David-Barton type identities and alternating run polynomials

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Abstract

In this paper, we first consider an alternate formulation of the David-Barton identity which relates the alternating run polynomials to Eulerian polynomials. By using this alternate formulation, we see that for any γ-positive polynomial, there exists a David-Barton type identity. We then consider the joint distribution of cycle runs and cycles over the set of permutations. Furthermore, we introduce the definition of semi-γ-positive polynomial. The γ-positivity of a polynomial $f(x)$ is a sufficient (not necessary) condition for the semi-γ-positivity of $f(x)$. We show that the alternating run polynomial of dual Stirling permutations is semi-γ-positive but not γ-positive.

Keywords: Alternating runs, Eulerian polynomials, Semi-γ-positivity, Stirling permutations
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1. Introduction

This paper is divided into three parts. The first part concerns the connection between the theory of γ-positivity and David-Barton type identities. The second part concerns a bivariate polynomial which contains the alternating run polynomial as a special case. In the third part, we first introduce the definition of semi-γ-positivity, and then we study the semi-γ-positivity of the alternating run polynomials of dual Stirling permutations.

The enumeration of permutations by number of alternating runs was first studied by André [1]. Knuth [24, Section 5.1.3] discussed this topic in connection to sorting and searching. Over the past few decades, the study of alternating run statistic was initiated by David and Barton [14, 157-162]. In the following, we present a survey of this topic.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]=\{1,2,\ldots,n\}$. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$. A *descent* of $\pi\in\mathfrak{S}_n$ is an index $i\in[n-1]$ such that $\pi(i)>\pi(i+1)$. The classical *Eulerian polynomial* is defined by $A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{des}(\pi)+1}$, where $\operatorname{des}(\pi)$ is the number of descents of π. The exponential generating function of $A_n(x)$ is given as follows:

$$A(x,z)=1+\sum_{n=1}^{\infty}A_n(x)\frac{z^n}{n!}=\frac{1-x}{1-xe^{z(1-x)}}.$$

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An *alternating run* of π is a maximal consecutive subsequence that is increasing or decreasing (see [1, 27]). Let $\operatorname{altrun}(\pi)$ be the number of alternating runs of π. For example, if $\pi = 324156$, then $\operatorname{altrun}(\pi) = 4$. We define $R_{n,k} = \#\{\pi \in \mathfrak{S}_n : \operatorname{altrun}(\pi) = k\}$. It is well known that $R_{n+1,k} = kR_{n,k} + 2R_{n,k-1} + (n-k+1)R_{n,k-2}$, with the initial conditions $R_{1,0} = 1$ and $R_{1,k} = 0$ for $k \geq 1$ (see [1]). For $n \geq 1$, let $R_n(x) = \sum_{k=0}^{n-1} R_{n,k}x^k$ be the *alternating run polynomials*. By solving a differential equation, David and Barton [14, 157-162] obtained a remarkable identity:

$$R_n(x) = \left(\frac{1+x}{2}\right)^{n-1} (1+w)^{n+1} A_n\left(\frac{1-w}{1+w}\right) \tag{1}$$

for $n \geq 2$, where $w = \sqrt{\frac{1-x}{1+x}}$.

Since Eulerian polynomial is one of the main topics of combinatorics, the identity (1) arouses growing interest in the study of alternating run polynomials. In [4], Bóna and Ehrenborg proved that $R_{n,k}^2 \geq R_{n,k-1}R_{n,k+1}$. By using (1), Wilf derived the real-rootedness of $R_n(x)$ (see [5, Theorem 1.42]). Canfield and Wilf [8] presented an asymptotic formula for $R_{n,k}$. In [26], an explicit formula for $R_{n,k}$ was obtained by combining the derivative polynomials of tangent function and the following result of Carlitz [9]:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} R_{n+1,k}x^{n-k} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin(z\sqrt{1-x^2})}{x-\cos(z\sqrt{1-x^2})}\right)^2.$$

By using (1), one can prove that $R_n(x)$ has the zero $x = -1$ with the multiplicity $\lfloor \frac{n}{2} \rfloor - 1$, which can also be obtained based on the recurrence relation of $R_n(x)$ (see [31]). Chow and Ma [13] established connections between alternating run statistic and peak statistics of signed permutations. By generalizing a reciprocity formula of Gessel, Zhuang [40] obtained generating function for permutation statistics that are expressible in terms of alternating runs. Subsequently, Zhuang [41] proved several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials. Josuat-Vergès and Pang [23] showed that alternating runs can be used to define subalgebras of Solomon's descent algebra.

The alternating run polynomials are closely related to up-down run polynomials. An *up-down run* of a permutation $\pi \in \mathfrak{S}_n$ is an alternating run of π endowed with a 0 in the front (see [15, 27]). Let $\operatorname{udrun}(\pi)$ be the number of up-down runs of π. For example, if $\pi = 324156$, then $\operatorname{udrun}(\pi) = 5$. We define $T_{n,k} = \#\{\pi \in \mathfrak{S}_n : \operatorname{udrun}(\pi) = k\}$. The numbers $T_{n,k}$ satisfy the following recurrence relation

$$T_{n+1,k} = kT_{n,k} + T_{n,k-1} + (n-k+2)T_{n,k-2}, \tag{2}$$

with $T_{0,k} = \delta_{0,k}$ (see [15]). Let $T_n(x) = \sum_{k=0}^{n} T_{n,k}x^k$ be the *up-down run polynomials*. In [27], several convolution formulas for the polynomials $R_n(x)$ and $T_n(x)$ are obtained by using context-free grammars. In particular,

$$R_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} T_k(x)T_{n-k}(x).$$

The concept of alternating subsequences in permutations was introduced by Stanley [37]. An *alternating subsequence* of π is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying $\pi(i_1)>\pi(i_2)<\cdots\pi(i_k)$, where $i_1<i_2<\cdots<i_k$. Denote by $\operatorname{as}(\pi)$ the length of the longest alternating subsequence of π. Clearly, $\operatorname{as}(\pi)=\operatorname{udrun}(\pi)$. Thus $T_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{as}(\pi)}$. Following [5, Section 1.3.2]: we have $T_n(x)=\frac{1}{2}(1+x)R_n(x)$ for $n\geq 2$. Set $\rho=\sqrt{1-x^2}$. Stanley [37, Theorem 2.3] showed that

$$T(x,z)=\sum_{n=0}^{\infty}T_n(x)\frac{z^n}{n!}=(1-x)\frac{1+\rho+2xe^{\rho z}+(1-\rho)e^{2\rho z}}{1+\rho-x^2+(1-\rho-x^2)e^{2\rho z}}. \tag{3}$$

By using (3), Stanley [37] obtained an explicit formula for $T_{n,k}$.

Stirling permutations were introduced by Gessel and Stanley [20]. A *Stirling permutation* of order n is a permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ such that for each i, $1\leq i\leq n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma=\sigma_1\sigma_2\cdots\sigma_{2n}\in\mathcal{Q}_n$. An occurrence of a *left ascent-plateau* is an index i such that $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, where $i\in\{1,2,\ldots,2n-1\}$ and $\sigma_0=0$. Let $\operatorname{lap}(\sigma)$ be the number of left ascent-plateaus of σ. Let $N_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\operatorname{lap}(\sigma)}$. Following [28, Theorem 3], the exponential generating function of $N_n(x)$ is given as follows:

$$N(x,z)=\sum_{n=0}^{\infty}N_n(x)\frac{z^n}{n!}=\sqrt{\frac{1-x}{1-xe^{2z(1-x)}}}.$$

Motivated by the fact that $A(x,2z)=N^2(x,z)$. It is natural to explore the alternating run statistic on Stirling permutations or its variations.

This paper is organized as follows. In Section 2, we show that for any γ-positive polynomial, there exists a David-Barton type identity. In Section 3, we consider a bivariate polynomial $R(x,q)$ such that $R_n(x,1)=T_n(x)$, $R_n(x,2)=R_{n+1}(x)$. In Section 4, we show that the alternating run polynomial of dual Stirling permutations is semi-γ-positive but not γ-positive.

2. γ-positivity and David-Barton type identities

Let Δ be a simplicial complex of dimension $n-1$. The *f-vector* of Δ is the sequence of integers $(f_{-1},f_0,f_1,\ldots,f_{n-1})$, where f_i is the number of faces with $i+1$ vertices in Δ. The f-polynomial and h-polynomial of Δ are respectively defined as $f(x)=\sum_{i=0}^n f_{i-1}x^i$, and

$$h(x)=(1-x)^nf\left(\frac{x}{1-x}\right)=\sum_{i=0}^n f_{i-1}x^i(1-x)^{n-i}=\sum_{i=0}^n h_ix^i.$$

The sequence of integers $(h_0,h_1,\ldots,h_n)$ is called the *h-vector* of Δ. If $h(x)$ is symmetric, i.e., $h_i=h_{n-i}$ for any $0\leq i\leq n$, then $h(x)$ can be expanded as $h(x)=\sum_{i=0}^{\lfloor d/2\rfloor}\gamma_ix^i(1+x)^{n-2i}$. The sequence of integers $(\gamma_0,\gamma_1,\ldots,\gamma_{\lfloor d/2\rfloor})$ is called the *γ-vector* of Δ. Following Gal [18], we say that $h(x)$ is *γ-positive* if $\gamma_i\geq 0$ for all $0\leq i\leq\lfloor\frac{n}{2}\rfloor$. The γ-positivity provides a natural approach to study symmetric and unimodal polynomials (see [6, 12, 25] for instance). We refer the reader to Athanasiadis's survey article [3] for details.

2.1. Main result

We now give our first main result.

Theorem 1. *Let*

$$M_n(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} m(n,k) x^k (1+x)^{n+\delta-2k}$$

be a symmetric polynomial, where δ is a fixed integer. Set $w = \sqrt{\frac{1-x}{1+x}}$. Then

$$N_n(x) = \left(\frac{1+x}{2}\right)^{n-\delta} (1+w)^{n+\delta} M_n\left(\frac{1-w}{1+w}\right) \tag{4}$$

if and only if

$$N_n(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \frac{1}{2^{k-2\delta}} m(n,k) x^k (1+x)^{n-\delta-k}. \tag{5}$$

Proof. Set $\alpha = \frac{1+x}{2}$. Note that

$$1-w^2 = \frac{x}{\alpha},\ \frac{1-w}{1+w} = \frac{1-w^2}{(1+w)^2} = \frac{1}{(1+w)^2}\frac{x}{\alpha},\ 1+\frac{1-w}{1+w} = \frac{2}{1+w}.$$

It follows from (4) that

$$\begin{aligned} N_n(x) &= \alpha^{n-\delta}(1+w)^{n+\delta} \sum_k m(n,k) \frac{1}{(1+w)^{2k}} \frac{x^k}{\alpha^k} \left(\frac{2}{1+w}\right)^{n+\delta-2k} \\ &= \sum_k m(n,k) x^k \alpha^{n-\delta-k} 2^{n+\delta-2k} \\ &= \sum_k m(n,k) x^k \left(\frac{1+x}{2}\right)^{n-\delta-k} 2^{n+\delta-2k} \\ &= \sum_k \frac{1}{2^{k-2\delta}} m(n,k) x^k (1+x)^{n-\delta-k}, \end{aligned}$$

and vice versa. This completes the proof. □

Let $m_n(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} m(n,k) x^k$. Without loss of generality, assume that $m_n(x)$ has only simple and real non-positive zeros, i.e.,

$$m_n(x) = c \prod_{k=0}^{\lfloor (n+\delta)/2 \rfloor} (x - r_i),$$

where $c = m(n, \lfloor (n+\delta)/2 \rfloor)$ and $r_1 < r_2 < \cdots < r_{\lfloor (n+\delta)/2 \rfloor} \le 0$. From Theorem 1, we see that

$$M_n(x) = (1+x)^{n+\delta} m_n\left(\frac{x}{(1+x)^2}\right) = c(1+x)^{n+\delta} \prod_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \left(\frac{x}{(1+x)^2} - r_i\right).$$

Note that

$$x - r_i(1+x)^2 = -r_i\left(x + \frac{2r_i - 1 - \sqrt{1-4r_i}}{2r_i}\right)\left(x + \frac{2r_i - 1 + \sqrt{1-4r_i}}{2r_i}\right).$$

Therefore, $M_n(x)$ has only simple and real non-positive zeros. Furthermore, we see that

$$\begin{aligned} N_n(x) &= \left(\frac{1+x}{2}\right)^{n-\delta} (1+w)^{n+\delta} \left(\frac{2}{1+w}\right)^{n+\delta} m_n\left(\frac{1-w^2}{4}\right) \\ &= 2^{2\delta}\,(1+x)^{n-\delta}\, m_n\left(\frac{x}{2(1+x)}\right) \\ &= 2^{2\delta} c\,(1+x)^{n-\delta} \prod_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \left(\frac{x}{2(1+x)} - r_i\right) \\ &= 2^{2\delta} c(1+x)^{\lceil (n+\delta)/2 \rceil - 2\delta} \prod_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \left(\frac{x}{2} - r_i(1+x)\right). \end{aligned}$$

So the following result is immediate.

Proposition 2. *Let $M_n(x)$ and $N_n(x)$ be two polynomials given in Theorem 1. If $M_n(x)$ has only simple and real non-positive zeros, then $N_n(x)$ has only real non-positive zeros and $N_n(x)$ has $\lfloor \frac{n+\delta}{2} \rfloor$ simple zeros and the zero $x=-1$ with the multiplicity $\lceil \frac{n+\delta}{2} \rceil - 2\delta$.*

According to Theorem 1, if the h-polynomial of a simplicial complex Δ of dimension $n+\delta-1$ has the following expansion:

$$h(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \gamma_k x^k (1+x)^{n+\delta-2k}.$$

then one can define an associated polynomial $h^*(x)$ as follows:

$$h^*(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \frac{1}{2^{k-2\delta}} \gamma_k x^k (1+x)^{n-\delta-k} = \sum_{k=0}^{n-\delta} h_k^* x^k.$$

Since the polynomials $h(x)$ and $h^*(x)$ occur in pairs, we call $h^*(x)$ the *h-polynomial of the second kind*. It would be interesting to explore combinatorial and topological significance of $h^*(x)$.

In the following subsections, we shall present several applications of Theorem 1.

2.2. Eulerian polynomials

The γ-positivity of Eulerian polynomials was first observed by Foata and Schützenberger [16]. They found that

$$A_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a(n,k) x^k (1+x)^{n+1-2k},$$

where the numbers $a(n,k)$ satisfy the recurrence $a(n,k) = ka(n-1,k) + (2n-4k+4)a(n-1,k-1)$, with the initial conditions $a(1,1)=1$ and $a(1,k)=0$ for $k \neq 1$ (see [12, 32] for instance). An index $i \in [n]$ is a *peak* (resp. *double descent*) of π if $\pi(i-1) < \pi(i) > \pi(i+1)$ (resp. $\pi(i-1) > \pi(i) > \pi(i+1)$), where $\pi(0) = \pi(n+1) = 0$. The number $a(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k peaks and without double descents (see [3, 6, 17] for instance).

By using the David-Barton identity (1) and Theorem 1, we get the following result.

Proposition 3. *For $n \geq 2$, we have*

$$R_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{1}{2^{k-2}} a(n,k) x^k (1+x)^{n-1-k}.$$

Note that $\lceil \frac{n+1}{2} \rceil - 2 = \lfloor \frac{n}{2} \rfloor - 1$. Using Proposition 2, we get the following corollary.

Corollary 4. *For any $n \geq 2$, $(1+x)^{\lfloor n/2 \rfloor - 1}$ divides $R_n(x)$.*

2.3. Eulerian polynomials of type B

Let $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let B_n be the hyperoctahedral group of rank n. Elements of B_n are signed permutations of $\pm[n]$ with the property that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. In the sequel, we always assume that signed permutations in B_n are prepended by 0. That is, we identify a signed permutation $\pi = \pi(1) \cdots \pi(n)$ with the word $\pi(0)\pi(1) \cdots \pi(n)$, where $\pi(0) = 0$. A *type B descent* is an index $i \in \{0, 1, \ldots, n-1\}$ such that $\pi(i) > \pi(i+1)$. Let $\operatorname{des}_B(\pi)$ be the number of type B descents of π. The *type B Eulerian polynomials* are defined by $B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)}$. It is well known that $B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k) x^k (1+x)^{n-2k}$, where the numbers $b(n,k)$ satisfy the recurrence relation

$$b(n,k) = (1+2k) b(n-1,k) + 4(n-2k+1) b(n-1,k-1), \tag{6}$$

with the initial conditions $b(1,0) = 1$ and $b(1,k) = 0$ for $k \neq 0$ (see [12, 32]). An index $i \in [n-1]$ is a *left peak* of π if $\pi(i-1) < \pi(i) > \pi(i+1)$, where $\pi(0) = 0$. The number $b(n,k)/4^k$ counts permutations in $\mathfrak{S}_n$ with k left peaks (see [3, 12, 32] for instance)

Define

$$b_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^k} b(n,k) x^k (1+x)^{n-k}. \tag{7}$$

Then by Theorem 1, we get the following result.

Proposition 5. *For $n \geq 1$, we have*

$$b_n(x) = \left(\frac{1+x}{2} \right)^n (1+w)^n B_n \left(\frac{1-w}{1+w} \right).$$

Corollary 6. *For any $n \geq 2$, $(1+x)^{\lceil n/2 \rceil}$ divides $b_n(x)$.*

Combining (6) and (7), we see that the polynomials $b_n(x)$ satisfy the recurrence relation

$$b_{n+1}(x) = (1 + x + 2nx^2) b_n(x) + 2x(1-x^2) b_n'(x), \tag{8}$$

with the initial conditions $b_0(x) = 1$, $b_1(x) = 1 + x$. For $n \geq 1$, we define $b_n(x) = \frac{1+x}{x} c_n(x)$. By using (8), it is easy to verify that $c_{n+1}(x) = (2nx^2 + 3x - 1) c_n(x) + 2x(1-x^2) c_n'(x)$. Let $\widehat{B}_n$ be the set of signed permutations in B_n with $\pi(1) > 0$. According to [39, Theorem 4.3.1], we have

$$c_n(x) = \sum_{\pi \in \widehat{B}_n} x^{\operatorname{altrun}(\pi)}.$$

2.4. Derangement polynomials

We say that a permutation $\pi \in \mathfrak{S}_n$ is a *derangement* if $\pi(i) \neq i$ for each $i \in [n]$. Let $\mathcal{D}_n$ be the set of all derangements in $\mathfrak{S}_n$. Let $d_n(x) = \sum_{\pi\in\mathcal{D}_n} x^{\mathrm{exc}\,(\pi)}$ be the *derangement polynomials*, where $\mathrm{exc}\,(\pi) = \#\{i \in [n-1] \mid \pi(i) > i\}$. The derangement polynomials satisfy the recurrence relation $d_{n+1}(x) = nxd_n(x)+x(1-x)d_n'(x)+nxd_{n-1}(x)$, with the initial conditions $d_0(x) = 1, d_1(x) = 0$ and $d_2(x) = x$ (see [7]). A *double excedance* of $\pi \in \mathfrak{S}_n$ is any index $i \in [n-1]$ such that $\pi(i) > i > \pi^{-1}(i)$. Following [2, Equations (1.3) and (3.2)] and [33, Section 5], we have $d_n(x) = \sum_{i=1}^{\lfloor n/2\rfloor} \xi(n,i)x^i(1+x)^{n-2i}$, where $\xi(n,i)$ is the number of derangements $\pi \in \mathcal{D}_n$ with i excedances and no double excedances. Several refinements of this interpretation were studied by Shin and Zeng [34, 35].

Define

$$e_n(x) = \sum_{i=1}^{\lfloor n/2\rfloor} \frac{1}{2^i}\xi(n,i)x^i(1+x)^{n-i}.$$

Then by Theorem 1, we get the following result.

Proposition 7. *For $n \geq 2$, we have*

$$e_n(x) = \left(\frac{1+x}{2}\right)^n (1+w)^n d_n\left(\frac{1-w}{1+w}\right).$$

Corollary 8. *For any $n \geq 2$, $(1+x)^{\lceil n/2\rceil}$ divides $e_n(x)$.*

For $n \geq 2$, set

$$e_n(x) = \frac{(1+x)^{\lceil n/2\rceil}}{2^{\lfloor n/2\rfloor}} h_n(x).$$

The first few $h_n(x)$ are $h_2(x) = h_3(x) = x$, $h_4(x) = x(2+7x)$ and $h_5(x) = 2x(1+10x)$.

2.5. Descent polynomials of involutions

Let $\mathcal{I}_n$ be the set of involutions in $\mathfrak{S}_n$. The descent polynomial of involutions is defined by $I_n(t) = \sum_{\pi\in\mathcal{I}_n} t^{\mathrm{des}\,(\pi)}$. The first few polynomials $I_n(x)$ are $I_1(x) = 1$, $I_2(x) = 1+x$, $I_3(x) = 1+2x+x^2$ and $I_4(x) = 1+4x+4x^2+x^3$. Very recently, Wang [38] proved a conjecture of Guo and Zeng [21] which states that there are nonnegative integers $i(n,k)$ such that

$$I_n(x) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} i(n,k)x^k(1+x)^{n-1-2k}.$$

We define

$$i_n(x) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \frac{1}{2^{k+2}} i(n,k)x^k(1+x)^{n+1-k}.$$

Then by Theorem 1, we get the following result.

Proposition 9. *For $n \geq 2$, we have*

$$i_n(x) = \left(\frac{1+x}{2}\right)^{n+1} (1+w)^{n-1} I_n\left(\frac{1-w}{1+w}\right).$$

For $n \geq 2$, set

$$i_n(x) = \frac{(1+x)^{\lfloor n/2 \rfloor +1}}{2^{\lfloor n/2 \rfloor +1}} j_n(x).$$

The first few $j_n(x)$ are $j_2(x) = 1+x,\ j_3(x) = (1+x)^2$ and $j_4(x) = (1+x)(2+3x)$.

3. The joint distribution of cycle runs and cycles

Let V be an alphabet whose letters are regarded as independent commutative indeterminates. Following Chen [10], a *context-free grammar* G over V is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V, see [11, 30] for details. The formal derivative $D := D_G$ with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the relations: $D(u+v) = D(u) + D(v),\ D(uv) = D(u)v + uD(v)$. For a constant c, we have $D(c) = 0$.

We now recall the grammatical interpretations of the numbers $T_{n,k}$ and $R_{n,k}$.

Proposition 10 ([27]). *We have the following result:*

(i) If $G = \{a \to ab,\ b \to bc,\ c \to b^2\}$, then $D^n(a) = a\sum_{k=0}^{n} T_{n,k} b^k c^{n-k}$;

(ii) If $G = \{a \to 2ab,\ b \to bc,\ c \to b^2\}$, then $D^n(a) = a\sum_{k=0}^{n} R_{n+1,k} b^k c^{n-k}$.

Motivated by Proposition 10, it is natural to consider the grammar

$$G_0 = \{a \to qab,\ b \to bc,\ c \to b^2\}. \tag{9}$$

Note that $D_{G_0}(a) = qab,\ D^2_{G_0}(a) = a(q^2b^2 + qbc)$. By induction, it is easy to verify that

$$D^n_{G_0}(a) = a\sum_{k=0}^{n} R_{n,k}(q) b^k c^{n-k}. \tag{10}$$

It follows from (9) that $D^{n+1}_{G_0}(a) = a\sum_k R_{n,k}(q)\left(kb^kc^{n-k+1} + qb^{k+1}c^{n-k} + (n-k)b^{k+2}c^{n-k-1}\right)$, which leads to the recurrence relation

$$R_{n+1,k}(q) = kR_{n,k}(q) + qR_{n,k-1}(q) + (n-k+2)R_{n,k-2}(q). \tag{11}$$

Let $R_n(x,q) = \sum_{k=0}^{n} R_{n,k}(q)x^k$ be the *q-alternating run polynomials*. Then from Propsition 10, we get the following result.

Proposition 11. *For $n \geq 0$, we have $R_n(x,1) = T_n(x)$ and $R_n(x,2) = R_{n+1}(x)$.*

The first few $R_n(x,q)$ are $R_0(x,q) = 1, R_1(x,q) = qx, R_2(x,q) = qx(1+qx)$ and $R_3(x,q) = qx(1 + 3qx + x^2 + q^2x^2)$. We define

$$R(x,q;z) = \sum_{n=0}^{\infty} R_n(x,q)\frac{z^n}{n!}.$$

Theorem 12. *We have $R(x,q;z) = T^q(x,z)$, where $T(x,z)$ is given by* (3)*. Therefore,*

$$\sum_{n=0}^{\infty} D_{G_0}^n(a)\frac{z^n}{n!} = aR\left(\frac{b}{c}, q; cz\right) = aT^q\left(\frac{b}{c}, cz\right). \tag{12}$$

Proof. By rewriting (11) in terms of generating function $R(x,q;z)$, we obtain

$$(1-x^2z)\frac{\partial}{\partial z}R(x,q;z) = x(1-x^2)\frac{\partial}{\partial x}R(x,q;z) + qxR(x,q;z). \tag{13}$$

It is routine to check that the generating function $T^q(x,z)$ satisfies (13). Also, this generating function gives $T^q(0,z) = T^q(x,0) = 1$. Hence $R(x,q;z) = T^q(x,z)$. Then (12) follows immediately from (10). □

In the rest of this section, we present a combinatorial interpretation of $R_n(x,q)$. We say that $\pi \in \mathfrak{S}_n$ is a *circular permutation* if it has only one cycle. Let $A = \{x_1, \ldots, x_k\}$ be a finite set of positive integers, and let $\mathcal{C}_A$ be the set of all circular permutations of A. Let $w \in \mathcal{C}_A$. We will write w by using its canonical presentation $w = y_1y_2\cdots y_k$, where $y_1 = \min A$, $y_i = w^{i-1}(y_1)$ for $2 \le i \le k$ and $y_1 = w^k(y_1)$. A *cycle peak* (resp. *cycle double ascent*, *cycle double descent*) of w is an entry y_i, $2 \le i \le k$, such that $y_{i-1} < y_i > y_{i+1}$ (resp. $y_{i-1} < y_i < y_{i+1}$, $y_{i-1} > y_i > y_{i+1}$), where we set $y_{k+1} = \infty$. Let $\operatorname{cpk}(w)$ (resp. $\operatorname{cdasc}(w)$, $\operatorname{cddes}(w)$) be the number of cycle peaks (resp. cycle double ascents, cycle double descents) of w. We define the number of *cycle runs* $\operatorname{crun}(w)$ of w to be the number of alternating runs of the word $y_1y_2\cdots y_k\infty$. It is clear that $\operatorname{crun}(w) = 2\operatorname{cpk}(w) + 1$.

In the following discussion we always write $\pi \in \mathfrak{S}_n$ in standard cycle decomposition, where the cycles are written in increasing order of their smallest entry and each of these cycles is expressed in canonical presentation. Moreover, we always assume that each cycle is appended by ∞. Let $\operatorname{cyc}(\pi)$ be the number of cycles of $\pi \in \mathfrak{S}_n$. Assume that $\operatorname{cyc}(\pi) = k$ and $\pi = w_1w_2\cdots w_k$, where w_i is the ith cycle of π. Let $\operatorname{crun}(\pi) = \sum_{i=1}^k \operatorname{crun}(w_i)$ be the number of cycle runs of π. In particular, $\operatorname{crun}((1)(2)\cdots(n)) = \operatorname{crun}((1\infty)(2\infty)\cdots(n\infty)) = n$.

Following [11], a *grammatical labeling* is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar. We can now present the second main result.

Theorem 13. *For $n \ge 1$, we have*

$$R_n(x,q) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{crun}(\pi)} q^{\operatorname{cyc}(\pi)}. \tag{14}$$

Proof. Recall the grammar (9). We introduce a grammatical labeling of π as follows:

(L_1) Put a subscript label q at the end of each cycle of π;

(L_2) Put a superscript label a at the end of π;

(L_3) Put a superscript label b just before each ∞;

(L_4) If $\pi(i)$ is a cycle peak, then put a superscript label b just before $\pi(i)$ and a superscript label b right after $\pi(i)$;

(L_5) If $\pi(i)$ is a cycle double ascent, then put a superscript label c just before $\pi(i)$;

(L_6) If $\pi(i)$ is a cycle double descent, then put a superscript label c right after $\pi(i)$.

It should be noted that $\mathrm{cpk}\,(\pi)$ equals the number of "b"s right after cycle peaks. The weight of π is the product of its labels. Note that $\mathfrak{S}_1 = \{(1^b\infty)_q^a\}$ and $\mathfrak{S}_2 = \{(1^b\infty)_q(2^b\infty)_q^a,\ (1^c2^b\infty)_q^a\}$. Then the weight of $(1^b)_q^a$ is given by $D_{G_0}(a)$, and the sum of weights of the elements in $\mathfrak{S}_2$ is given by $D_{G_0}^2(a)$. Let $r_n(i,j) = \{\pi \in \mathfrak{S}_n : \mathrm{crun}\,(\pi) = i,\ \mathrm{cyc}\,(\pi) = j\}$. Suppose we get all labeled permutations in $r_{n-1}(i,j)$, where $n \geq 3$. Let π be obtained from $\pi' \in r_{n-1}(i,j)$ by inserting the entry n. We distinguish four cases:

(c_1) If we insert n as a new cycle, then $\pi \in r_n(i+1, j+1)$. This case corresponds to the substitution rule $a \to qab$;

(c_2) If we insert n just before a ∞, then $\pi \in r_n(i,j)$. This case corresponds to the substitution rule $b \to bc$;

(c_3) If we insert n just before or right after a cycle peak, then $\pi \in r_n(i,j)$. This case corresponds to the substitution rule $b \to bc$;

(c_4) If we insert n just before a cycle double ascent or right after a cycle double descent, then $\pi \in r_n(i+2, j)$. This case corresponds to the substitution rule $c \to b^2$.

In each case, the insertion of n corresponds to one substitution rule in the grammar (9). It is easy to check that the action of D_{G_0} on elements of $\mathfrak{S}_{n-1}$ generates all elements of $\mathfrak{S}_n$. By induction, we obtain a constructive proof of (14). This completes the proof. □

Combining Proposition 11 and Theorem 13, we get the following corollary.

Corollary 14. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{udrun}\,(\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{crun}\,(\pi)}. \tag{15}$$

We end this section by giving an inductive proof of (15). Let

$$\widetilde{T}_{n,k} = \#\{\pi \in \mathfrak{S}_n :\ \mathrm{crun}\,(\pi) = k\}.$$

There are three ways by which a permutation $\pi' \in \mathfrak{S}_{n+1}$ with k cycle runs can be obtained from $\pi \in \mathfrak{S}_n$ by inserting the entry $n+1$:

(i) If $\mathrm{crun}\,(\pi) = k$, then we can put $n+1$ right after the end of each ascending run which does not end with ∞, and just before the beginning of each descending run. Moreover, if an ascending run ends with ∞, then we put $n+1$ just before ∞. This gives $k\widetilde{T}_{n,k}$ possibilities.

(ii) If $\mathrm{crun}\,(\pi)=k-1$, then we insert $n+1$ as a new cycle $(n+1)$. This gives $\widetilde{T}_{n,k-1}$ possibilities.

(iii) If $\mathrm{crun}\,(\pi)=k-2$, then we can insert $n+1$ into the remaining $n-(k-2)=n-k+2$ positions. This gives $(n-k+2)\widetilde{T}_{n,k-2}$ possibilities.

Clearly, $\widetilde{T}_{1,1}=1$. Recall the recurrence relation (2). We see that $\widetilde{T}_{n,k}$ satisfy the same recurrence relation and initial conditions as $T_{n,k}$, so they agree.

4. γ-positivity and Semi-γ-positivity

Let $g(x)=\sum_{i=0}^{2n} g_i x^i$ be a symmetric polynomial. Note that

$$g(x)=\sum_{i=0}^{n}\gamma_i x^i(1+x)^{2(n-i)}=\sum_{i=0}^{n}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}2^{\ell}\gamma_i x^{i+\ell}(1+x^2)^{n-i-\ell}.$$

Hence $g(x)$ can be expanded as $g(x)=\sum_{k=0}^{n}\lambda_k x^k(1+x^2)^{n-k}$. It is clear that if $\gamma_i\geq 0$ for all $0\leq i\leq n$, then $\lambda_k\geq 0$ for all $0\leq k\leq n$. Furthermore, we have

$$\begin{aligned} g(x)&=\sum_{k=0}^{\lfloor n/2\rfloor}\lambda_{2k}x^{2k}(1+x^2)^{n-2k}+\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\lambda_{2k+1}x^{2k+1}(1+x^2)^{n-2k-1}\\ &=g_1(x^2)+xg_2(x^2). \end{aligned}$$

Similarly, if $h(x)=\sum_{i=0}^{2n+1} h_i x^i$ a symmetric polynomial, then we have

$$\begin{aligned} h(x)&=\sum_{i=0}^{n}\beta_i x^i(1+x)^{2n+1-2i}\\ &=(1+x)\sum_{i=0}^{n}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}2^{\ell}\beta_i x^{i+\ell}(1+x^2)^{n-i-\ell}. \end{aligned}$$

Hence $h(x)$ can be expanded as $h(x)=(1+x)\sum_{k=0}^{n}\mu_k x^k(1+x^2)^{n-k}$.

Definition 15. *If $f(x)=(1+x)^{\nu}\sum_{k=0}^{n}\lambda_k x^k(1+x^2)^{n-k}$ and $\lambda_k\geq 0$ for all $0\leq k\leq n$, then we say that $f(x)$ is semi-γ-positive, where $\nu=0$ or $\nu=1$.*

The γ-positivity of a polynomial $f(x)$ is a sufficient (not necessary) condition for the semi-γ-positivity of $f(x)$. From the above discussion, we can conclude the following result.

Proposition 16. *If $f(x)=(1+x)^{\nu}\left(f_1(x^2)+xf_2(x^2)\right)$ is a semi-γ-positive polynomial, then $f_1(x)$ and $f_2(x)$ are both γ-positive polynomials.*

In the rest of this section, we give a polynomial which is semi-γ-positive but not γ-positive. Let σ be a Stirling permutation of order n. Let Φ be the injection which maps each first occurrence of entry j in σ to $2j$ and the second occurrence of j to $2j-1$, where $j\in[n]$. For example, $\Phi(221331)=432651$. Let $\Phi(\mathcal{Q}_n)=\{\pi\mid\sigma\in\mathcal{Q}_n,\Phi(\sigma)=\pi\}$ be the set of *dual Stirling permutations* of order n. Clearly, $\Phi(\mathcal{Q}_n)$ is a subset of $\mathfrak{S}_{2n}$. For $\pi\in\Phi(\mathcal{Q}_n)$, we see that the entry $2j$ is to the left of $2j-1$, and all entries in π between $2j$ and $2j-1$ are larger than $2j$, where $1 \leq j \leq n$. Moreover, any dual Stirling permutation always ends with a descending run. The *alternating run polynomials of dual Stirling permutations* are defined by

$$F_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\mathrm{altrun}\,(\pi)}.$$

Let $F_n(x) = \sum_{k=1}^{2n-1} F_{n,k} x^k$. According to [29], the numbers $F_{n,k}$ satisfy the recurrence relation

$$F_{n+1,k} = kF_{n,k} + F_{n,k-1} + (2n-k+2)F_{n,k-2}. \tag{16}$$

with the initial conditions $F_{0,0} = 1,\ F_{1,1} = 1$ and $F_{n,0} = 0$ for $n \geq 1$. It follows from (16) that

$$F_{n+1}(x) = (x + 2nx^2)F_n(x) + x(1-x^2)F_n'(x).$$

The first few $F_n(x)$ are $F_1(x) = x, F_2(x) = x + x^2 + x^3$, and $F_3(x) = x + 3x^2 + 7x^3 + 3x^4 + x^5$. Let $r(x) = \sqrt{\frac{1+x}{1-x}}$. By induction, it is easy to verify that

$$\left(x\frac{d}{dx}\right)^n r(x) = \frac{r(x)F_n(x)}{(1-x^2)^n}.$$

We now recall another combinatorial interpretation of $F_n(x)$. An occurrence of an *ascent-plateau* of $\sigma \in \mathcal{Q}_n$ is an index i such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{2, 3, \ldots, 2n-1\}$. Let $\mathrm{ap}\,(\sigma)$ be the number of ascent-plateaus of σ. Thus, if $\sigma_1 = \sigma_2$, then $\mathrm{lap}\,(\sigma) = \mathrm{ap}\,(\sigma) + 1$; if $\sigma_1 < \sigma_2$, then $\mathrm{lap}\,(\sigma) = \mathrm{ap}\,(\sigma)$. The number of *flag ascent-plateaus* of σ is defined by

$$\mathrm{fap}\,(\sigma) = \begin{cases} 2\mathrm{ap}\,(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2; \\ 2\mathrm{ap}\,(\sigma), & \text{otherwise.} \end{cases}$$

Hence $\mathrm{fap}\,(\sigma) = \mathrm{ap}\,(\sigma) + \mathrm{lap}\,(\sigma)$. It is easy to check that $\mathrm{fap}\,(\sigma) = \mathrm{altrun}\,(\Phi(\sigma))$ for any $\sigma \in \mathcal{Q}_n$ (see [30, Section 3]). For $n \geq 1$, we have

$$F_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{fap}\,(\sigma)}.$$

There is a grammatical interpretation of $F_n(x)$.

Lemma 17 ([29, 30])**.** *If*

$$G_1 = \{x \to xyz, y \to yz^2, z \to y^2 z\}, \tag{17}$$

then we have

$$D_{G_1}^n(x) = x \sum_{k=0}^{2n-1} F_{n,k} y^k z^{2n-k}. \tag{18}$$

Proposition 18. *The polynomials $F_n(x)$ are not γ-positive for $n \geq 2$.*

Proof. Consider a change of the grammar (17). Set $a = yz$ and $b = y + z$. Then we have $D(x) = xa,\ D(a) = a(b^2 - 2a),\ D(b) = ab$. If $G_2 = \{x \to xa,\ a \to a(b^2 - 2a),\ b \to ab\}$, then by induction, we see that there exist integers $\gamma_{n,k}$ such that

$$D^n_{G_2}(x) = x \sum_{k=0}^{n} \gamma_{n,k} a^k b^{2n-2k}. \tag{19}$$

Note that $D^{n+1}_{G_2}(x) = x \sum_k \gamma_{n,k} a^k b^{2n-2k} \big(kb^2 + (1 + 2n - 4k)a\big)$. By comparing the coefficients of $a^k b^{2n-2k+2}$, we get that the numbers $\gamma_{n,k}$ satisfy the recurrence relation

$$\gamma_{n+1,k} = k\gamma_{n,k} + (2n - 4k + 5)\gamma_{n,k-1}, \tag{20}$$

with $\gamma_{1,1} = 1$ and $\gamma_{1,k} = 0$ for $k \neq 1$. It is clear that $\gamma_{n,0} = 0$ for $n \geq 1$. By using (19), upon taking $a = yz$ and $b = y + z$, we get

$$D^n_{G_1}(x) = x \sum_{k=0}^{n} \gamma_{n,k} (yz)^k (y + z)^{2n-2k}. \tag{21}$$

Then comparing (21) with (18), we get

$$F_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k (1 + x)^{2n-2k}.$$

Note that $\gamma_{n+1,n+1} = -(2n - 1)\gamma_{n,n}$. By induction, we obtain $\gamma_{n+1,n+1} = (-1)^n (2n - 1)!!$ for $n \geq 1$. Note that $\gamma_{n+1,n} = n\gamma_{n,n} - (2n - 5)\gamma_{n,n-1} = (-1)^{n-1}(2n - 3)!!n - (2n - 5)\gamma_{n,n-1}$ for $n \geq 2$. Then by induction, it is easy to verify that $\operatorname{sgn} \gamma_{n+1,n} = (-1)^{n-1}$ for $n \geq 1$, so we have

$$\operatorname{sgn}(\gamma_{n+1,n}\gamma_{n+1,n+1}) = -1.$$

Therefore, the polynomial $F_n(x)$ is not γ-positive for any $n \geq 2$. This completes the proof. □

Let $\gamma_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k$. It follows from (20) that

$$\gamma_{n+1}(x) = (2n + 1)x\gamma_n(x) + x(1 - 4x)\gamma_n'(x).$$

The first few $\gamma_n(x)$ are $\gamma_0(x) = 1,\ \gamma_1(x) = x,\ \gamma_2(x) = x - x^2,\ \gamma_3(x) = x - x^2 + 3x^3$.

We can now present the third main result of this paper.

Theorem 19. *The polynomial $F_n(x)$ is semi-γ-positive. More precisely, we have*

$$F_n(x) = \sum_{k=0}^{n} f_{n,k} x^k (1 + x^2)^{n-k},$$

where the numbers $f_{n,k}$ satisfy the recurrence relation

$$f_{n+1,k} = kf_{n,k} + f_{n,k-1} + 4(n - k + 2)f_{n,k-2}, \tag{22}$$

with $f_{0,0} = 1$ and $f_{n,0} = 0$ for $n \geq 1$. Let $f_n(x) = \sum_{k=0}^{n} f_{n,k} x^k$. Then

$$f(x, z) = \sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!} = \sqrt{T(2x, z)}, \tag{23}$$

where $T(x, z)$ is given by (3).

Proof. We first consider the grammar (17). Note that $D(x) = xyz,\ D(yz) = yz(y^2 + z^2)$, and $D(y^2 + z^2) = 4y^2z^2$. Set $u = yz$ and $v = y^2 + z^2$. Then we have $D(x) = xu,\ D(u) = uv$ and $D(v) = 4u^2$. Consider

$$G_3 = \{x \to xu,\ u \to uv,\ v \to 4u^2\}. \tag{24}$$

By induction we see that there exist nonnegative integers $f_{n,k}$ such that

$$D_{G_3}^n(x) = x \sum_{k=0}^{n} f_{n,k} u^k v^{n-k}. \tag{25}$$

Note that $D_{G_3}^{n+1}(x) = x \sum_k f_{n,k} \left(u^{k+1}v^{n-k} + ku^k v^{n-k+1} + 4(n-k)u^{k+2}v^{n-k-1}\right)$. By comparing the coefficients of $u^k v^{n+1-k}$, we get (22). Moreover, it follows from (24) that $f_{0,0} = 1$ and $f_{n,0} = 0$ for $n \geq 1$. By using (25), upon taking $u = yz$ and $v = y^2 + z^2$, we get

$$D_{G_1}^n(x) = x \sum_{k=0}^{n} f_{n,k} (yz)^k (y^2 + z^2)^{n-k}. \tag{26}$$

By comparing (26) with (18), we get

$$F_n(x) = \sum_{k=0}^{n} f_{n,k} x^k (1 + x^2)^{n-k}. \tag{27}$$

We now consider a change of the grammar (9). Set $q = \frac{1}{2},\ a = x,\ b = 2u,\ c = v$. Then $D(x) = xu,\ D(u) = uv,\ D(v) = 4u^2$, which are the substitution rules in the grammar (24). It follows from (12) that

$$\sum_{n=0}^{\infty} D_{G_3}^n(x) \frac{z^n}{n!} = x \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{n,k} u^k v^{n-k} \frac{z^n}{n!} = xR\left(\frac{2u}{v}, \frac{1}{2}; vz\right),$$

which leads to $f(x, z) = R(2x, 1/2; z) = \sqrt{T(2x, z)}$. This completes the proof. □

The first few $f_n(x)$ are $f_1(x) = x,\ f_2(x) = x + x^2,\ f_3(x) = x + 3x^2 + 5x^3$. By using (23), it is not hard to verify that

$$\sum_{n=0}^{\infty} f_{n,n} \frac{x^n}{n!} = \sqrt{\frac{1 + \tan x}{1 - \tan x}}.$$

The numbers $f_{n,n}$ appear as A012259 in [36]. Combining (23) and (27), we immediately get the following result.

Corollary 20. *We have*

$$F(x, z) = \sum_{n=0}^{\infty} F_n(x) \frac{z^n}{n!} = \sqrt{T\left(\frac{2x}{1 + x^2}, (1 + x^2)z\right)}.$$

Equivalently, we have

$$\sum_{\pi \in \mathfrak{S}_n} (2x)^{\operatorname{udrun}(\pi)} (1 + x^2)^{n - \operatorname{udrun}(\pi)} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma \in \mathcal{Q}_k} x^{\operatorname{fap}(\sigma)} \sum_{\sigma' \in \mathcal{Q}_{n-k}} x^{\operatorname{fap}(\sigma')}. \tag{28}$$

It would be interesting to give a combinatorial proof of (28).

5. Concluding remarks

In this paper, we show that for any γ-positive polynomial, there exists a David-Barton type identity. Based on the survey article [3], one can derive several David-Barton type identities. According to *Hermite-Biehler theorem* of zeros of polynomials [19, p. 228], if $F(x) = f(x^2)+xg(x^2)$ is Hurwitz stable, then both $f(x)$ and $g(x)$ have only nonpositive zeros. From Proposition 16, we see that semi-γ-positivity may be seen as a dual of Hurwitz stability. It would be interesting to explore combinatorial, algebraic and topological significance of semi-γ coefficients.

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