} \neq 0 \) for all \( k \in \mathbb{N}_0 \). In addition, \( \mathcal{L} \) is strictly positive if and only if \( s_k \in \mathbb{R} \) and \( t_{k+1} > 0 \) for all \( k \in \mathbb{N}_0 \).

Aigner discovered an interesting connection between recursive matrices and coefficients of their monic OPS:

Theorem 2.5 ([2]). Let \( y = (y_n)_{n \geq 0} \) be a sequence. Then the following are equivalent:

(i) \( y \) is a Catalan-like number sequence corresponding to \( (\sigma, \tau) \);

(ii) \( \Delta_0(y) \neq 0 \) for all \( m \in \mathbb{N}_0 \);

(iii) There exists a recursive matrix \( R_{\sigma, \tau} = [r_{n,k}]_{n, k \geq 0} \) such that \( r_{n,0} = y_n \) for all \( n \in \mathbb{N}_0 \);

(iv) There exists a monic OPS, \( (P_n(x))_{n \geq 0} \), with respect to \( \mathcal{L}_y \);

(v) There exist sequences \( \sigma = (s_k)_{k \geq 0} \) and \( \tau = (t_k)_{k \geq 1} \) with \( t_k \neq 0 \) for all \( k \in \mathbb{N}_0 \) satisfying (2.2);

(vi) There exists a quasi-definite Riesz functional \( \mathcal{L}_y \).

If a sequence \( y \) is said to be positive (resp. strictly positive) if the Hankel matrix \( H_m(y) \) is positive semidefinite (resp. positive definite) for all \( n \in \mathbb{N}_0 \). Observe that if a sequence is positive but not strictly, then the Hankel matrix \( H_m(y) \) is positive semidefinite for all \( n \in \mathbb{N}_0 \) and at least one of them is singular. In fact, once one of the finite Hankel matrix is singular, all the following ones are also singular. The strict positivity of a sequence leads to an infinite support of a representing measure as in the next result:

Theorem 2.6 ([26], Theorem 12a). A necessary and sufficient condition that there exists a nonnegative Borel measure \( \mu \) with \( |\text{supp}(\mu)| = \infty \) (resp. \( |\text{supp}(\mu)| < \infty \)) such that

\[
y_k = \int_{\mathbb{R}} x^k d\mu \quad \text{for all} \quad k \in \mathbb{N}_0
\]

is that the sequence \( y \) is strictly positive (resp. positive but not strictly).
Our main concern is about the existence of a representing measure for a Catalan-like number sequence; combining the results so far, we can easily prove the following theorem:

**Theorem 2.7.** Let \( y = (y_n)_{n \geq 0} \) be a sequence. Then the following are equivalent:

(i) \( y \) is a strictly positive Catalan-like number sequence;
(ii) \( \Delta_m(y) > 0 \) for all \( m \in \mathbb{N}_0 \);
(iii) \( y \) admits a \( K \)-representing measure \( \mu \) with \( K \subseteq \mathbb{R} \) such that \( \text{supp}(\mu) = \infty \);
(iv) There exists a recursive matrix \( R^{\sigma, \tau} = [r_{n,k}]_{n,k \geq 0} \) such that \( r_{n,0} = y_n \) and \( t_k > 0 \) for all \( n, k \);
(v) There exists a monic \( \text{OPS} \), \( \{P_n(x)\}_{n \geq 0} \), with respect to a strictly positive \( \mathcal{L}_y \);
(vi) There exist sequences \( \sigma = (s_n)_{n \geq 0} \) and \( \tau = (t_k)_{k \geq 1} \) with \( t_k > 0 \) satisfying (1.3);
(vii) There exist sequences \( \sigma = (s_k)_{k \geq 0} \) and \( \tau = (t_k)_{k \geq 1} \) with \( t_k > 0 \) satisfying (2.2);
(viii) There exists a strictly positive Riesz functional \( \mathcal{L}_y \) satisfying (2.5).

**Proof.** Note that \( y \) is the Catalan-like number sequence if and only if \( \Delta_m(y) \neq 0 \) for all \( m \in \mathbb{N}_0 \). By the definition of a positive sequence, (i) and (ii) are equivalent. By Theorem 2.6, (i) and (iii) are equivalent. Applying Theorem 2.5, it is easy to check the remainder. \( \square \)

When the Riesz functional of a sequence is positive, there is an intimate relationship between the zeros of the corresponding orthogonal polynomials and the support of a representing measure for the sequence. We maintain the hypothesis that the Riesz functional \( \mathcal{L}_y \) is positive with a representing measure \( \mu \). For proofs of each theorem and proposition, see [10] and references therein.

We now need to define notations that will be used in the sequel; let us denote the zeros of \( P_n(x) \) by \( x_{n,1} \) with

\[
x_{n,1} < x_{n,2} < \cdots < x_{n,n}.
\]

We denote

\[
U = \{\xi_i | i = 1, 2, 3, \ldots\}, \quad V = \{\eta_j | j = 1, 2, 3, \ldots\},
\]

where

\[
\xi_i = \lim_{n \to \infty} x_{n,i}, \quad \eta_j = \lim_{n \to \infty} x_{n,n-j+1} \text{ for all } i, j.
\]

Since \( \xi_{i-1} \leq \xi_i \leq \eta_{j-1} \), we define

\[
\xi = \begin{cases} 
-\infty, & \text{if } \xi_i = -\infty \text{ for all } i; \\
\lim_{n \to \infty} \xi_i, & \text{otherwise}.
\end{cases}
\]

\[
\eta = \begin{cases} 
+\infty, & \text{if } \eta_j = +\infty \text{ for all } j; \\
\lim_{n \to \infty} \eta_j, & \text{otherwise}.
\end{cases}
\]

Taking \( \xi_0 = -\infty \) and \( \eta_0 = +\infty \), we have

\[
-\infty = \xi_0 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \xi \leq \eta \leq \cdots \leq \eta_2 \leq \eta_1 \leq \eta_0 = +\infty.
\]

Note that when \( \xi \) and \( \eta \) are both finite, it is known that \( \text{supp}(\mu) = U \cup S \cup V \), where \( S \) is a set contained in \( [\xi, \eta] \). For more information, see [10, p. 63].

**Definition 2.1.** The closed interval, \( [\xi_1, \eta_1] \), is called the true interval of orthogonality of the \( \text{OPS} \) (of \( \mathcal{L}_y \)).

The true interval of orthogonality is the smallest closed interval that contains all of the zeros of all \( P_n(x) \). Obviously, the true interval of orthogonality is the smallest closed interval that includes \( \text{supp}(\mu) \) (see [10, Theorem 3.2]).

We pose to introduce a notation that is of great utility in the analysis of the 3-term recurrence relation of an \( \text{OPS} \):

**Definition 2.2.** A sequence \( y = (y_n)_{n \geq 0} \) is called a chain sequence if there exists a sequence \( (g_k)_{k \geq 0} \) such that

(i) \( 0 \leq g_0 < 1 \);
(ii) \( 0 < g_{n+1} < 1 \) for all \( n \in \mathbb{N}_0 \);
(iii) \( y_n = (1 - g_n)g_{n+1} \) for all \( n \in \mathbb{N}_0 \).

Such a sequence \( (g_k)_{k \geq 0} \) is called a parameter sequence for \( y \), and \( g_0 \) is called an initial parameter.

The theory of chain sequences can explain a connection between the true interval of orthogonality, \( [\xi_1, \eta_1] \), and the sequences \( \sigma = (s_k)_{k \geq 0} \) and \( \tau = (t_k)_{k \geq 1} \) in (2.2). To introduce coming results, we define a notation

\[
\alpha_n(x) := \frac{t_{n+1}}{(s_n - x)(s_{n+1} - x)} \quad \text{for all } n \in \mathbb{N}_0.
\]
**Theorem 2.8** ([10]). Assume that the sequences \( \sigma = (s_k)_{k \geq 0} \) and \( \tau = (t_k)_{k \geq 1} \) are as in (2.2) and that \( \xi_1, \eta_1, \) and \( \alpha(x) \) are as above. Then the following hold:

(i) \( \xi_1 \geq a \) if and only if \( s_k > a \) for all \( k \in \mathbb{N}_0 \) and \( (\alpha_n(a))_{n \geq 0} \) is a chain sequence.

(ii) \( \eta_1 \leq b \) if and only if \( t_k < b \) for all \( k \in \mathbb{N}_0 \) and \( (\alpha_n(b))_{n \geq 0} \) is a chain sequence.

Even though the proof of the proceeding theorem is omitted, the readers can refer to [10, Chapter IV]. We will use Theorem 2.8 to prove Theorem 3.1, which is one of our main results.

3. Catalan-like number sequences with a unique representing measure

As a main result, we give an affirmative answer to the question of [13], we will show that many well-known Catalan-like number sequences are Hausdorff moment sequences.

We denote by \( y(p, s; q, t) \) the Catalan-like number sequence corresponding to \( \sigma = (p, s, s, \ldots) \) and \( \tau = (q, t, t, \ldots). \)

Remark that many well-known Catalan-like number sequences have such corresponding sequences. For readers’ convenience, we restate Theorem 1.1 with more details:

**Theorem 3.1.** If a Catalan-like number sequence \( y \equiv y(p, s; q, t) \) satisfies

(i) \( q > 0, \ t > 0, \)

(ii) \( q \leq \sqrt{t}(p-s+2\sqrt{t}), \ q \leq -\sqrt{t}(p-s-2\sqrt{t}), \)

(iii) \( \max(q, t) < s + 2\sqrt{t}, \)

then it is a Hausdorff moment sequence such that the support of the representing measure is contained in \([s - 2\sqrt{t}, s + 2\sqrt{t}]. \)

In fact, \([s - 2\sqrt{t}, s + 2\sqrt{t}] \) is the smallest interval including the support. Furthermore, if \( y \) additionally holds the condition, \( s \geq 2\sqrt{t} \), then it is also Stieltjes moment sequence as well.

**Proof.** It follows from Theorems 2.4 and 2.7 (vi) that \( y \) admits a representing measure \( \mu. \) Let \([\xi_1, \eta_1]\) be the true interval of orthogonality of the Riesz functional \( \mathcal{L}_y \) which is defined as above. Now note that

\[
\alpha_n(x) = \begin{cases} 
\frac{q}{(p-x)(s-x)} - (1-g_0)g_1, & \text{if } n = 0; \\
\frac{t}{(s-x)^2} - (1-g_0)g_{n+1}, & \text{if } n \geq 1.
\end{cases}
\]

To verify \( \xi_1 \geq s - 2\sqrt{t} \) and \( \eta_1 \leq s + 2\sqrt{t} \) (that is, \( \text{supp}(\mu) \subseteq [\xi_1, \eta_1] \)), it is enough to show that \( \alpha_n(s - 2\sqrt{t}) \) and \( \alpha_n(s + 2\sqrt{t}) \) are chain sequences by Theorem 2.8.

First, let \((g_n)_{n \geq 0}\) be a sequence given as

\[
1 - \frac{q}{\sqrt{t}(p-s+2\sqrt{t})} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}, \ldots
\]

By the condition (ii), it is true that \( 0 \leq g_0 < 1 \) and \( 0 < g_{n+1} < 1 \) for all \( n \in \mathbb{N}_0. \) Since \( \alpha_n(s - 2\sqrt{t}) = (1 - g_n)g_{n+1} \) for all \( n \in \mathbb{N}_0, \) \( \alpha_n(s - 2\sqrt{t}) \) is a chain sequence with the parameter sequence \((g_n)_{n \geq 0}\).

Similarly, let \((h_n)_{n \geq 0}\) be a sequence given as

\[
1 + \frac{q}{\sqrt{t}(p-s-2\sqrt{t})} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}, \ldots
\]

By the conditions (i) and (ii), it is true that \( 0 \leq h_0 < 1 \) and \( 0 < h_{n+1} < 1 \) for all \( n \in \mathbb{N}_0. \) Since \( \alpha_n(s + 2\sqrt{t}) = (1 - h_n)h_{n+1} \)

for all \( n \in \mathbb{N}_0, \) \( \alpha_n(s + 2\sqrt{t}) \) is a chain sequence with the chain sequence \((h_n)_{n \geq 0}. \)

Next using the fact in [10, p. 121], we have \( \xi = s - 2\sqrt{t} \) and \( \eta = s + 2\sqrt{t}. \) Thus, \( s - 2\sqrt{t} \leq \xi_1 \leq \xi = s - 2\sqrt{t} \) and \( s + 2\sqrt{t} = \eta \leq \eta_1 \leq s + 2\sqrt{t}. \)

Finally, the additional condition \( s \geq 2\sqrt{t} \) implies \( s - 2\sqrt{t} \geq 0, \) so \([\xi_1, \eta_1] \subset [0, \infty). \) \( \square \)

**Theorem 3.13** in [22] states that a sequence \( y = (y_n)_{n \geq 0} \) is an \([a, b]\)-moment sequence if and only if \( H_m(y) \) and \( (a+b)H_m(Ey) - H_m(E(y)) - abH_m(y) \) are positive semidefinite for all \( m \in \mathbb{N}_0, \) where \( Ey \) denotes the shifted sequence, that is, \( Ey \equiv E(y_{(n)})_{n \geq 0} = (y_{(n)+1})_{n \geq 0}. \) Combining Theorem 3.1 with this result, we can specify the location of the support of the representing measure for Catalan-like number sequences.

**Corollary 3.1.** The following items can treat the Catalan numbers \( C_n, \) the Motzkin numbers \( M_n, \) the central Delannoy numbers \( D_n, \) and the hexagonal numbers \( h_n \) in order:

(i) When \( y = y_n(p, 2; q, 1) \) satisfies \( 0 < q \leq p \leq 4 - q \) (this inequality forms a triangular region on \((p, q)\)-plane), then it has a representing measure with the support contained in \([0, 4], \) which is equivalent to \( H_m(y) \geq 0 \) and \( H_m(4Ey - E(y)) \geq 0 \) for all \( m \in \mathbb{N}_0. \)
Example 3.1. The Catalan–like number sequences in Example 1.1 are Hausdorff moment sequences.

(1) The Catalan numbers $C_n$, the shifted Catalan numbers $C_{n+1}$, and the central binomial coefficients $\binom{2n}{n}$ have the representing measures with a compact support contained in $[0, 4]$. For example, $C_n$ and $\binom{2n}{n}$ are uniquely represented by

$$C_n = \int_0^4 x^n \left( \frac{1}{2\pi} \frac{\sqrt{4 - x}}{x} \right) dx \quad \text{and} \quad \binom{2n}{n} = \int_0^4 x^n \left( \frac{1}{\pi \sqrt{x(4 - x)}} \right) dx,$$

respectively. For more information about integral representations of the Catalan numbers, see [19].

(2) The Motzkin numbers $M_n$, the central trinomial coefficients $T_n$, and the Riordan numbers $R_n$ have the representing measures with a compact support contained in $[-1, 3]$. For example, $M_n$ and $T_n$ are represented by

$$M_n = \int_{-1}^3 x^n \left( \frac{1}{2\pi} \frac{\sqrt{1 - x}(1 + x)}{x} \right) dx \quad \text{and} \quad T_n = \int_{-1}^3 x^n \left( \frac{1}{\pi \sqrt{(1 - x)(1 + x)}} \right) dx,$$

respectively (see [18]).

(3) The central Delannoy numbers $D_n$ and the large Schröder numbers $r_n$ have the representing measures with a compact support contained in $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$. For example, $D_n$ is represented by

$$D_n = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} x^n \left( \frac{1}{\pi} \frac{1}{\sqrt{(3 + 2\sqrt{2} - x)(x - 3 + 2\sqrt{2})}} \right) dx.$$

This integral representations can be obtained from an integral form in [20, Theorem 1.3].

(4) The (restricted) hexagonal numbers $h_n$ have the representing measures with a compact support contained in $[1, 5]$. Many well-known combinatorial numbers can be expressed as integrals with $[a, b]$-representing measures. Remark that every Hausdorff moment sequence is determinate. Thus the Catalan–like number sequences in Example 1.1 are determinate, which means that for each sequence there is a unique measure, respectively. In [6], Berg and Szwarc proved that if a sequence $y$ satisfies $\Delta_m(y) > 0$ for $m < s$ while $\Delta_m(y) = 0$ for $m \geq s$, then all Hankel matrices are positive semidefinite, and in particular, $y$ is a Hamburger moment sequence with a discrete measure $\mu$ such that $\|\text{supp}(\mu)\| = s$. Note that by Theorem 2.5, it holds $\Delta_m(y) \neq 0$, $m \geq 0$ for any Catalan-like number sequence. Thus the supports of representing measures are not finite, which means that the representing measures for Catalan-like number sequences cannot be expressed as a combination of simple discrete measures. Although we do not know how to obtain the representing measures in general, Mnatsakanov [16] provided an approximation of the representing measure for a given Hausdorff moment sequence; to find such an approximation, we should check if a given sequence is a Hausdorff moment sequence and need to find an interval including the support. Our result will be helpful for the purpose.

4. New sequences derived from Catalan-like number sequences

4.1. Subsequences of Catalan-like number sequences

Example 4.1. Consider the following sequence $y = (y_n)_{n \geq 0}$.

1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, 0, 429, 0, ... .

It is easy to check that $(y_{2n})_{n \geq 0}$ is the Catalan–like number sequence corresponding to $(\sigma, \tau)$ with $\sigma = (0, 0, 0, \ldots)$ and $\tau = (1, 1, \ldots)$. Note that since $\det(H_m(y)) > 0$ for all $m \in \mathbb{N}_0$ and $\det(H_2(y)) = -1$, it follows that $(y_n)_{n \geq 0}$ is a Hamburger moment sequence but not a Stieltjes moment sequence.

This example motivates us to study subsequences of Catalan-like number sequences. In [4], Arya et al. showed that for a given Hamburger moment sequence $(y_n)_{n \geq 0}$, its subsequence $(y_{nk})_{k \geq 0}$ $(n_k = dk + \ell, d \in \mathbb{N}_0, \ell \in 2\mathbb{N}_0)$ is always
a Hamburger moment sequence. They also discovered a relationship between a Hamburger moment sequence and its subsequences obtained via a Cauchy transform. We may have a similar result for multisequences, see [5].

In the sequel we consider subsequences of Catalan-like number sequences which admit $K$-measures when $K = \mathbb{R}$, $K = [0, \infty)$, or $K = [a, b]$. By a subsequence of a sequence $(y_n)_{n \geq 0}$, we shall mean a sequence of the form $(y_{n_k})_{k \geq 0}$, where each $n_k \in \mathbb{N}_0$ and $n_0 < n_1 < \cdots$.

Since almost all of well-known Catalan-like number sequences are positive, some of subsequences can be selected to be positive; so, these subsequences in turn become classical $K$-moment sequences for some closed $K \subseteq \mathbb{R}$.

**Theorem 4.1.** Let $K \subseteq \mathbb{R}$ be a closed set. Let $d \in \mathbb{N}_0$ and let

\[
\ell \in \mathbb{N}_0, \quad \text{if } K \subseteq [0, \infty); \\
\ell \in 2\mathbb{N}_0, \quad \text{otherwise.}
\]

If $y = (y_n)_{n \geq 0}$ is a $K$-moment sequence, then the subsequence, $\tilde{y} = (\tilde{y}_k)_{k \geq 0}$, defined by $\tilde{y}_k = y_{dk+\ell}$ for all $k \in \mathbb{N}_0$, is a $\tilde{K}$-moment sequence, where $\tilde{K} = \{x^d | x \in K\}$.

**Proof.** Note that $\tilde{K}$ is a closed set. Since $y$ admits a $K$-representing measure, the Riesz functional $L_y$ of $y$ is $K$-positive. We will show that $L_y(p) \geq 0$ for all $p \in \mathbb{R}[x]$ with $p|_K \geq 0$. Let $p(x) = \sum p_kx^k \in \mathbb{R}[x]$ such that $p|_R \geq 0$. Since $q(x) := p(x^d)x^{\ell} \geq 0$ on $K$, it follows that

\[
L_y(p) = \sum p_k\tilde{y}_k = \sum p_ky_{dk+\ell} = L_y(q) \geq 0.
\]

Thus, $L_y$ is $\tilde{K}$-positive. Therefore, by **Theorem 2.1** $\tilde{y}$ admits a $\tilde{K}$-representing measure. $\Box$

Recall that if $(y_n)_{n \geq 0}$ is an $[a, b]$-moment sequence with $0 \leq a \leq b < +\infty$, then it is clearly both a Stieltjes moment sequence and a Hamburger moment sequence; similarly, if $(y_n)_{n \geq 0}$ is a Stieltjes moment sequence, then it must be a Hamburger moment sequence.

**Corollary 4.1.** For a sequence $y = (y_n)_{n \geq 0}$, a subsequence, $\tilde{y} = (\tilde{y}_k)_{k \geq 0}$, is defined by $\tilde{y}_k = y_{dk+\ell}$ for all $k \in \mathbb{N}_0$.

(i) If $y$ is an $[a, b]$-moment sequence with $0 \leq a \leq b < +\infty$, then $\tilde{y}$ is an $[a, b]$-moment sequence for all $d, \ell \in \mathbb{N}_0$.

(ii) If $y$ is a Stieltjes moment sequence, then $\tilde{y}$ is a Stieltjes moment sequence for all $d, \ell \in \mathbb{N}_0$.

(iii) If $y$ is a Hamburger moment sequence, then $\tilde{y}$ is a Hamburger moment sequence for all $d \in 2\mathbb{N}_0 + 1$ and $\ell \in 2\mathbb{N}_0$.

(iv) If $y$ is a Hamburger moment sequence, then $\tilde{y}$ is a Stieltjes moment sequence for all $d \in 2\mathbb{N}_0$ and $\ell \in 2\mathbb{N}_0$.

(v) If $y$ is a Hamburger moment sequence, then $\tilde{y}$ may or may not be a Hamburger moment sequence for all $d \in \mathbb{N}_0$ and $\ell \in 2\mathbb{N}_0 + 1$.

For an easier understanding, we may restate **Corollary 4.1** as follows: If $y = (y_n)_{n \geq 0}$ is an $(A)$-moment sequence, then the subsequence, $\tilde{y} = (\tilde{y}_n)_{n \geq 0}$, is a $(B)$-moment sequence with respect to the parity of $d$ and $\ell$ as in the following table

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\ell$</th>
<th>$(A)$</th>
<th>$(B)$</th>
<th>$d$</th>
<th>$\ell$</th>
<th>$(A)$</th>
<th>$(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>$(-\infty, \infty)$</td>
<td>N/A</td>
<td>$[0, \infty)$</td>
<td>odd</td>
<td>$(-\infty, \infty)$</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>odd</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$</td>
<td>$[a, b]$</td>
<td>odd</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$</td>
<td>$[a, b]$</td>
</tr>
<tr>
<td>odd</td>
<td>$(-\infty, \infty)$</td>
<td>$(-\infty, \infty)$</td>
<td>even</td>
<td>even</td>
<td>$(-\infty, \infty)$</td>
<td>$[0, \infty)$</td>
<td>$[a, b]$</td>
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<tr>
<td>even</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$</td>
<td>even</td>
<td>even</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$</td>
<td>$[a, b]$</td>
</tr>
</tbody>
</table>

**Proof.** By **Theorem 4.1** it is easy to check that (i)–(iv) are true. To verify (v), suppose $y$ is a Hamburger moment sequence which is also a Stieltjes moment sequence. Then it is obvious that $\tilde{y}$ is a Hamburger moment sequence. On the other hand, let us present a Hamburger moment sequence which is not a Stieltjes moment sequence; let $\mu = \frac{3}{\pi} \delta_{-1} + \frac{3}{\pi} \delta_2$ and let

\[
y_k = \int_{\mathbb{R}} x^d d\mu \quad \text{for all } k \in \mathbb{N}_0.
\]

Since $y$ is a Hamburger moment sequence, it holds $H_m(y) \geq 0$ for all $m \in \mathbb{N}_0$. However, its subsequence $\tilde{y}$ with $d = 1$ and $\ell = 1$ does not admit any $\mathbb{R}$-representing measure because $\det([\tilde{y}_{ij}]_{0 \leq i, j \leq 1}) = -\frac{3}{\pi} < 0$. $\Box$

**Theorem 4.2.** A necessary and sufficient condition for $n_k$ that a subsequence $(y_{n_k})_{n \geq 0}$ is a Stieltjes moment sequence for a Stieltjes moment sequence $(y_n)_{n \geq 0}$ is

\[n_k = dk + \ell \quad \text{for all } k \in \mathbb{N}_0,
\]

where $d, \ell \in \mathbb{N}_0$.  

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**Proof.** Let \( y = (y_k)_{k \in \mathbb{N}_0} \) be a Stieltjes moment sequence. Then it follows from Corollary 4.1 that \((y_n)_{k \geq 0}\) is also a Stieltjes moment sequence.

Conversely, suppose that \((y_n)_{k \geq 0}\) is a Stieltjes moment sequence whose \(n_k\) is not of the above form. Then there exist three consecutive terms \(n_i, n_{i+1}, n_{i+2} \in \mathbb{N}_0\) such that \(n_{i+1} - n_i \neq n_{i+2} - n_{i+1}\). First, we assume that \(n_{i+1} - n_i < n_{i+2} - n_{i+1}\). Set \(d = n_{i+1} - n_i\) and \(a = n_i\). Then, \(n_{i+1} = a + d\) and \(n_{i+2} = a + 2d + e\) for \(e \geq 1\).

Since \((y_n)_{k \geq 0}\) is a Stieltjes moment sequence, it follows that

\[
\begin{pmatrix}
  y_n \\
  y_{n+1} \\
  y_{n+2}
\end{pmatrix} =
\begin{pmatrix}
  y_a \\
  y_{a+d} \\
  y_{a+2d+e}
\end{pmatrix}
\]  

must be positive definite. However, there are many instances of \((y_n)_{k \geq 0}\) such that this matrix is not positive definite. Actually, the Stieltjes moment sequence with the representing \(\tilde{\mu} = \delta_{1/2}\) is such an example. This is a contradiction. When \(n_{i+1} - n_i > n_{i+2} - n_{i+1}\), a similar argument (may use \(\mu = \delta_2\)) will lead to a contradiction as well. \(\square\)

Note that unlike a Hamburger moment sequence, a Stieltjes moment sequence has the principal submatrix which is of the form (4.2), regardless of parity of \(n_k\).

Remark that although \(y = (y_n)_{n \geq 0}\) is a Catalan-like number sequence, its subsequence \((y_n)_{k \geq 0}\), defined by \(n_k = dk + \ell\) for all \(k \in \mathbb{N}_0\), is possibly not a Catalan-like number sequence. For example, the sequence in Example 4.1 is a Catalan-like number sequence, but its subsequence \((y_k)_{k \geq 0}\), defined by \(y_k = y_{2k+1}\), is not since \(\Delta_m(y) = 0\) for some \(m \in \mathbb{N}_0\).

A sequence \(y = (y_n)_{n \geq 0}\) is called a Stieltjes (resp. Hausdorff) Catalan-like number sequence if it is both a Stieltjes (resp. Hausdorff) moment sequence and a Catalan-like number sequence.

**Theorem 4.3.** If \(y = (y_n)_{n \geq 0}\) is a Stieltjes Catalan-like number sequence corresponding to \((\sigma, \tau)\), then the subsequence \(\tilde{y} = (y_n)_{k \geq 0}\), defined by \(n_k = dk + \ell\) for all \(k \in \mathbb{N}_0\), is a Stieltjes Catalan-like number sequence corresponding to \(\tilde{\sigma} = (\tilde{\sigma}_k)_{k \geq 0}\) and \(\tilde{\tau} = (\tilde{\tau}_k)_{k \geq 1}\), where

\[
\tilde{\sigma}_k = \frac{L_{\tilde{\sigma}}[x \tilde{P}_n^2(x)]}{L_{\tilde{\sigma}}[\tilde{P}_n^2(x)]} \quad \text{and} \quad \tilde{\tau}_k = \frac{L_{\tilde{\tau}}[\tilde{P}_n^2(x)]}{L_{\tilde{\tau}}[\tilde{P}_{n-1}^2(x)]}
\]  

such that

\[
\tilde{P}_n(x) = \frac{1}{\Delta_{n-1}(\tilde{y})} \det \begin{bmatrix}
  y_\ell & y_{d+\ell} & \cdots & y_{dn+\ell} \\
  y_{d+\ell} & y_2 & \cdots & y_{dn+1+\ell} \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{dn+1+\ell} & y_{dn+\ell} & \cdots & y_{d(n+1)+\ell} \\
  1 & x & \cdots & x^n
\end{bmatrix}
\]

**Proof.** By Corollary 4.1 (i), the subsequence \(\tilde{y}\) is a Stieltjes moment sequences. Since \(y\) is a Stieltjes Catalan-like number sequence, it is a positive sequence and a Catalan-like number sequence. By Theorem 2.7 (ii), it holds that \(H_m(y) > 0\) for all \(m \in \mathbb{N}_0\). Let \(\lambda_m\) and \(\tilde{\lambda}_m\) be the smallest eigenvalue of \(H_m(y)\) and \(H_m(\tilde{y})\). Since \(\lambda_m > 0\) for all \(m \in \mathbb{N}_0\), it follows from the Cauchy interlacing theorem that \(\tilde{\lambda}_m > 0\) for all \(m \in \mathbb{N}_0\). Finally, the explicit forms of \(\tilde{\sigma}\) and \(\tilde{\tau}\) can be obtained by Theorems 2.2 and 2.3. \(\square\)

### 4.2. Linear combinations of terms in Catalan-like number sequences

Now we consider new sequences which are linear combinations of consecutive terms in Catalan-like number sequences. Let \(g(x) = \sum_{k=0}^d g_k x^k \in \mathbb{R}[x]\). For a sequence \(y = (y_k)_{k \geq 0}\), we define a new sequence \(T_g(y) = (T_g(y)_n)_{n \geq 0}\) by

\[
T_g(y)_n = \sum_{k=0}^d g_k y_{n+k}
\]

**Theorem 4.4.** Let \(g(x) = \sum_{k=0}^d g_k x^k \in \mathbb{R}[x]\) such that \(g|_{[a,b]} \geq 0\). If \(y\) is an \([a, b]\)-moment sequence, so is \(T_g(y)\). The representing measure for \(T_g(y)\) is \(gd\mu\), where \(\mu\) is a representing measure for \(y\).

**Proof.** Since \(y\) is an \([a, b]\)-moment sequence, there exists a nonnegative measure \(\mu\) such that

\[
y_k = \int_{a}^{b} x^k d\mu \quad \text{for all} \quad k \in \mathbb{N}_0.
\]
Thus it follows that
\[ T_g(y)_n = \sum_{k=0}^{b} g_k y_{n+k} = \int_a^b x^a g(x) d\mu = \int_a^b x^a d\tilde{\mu}, \]
where \( \tilde{\mu} = g d\mu \). This means that \( T_g(y) \) is an \([a, b]\)-moment sequence. \( \Box \)

Observe that \( T_g(y) \) is a Hausdorff moment sequence, and hence its representing measure is unique. Some results about a representing measure of subsequences can be seen in [4].

**Example 4.2.** Let \( y = (y_k)_{k \geq 0} \) be an \([a, b]\)-moment sequence. Then the following are also \([a, b]\)-moment sequences.

(i) \( T_g(y) = (\alpha y_n + \beta y_{n+1})_{n \geq 0} \) with \( g(x) = \alpha + \beta x \) such that \( g|_{[a, b]} \geq 0 \).

(ii) \( T_g(y) = -(a+b)y_n - y_{n+1} - y_{n+2} \) \( \geq 0 \) with \( g(x) = -(x-a)(x-b) \).

(iii) \( T_g(y) = (a^2 b y_n - (a^2 + 2ab)y_{n+1} + (2a + b)y_{n+2} - y_{n+3}) \) \( \geq 0 \) with \( g(x) = -(x-a)^2(x-b) \).

**Example 4.3.** Consider new sequences of Catalan numbers as follows:

(i) **(Translation)** The subsequence of Catalan numbers, \( (C_{n+\ell})_{n \geq 0} \), is represented by
\[ C_{n+\ell} = \int_0^4 x^n \left( \frac{x^\ell}{2\pi} \frac{4-x}{x} \right) dx. \]

(ii) **(Moving with \( d \) steps)** The subsequence of Catalan numbers, \( (C_{dn})_{n \geq 0} \), is represented by
\[ C_{dn} = \int_0^{4d} x^n \left( \frac{d x^{d-\ell}}{2\pi} \frac{4-d\sqrt{x}}{d\sqrt{x}} \right) dx. \]

(iii) **(Linear combinations)** The new sequence \( \tilde{C} := (\tilde{C}_n)_{n \geq 0} = (4C_{n+1} - C_{n+2})_{n \geq 0} \) is represented by
\[ \tilde{C}_n = \int_0^4 x^n \left( \frac{\sqrt{x}(4-x)^3}{2\pi x} \right) dx. \]

Note that since they are all \([0, 4]\)-Hausdorff moment sequences, their representing measures must be unique, respectively.

5. **Final remarks**

(i) For \( g(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{R}[x] \) satisfying \( g|_{[a, b]} \geq 0 \), we have seen that \( T_g(y) \) is an \([a, b]\)-moment sequence for any \([a, b]\)-moment sequence \( y = (y_k)_{k \geq 0} \). Note that the condition, \( g|_{[a, b]} \geq 0 \), is essential for the existence of a representing measure supported on \([a, b] \). Thus, one can ask what is a necessary and sufficient condition for a function \( g \in \mathbb{R}[x] \) that \( T_g(y) \) is an \([a, b]\)-moment sequence for an \([a, b]\)-moment sequence \( y \).

(ii) The Fine number \( F_n \) is corresponding to \( \sigma = (0, 2, 2, . . .) \) and \( \tau = (1, 1, 1, 1, . . .) \), which is clearly in the form of \( \gamma(p, s; q; t) \). We know from **Theorem 2.8** that \( F_n \) is a Hamburger moment sequence. However, \( F_n \) does not satisfy the numerical conditions in **Theorem 3.1**, and hence we do not know whether it is a Hausdorff moment sequence. On the other hand, some well-known counting numbers do not have the form of \( \gamma(p, s; q; t) \); for instance, the Bell number \( B_n \) is generated by \( \sigma = \tau = (1, 2, 3, 4, . . .) \). It is known that \( B_n \) is a Stieltjes moment sequence, but it is not a Hausdorff moment sequence; for, \( \tau \) is not bounded above, so it follows from **Theorem 2.8** (ii) that there is no \( b \in \mathbb{R} \) such that \( \eta_1 \leq b \). It would be interesting to know when a Stieltjes Catalan-like moment sequence becomes a Hausdorff moment sequence.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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