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The asymptotic behavior of some indices of iterated line graphs of regular graphs

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ABSTRACT

In this paper, we consider the asymptotic behavior of the number of spanning trees and the Kirchhoff index of iterated line graphs and iterated para-line graphs (or clique-inserted graphs) of a regular graph *G*. We show that the asymptotic behavior of these indices (except the Kirchhoff index of the iterated para-line graphs) is independent of the structure of the regular graph *G*.

1. Introduction

A line graph is a classical unary operation of graphs with finite number and infinite number of vertices. Its basic properties can be found in any text book on graph theory (see for example [1,3,6]). The concept of the para-line graph (or clique-inserted graph) of a graph is relatively new. The definition of a para-line graph was first introduced in [12]. The author obtained the spectrum of the para-line graph of a regular graph *G* with infinite number of vertices in terms of the spectrum of *G*. In [19], the para-line graph is called the clique-inserted graph. The authors showed that the limits of the spectrum of iterated para-line graphs of a regular graph *G* are a fractal, which is independent of the structure of *G*.

It is well known that the operation of a line graph can produce many new types of graphs. For example, taking the line graphs of honeycomb lattices with free, cylindrical, and toroidal boundaries, we get Kagomé lattices with free, cylindrical, and toroidal boundaries (see [16]). The line graph of a tetrahedron (respectively, cube and dodecahedron) graph is an octahedron (respectively, cuboctahedron and icosidodecahedron) graph. Clearly, we can go further, but iterated line graphs are not polyhedron graphs (3-connected planar graphs). On the other hand, the para-line graphs of honeycomb lattices with free, cylindrical, and toroidal boundaries are the 3.12.12 lattices with free, cylindrical, and toroidal boundaries (see [16]). This iterated procedure can be carried out repeatedly to produce a series of cubic lattices. For cubic polyhedron graphs such as the tetrahedron, cube, and dodecahedron, the iterated procedure produces infinite families of truncated polyhedron graphs.

In this paper, we discuss the growth rate of the number of spanning trees and the Kirchhoff index of iterated line graphs and iterated para-line graphs of a regular graph *G*. The study of enumeration of spanning trees has a long history in both mathematics and physics. Motivated by problems in the real world, the classical work of Kirchhoff [9] and

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Cayley [4] considered the problem of enumeration of spanning trees of graphs. Now, this has become basic material in many text books on graph theory. The concept of the Kirchhoff index appeared relatively late. It had been introduced in chemistry by Klein and Randić [10]. For further work on the Kirchhoff index, see, for example, [5,8,15,17,18,20–23].

In Section 2, we determine the growth rate of the number of spanning trees of iterated line graphs and iterated para-line graphs of an r-regular graph G. We prove that their growth rates are independent of the structure of G and only dependent on r and the number of vertices of G. In Section 3, we determine the growth rate of the Kirchhoff index of iterated line graphs and iterated para-line graphs of a regular graph. We show that the growth rate of the Kirchhoff index of iterated line graphs of an r-regular graph G is independent of the structure of G and only dependent on r and the number of vertices of G; however, the growth rate of the Kirchhoff index of iterated para-line graphs of an r-regular graph G is indeed dependent on the structure of G.

2. The number of spanning trees

The line graph $L(G)$ of a graph $G = (V(G), E(G))$ has vertex set $V(L(G)) = E(G)$, and two vertices e and f are adjacent if and only if two edges e and f in G have a common vertex. Set $L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$, $k = 1, 2, \ldots$. Denote by $t(G)$ the number of spanning trees of a connected graph G.

Let G be an r-regular graph with n vertices and m edges. Obviously, the graph $L^k(G)$ is regular. For convenience, let $L^k(G)$ be an r_k-regular graph with n_k vertices and m_k edges, $k = 0, 1, \ldots$, where $n_0 = n$ and $m_0 = m$. It is not difficult to prove inductively on k the following results:

$$r_k = 2^k r - 2^{k+1} + 2, \tag{1}$$

$$n_k = n \prod_{i=0}^{k-1} (2^{i-1} r - 2^i + 1), \tag{2}$$

$$m_k = n \prod_{i=0}^{k} (2^{i-1} r - 2^i + 1). \tag{3}$$

In particular, $m_k = n_{k+1}$, $k = 0, 1, \ldots$.

For the enumeration of spanning trees of the line graph $L(G)$ of a regular graph, we have the following result.

Lemma 2.1 (*Sachs [1]*). *Let G be a connected simple r-regular graph. Then the number of spanning trees of line graph $L(G)$ of G is given by*

$$t(L(G)) = r^{m-n-1} 2^{m-n+1} t(G),$$

where n and m are the numbers of vertices and edges of G ($m = \frac{nr}{2}$), and $t(G)$ is the number of spanning trees of G.

Lemma 2.2. *Let $\{y_k\}_{k\geq 0}$, $\{f_k\}_{k\geq 0}$, and $\{g_k\}_{k\geq 0}$ be three sequences satisfying the following recurrence relation:*

$$y_{k+1} = f_k y_k + g_k, \quad k \geq 0. \tag{4}$$

Then

$$y_{k+1} = \left(y_0 + \sum_{i=0}^{k} h_i\right) \prod_{j=0}^{k} f_j,$$

where $h_k = s_{k+1} g_k$, $s_{k+1} = (\prod_{i=0}^{k} f_i)^{-1}$, and $s_0 = 1$.

Proof. By (4), we have

$$s_{k+1} y_{k+1} = s_{k+1}(f_k y_k + g_k) = s_k y_k + s_{k+1} g_k.$$

Let $z_k = s_k y_k$. Note that $h_k = s_{k+1} g_k$. Then $z_{k+1} = z_k + h_k$. By induction on k, we can easily prove the following formula.

$$z_{k+1} = y_0 + \sum_{i=0}^{k} h_i.$$

Hence $y_{k+1} = z_{k+1} s_{k+1}^{-1} = (y_0 + \sum_{i=0}^{k} h_i) \prod_{j=0}^{k} f_j$. The lemma thus follows. □

Theorem 2.3. *Let G be a connected simple r-regular graph with n vertices. Then*

$$t(L^k(G)) \sim 2^{kn \prod_{i=0}^{k-1} (2^{i-1} r - 2^i + 1)}, \quad (k \to \infty).$$

Hence the asymptotic value of the number of spanning trees of iterated line graphs of a regular graph is independent of the structure of G.

Proof. Let r_k, n_k, and m_k be defined as in (1)–(3), where $r_0 = r$, $n_0 = n$, and $m_0 = m$. Then $L^k(G)$ is an r_k-regular graph with n_k vertices and m_k edges. By Lemma 2.1,

$$t(L^k(G)) = r_{k-1}^{m_{k-1}-n_{k-1}-1} 2^{m_{k-1}-n_{k-1}+1} t(L^{k-1}G), \quad k \geq 1.$$

Set $y_k = t(L^k(G))$, $f_k = r_k^{m_k-n_k-1} 2^{m_k-n_k+1}$, $k = 1, 2, \ldots$. Hence we have

$$y_{k+1} = f_k y_k, \qquad y_0 = t(G), \quad k \geq 0.$$

By Lemma 2.2,

$$y_k = t(G) \prod_{i=0}^{k-1} r_i^{m_i-n_i-1} 2^{m_i-n_i+1}.$$

Hence, by (1)–(3),

$$\begin{aligned}\log y_k &= \sum_{i=0}^{k-1} (m_i - n_i - 1) \log(2^i r - 2^{i+1} + 2) + \sum_{i=0}^{k-1} (m_i - n_i + 1) \log 2 + \log t(G) \\ &\leq \log(2^{k-1} r - 2^k + 2) \sum_{i=0}^{k-1} (m_i - n_i - 1) + \sum_{i=0}^{k-1} (m_i - n_i + 1) \log 2 + \log t(G).\end{aligned}$$

Note that $n_{k+1} = m_k$, $k \geq 0$. Hence $\sum_{i=0}^{k-1} (m_i - n_i - 1) = m_{k-1} - n_0 - k$. So we have

$$\log y_k \leq (m_{k-1} - n_0 - k)[\log(2^{k-1} r - 2^k + 2) + \log 2] + \log t(G). \tag{5}$$

On the other hand,

$$\log y_k \geq (m_{k-1} - n_{k-1} - 1) \log(2^{k-1} r - 2^k + 2). \tag{6}$$

By (5) and (6), it is not difficult to show the following result.

$$\lim_{k \to \infty} \frac{\log y_k}{k n_k} = \log 2. \tag{7}$$

Hence the theorem is immediate from (7). □

Now, we consider iterated para-line graphs. Let G be a simple graph. A para-line graph, denoted by $C(G)$, is defined as a line graph of the subdivision graph $S(G)$ (i.e., $S(G)$ is the graph obtained from G by inserting a vertex to every edge of G.) of G [12]. The para-line graph has also been called the clique-inserted graph [19]. Set $C^k(G) = C(C^{k-1}(G))$, $k \geq 1$, where $C^0(G) = G$. In [19], the authors proved the following result.

Lemma 2.4 ([19]). *Let G be an r-regular graph with n vertices. Then the number of spanning trees of the iterated para-line graphs $C^k(G)$ of G can be expressed by*

$$t(C^k(G)) = r^{ns-k} (r+2)^{ns+k} t(G),$$

where $s = (r/2 - 1)(r^k - 1)/(r - 1)$.

Theorem 2.5. *Let G be a simple r-regular graph with n vertices. Then*

$$t(C^k(G)) \sim (r^2 + 2r)^{n(r-2)r^k/(2(r-1))}, \quad (k \to \infty).$$

Hence the asymptotic value of the number of spanning trees of the iterated para-line graphs of a regular graph G is independent of the structure of G.

Proof. For convenience, set $n'_k = n \cdot r^k$, $m'_k = n \cdot r^{k+1}/2$, $k \geq 0$. Note that G is an r-regular graph with n vertices. By the definition of a para-line graph, it is not difficult to show that $C^k(G)$ is an r-regular graph with n'_k vertices and m'_k edges. By Lemma 2.4,

$$\lim_{k \to \infty} \frac{\log t(C^k(G))}{n \cdot r^k} = \lim_{k \to \infty} \frac{(ns - k) \log r + (ns + k) \log(r + 2) + \log t(G)}{n \cdot r^k},$$

where $s = (r/2 - 1)(r^k - 1)/(r - 1)$.

It is not difficult to prove the following limit:

$$\lim_{k \to \infty} \frac{\log t(C^k(G))}{n \cdot r^k} = \frac{r-2}{2(r-1)} \log(r^2 + 2r),$$

implying the theorem immediately. □

Let $\{L_n\}$ be a regular lattice sequence with v_n vertices, and let $t(L_n)$ be the number of spanning trees of L_n. In the study of statistical physics, to prove that

$$t(L_n) \sim \exp(v_n Z_c) \quad (n \to \infty)$$

and evaluate Z_c exactly are favorable topics (see, for example, [13] and the references cited therein.). By Theorem 2.5, we see that for the iterated para-line graph $C^k(G)$ of an r-regular graph G,

$$Z_c = \frac{r-2}{2(r-1)} \log(r^2 + 2r).$$

In particular, if $r = 3$, then we have $Z_c \approx 0.2940$. We may compare this result with that of some well-known 3-regular lattices, such as 3.12.12, 4.4.8, and hexagonal lattices with toroidal boundary conditions. For these three lattices, Shrock and Wu [13] showed that $Z_c \approx 0.7206, 0.7867$, and 0.80767, respectively. It is clear that the constant Z_c of the iterated para-line graphs of an r-regular graph G is evidently smaller than that of the above three lattices. On the other hand, by (7), we can see that, for the iterated line graphs $L^k(G)$ of an r-regular graph G $(r > 2)$, $Z_c = \infty$. This is "natural", since, for the degree r_k of $L^k(G)$, $r_k \to \infty$ if $k \to \infty$.

3. The Kirchhoff index

Klein and Randić [10] considered the so-called resistance distance between the vertices of a (molecular) graph G and its applications in chemistry; it is equal to the resistance between two respective vertices of an electrical network, constructed so as to correspond to G, and having the property that the resistance of each edge joining adjacent vertices is unity. The Kirchhoff index of a graph G, denoted by $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices [2,10]. That is,

$$Kf(G) = \sum_{\{u,v\} \subset V(G)} R(u, v),$$

where $R(u, v)$ equals the resistance distance between the vertices u and v of graph G. This index had been used earlier for other purposes [7]. At almost exactly the same time, Gutman and Mohar [8] and Zhu et al. [23] proved that, if G is a connected simple graph with n vertices, then

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i(G)}, \tag{8}$$

where $\mu_1(G), \mu_2(G), \ldots, \mu_{n-1}(G)$ are the non-zero Laplacian eigenvalues of G.

The following lemma is well known.

Lemma 3.1 ([1]). *Let G be a simple r-regular graph with n vertices and m edges. Suppose that the spectrum of G is $\{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$. Then the spectrum of $L(G)$ is $\{\lambda'_1 = 2r-2, \lambda'_2 = \lambda_2 + r - 2, \ldots, \lambda'_n = \lambda_n + r - 2, \lambda'_{n+1} = \lambda'_{n+2} = \cdots = \lambda'_m = -2\}$.*

Lemma 3.2. *Let G be a connected simple r-regular graph with n vertices and m edges, and let $L(G)$ be the line graph of G. Then*

$$Kf(L(G)) = \frac{1}{4} n(m-n) + \frac{r}{2} Kf(G).$$

Proof. Suppose that the eigenvalues of G are $\lambda_1 = r, \lambda_2, \ldots, \lambda_n$. Then $r - \lambda_2, r - \lambda_3, \ldots, r - \lambda_n$ are the non-zero Laplacian eigenvalues of G. By (8),

$$Kf(G) = n \sum_{i=2}^{n} \frac{1}{r - \lambda_i}. \tag{9}$$

On the other hand, by Lemma 3.1, the non-zero Laplacian eigenvalues of $L(G)$ are $2r-2-(\lambda_i+r-2) = r-\lambda_i$ $(i = 2, 3, \ldots, n)$ and $2r - 2 - (-2) = 2r$, with multiplicity $m - n$. Note that the number of vertices of $L(G)$ equals $\frac{nr}{2}$. Hence, by (8) and (9),

$$Kf(L(G)) = \frac{nr}{2} \left[\sum_{i=2}^{n} \frac{1}{r - \lambda_i} + \frac{m-n}{2r} \right] = \frac{r}{2} Kf(G) + \frac{1}{4} n(m-n).$$

So the lemma follows. □

Theorem 3.3. *Let G be a connected simple r-regular graph with n vertices. Then*

$$Kf(L^k(G)) \sim \frac{n^2(2^{k-2}r - 2^{k-1} + 1)}{4} \prod_{i=0}^{k-2} (2^{i-1}r - 2^i + 1)^2, \quad (k \to \infty).$$

Hence the asymptotic value of the Kirchhoff index of iterated line graphs of a regular graph is independent of the structure of G.

Proof. Let r_k, n_k, and m_k be defined as in (1)–(3), where $r_0 = r$, $n_0 = n$, and $m_0 = m$. Then $L^k(G)$ is an r_k-regular graph with n_k vertices and m_k edges. By Lemma 3.2,

$$Kf(L^k(G)) = \frac{1}{4}n_{k-1}(m_{k-1} - n_{k-1}) + \frac{r_{k-1}}{2}Kf(L^{k-1}G), \quad k \geq 1.$$

Set $y_k = Kf(L^k(G)), f_k = \frac{r_k}{2}, g_k = \frac{1}{4}n_k(m_k - n_k), k = 1, 2, \ldots$. Hence we have

$$y_{k+1} = f_k y_k + g_k, \qquad y_0 = Kf(G), \quad k \geq 0.$$

By Lemma 2.2,

$$\begin{aligned} Kf(L^{k+1}(G)) &= y_{k+1} = \left[Kf(G) + \sum_{i=0}^{k}\frac{1}{4}n_i(m_i - n_i)\prod_{j=0}^{i}\frac{2}{r_j}\right]\prod_{j=0}^{k}\frac{r_j}{2} \\ &= Kf(G)\prod_{j=0}^{k}\frac{r_j}{2} + \sum_{i=0}^{k-1}\frac{1}{4}n_i(m_i - n_i)\prod_{j=i+1}^{k}\frac{r_j}{2} + \frac{n_k(m_k - n_k)}{4}. \end{aligned} \tag{10}$$

Set

$$t_i = n_i(m_i - n_i)\prod_{j=i+1}^{k}\frac{r_j}{2}, \quad i = 0, 1, \ldots, k-1.$$

Note that $n_i = \frac{n}{2^i}\prod_{j=0}^{i-1} r_j$, $m_i = \frac{n}{2^{i+1}}\prod_{j=0}^{i} r_j$. Hence

$$m_i n_i = \frac{n^2 r_i}{2^{2i+1}}\prod_{j=0}^{i-1} r_j^2, \qquad n_i^2 = \frac{n^2}{2^{2i}}\prod_{j=0}^{i-1} r_j^2.$$

For $i = 0, 1, \ldots, k-1$, it is not difficult to show that

$$k \cdot \frac{t_i}{m_k n_k} = \frac{k \cdot 2^{k-i}}{\prod_{j=i}^{k-1} r_j} - \frac{k \cdot 2^{k-i+1}}{r_i \prod_{j=i}^{k-1} r_j} \longrightarrow 0 \quad (k \to \infty).$$

Set

$$t = \max_{0 \leq i \leq k-1} t_i.$$

Obviously,

$$\lim_{k \to \infty} \frac{kt}{m_k n_k} = 0.$$

By (10),

$$\frac{n_k(m_k - n_k)}{4m_k n_k} \leq \frac{Kf(L^{k+1}(G))}{m_k n_k} \leq \frac{Kf(G)\prod_{j=0}^{k}\frac{r_j}{2}}{m_k n_k} + \frac{kt}{4m_k n_k} + \frac{n_k(m_k - n_k)}{4m_k n_k}.$$

From the above identity, it is not difficult to prove the following result.

$$\lim_{k \to \infty} \frac{Kf(L^{k+1})(G)}{m_k n_k} = \frac{1}{4},$$

which implies the theorem immediately. □

Lemma 3.4 ([19]). *Let G be a simple r-regular graph with n vertices and m edges. Suppose that the eigenvalues of G are $\lambda_1 = r, \lambda_2, \ldots, \lambda_n$. Then the eigenvalues of the para-line graph $C(G)$ of G are $\frac{r-2\pm\sqrt{r^2+4(\lambda_i+1)}}{2}$ $(i = 1, 2, \ldots, n)$, -2, with multiplicity $m - n$, and 0, with multiplicity $m - n$.*

Lemma 3.5. *Let G be a simple r-regular graph with n vertices and m edges, and let $C(G)$ be the para-line graph of G. Then*

$$Kf(C(G)) = \frac{nr(m - n + 1)}{r + 2} + n(m - n) + r(r + 2)Kf(G).$$

Proof. In the proof of Lemma 3.2, we have

$$Kf(G)=n\sum_{i=2}^{n}\frac{1}{r-\lambda_i},$$

where $\lambda_1=r,\lambda_2,\ldots,\lambda_n$ are the eigenvalues of G. On the other hand, by Lemma 3.4, the non-zero Laplacian eigenvalues of $C(G)$ are $r+2$, with multiplicity $m-n$, r, with multiplicity $m-n$, $r-\frac{r-2+\sqrt{r^2+4(\lambda_i+1)}}{2}$ $(i=2,3,\ldots,n)$, and $r-\frac{r-2-\sqrt{r^2+4(\lambda_i+1)}}{2}$ $(i=1,2,\ldots,n)$. Note that the number of vertices of $C(G)$ equals nr. Hence, by (8),

$$\begin{aligned}Kf(C(G))&=nr\left[\sum_{i=2}^{n}\frac{1}{r-\frac{r-2+\sqrt{r^2+4(\lambda_i+1)}}{2}}+\sum_{i=1}^{n}\frac{1}{r-\frac{r-2-\sqrt{r^2+4(\lambda_i+1)}}{2}}+\frac{m-n}{r+2}+\frac{m-n}{r}\right]\\&=nr\left[\sum_{i=2}^{n}\frac{2}{r+2-\sqrt{r^2+4(\lambda_i+1)}}+\sum_{i=2}^{n}\frac{2}{r+2+\sqrt{r^2+4(\lambda_i+1)}}+\frac{m-n+1}{r+2}+\frac{m-n}{r}\right]\\&=nr\left[\sum_{i=2}^{n}\frac{r+2}{r-\lambda_i}+\frac{m-n+1}{r+2}+\frac{m-n}{r}\right]\\&=\frac{nr(m-n+1)}{r+2}+n(m-n)+r(r+2)Kf(G).\end{aligned}$$

So the lemma follows. □

Theorem 3.6. *Let G be a connected simple r-regular graph with n vertices. Then*

$$Kf(C^k(G))\sim\left[\frac{n^2(r-2)(r+1)}{2r(r+2)}+\frac{n}{(r+1)(r+2)}+Kf(G)\right]r^k(r+2)^k,\quad(k\to\infty).$$

$$Kf(C^k(G))\sim\left(\frac{n^2}{2}+Kf(G)\right)r^k(r+2)^k,\quad(k\to\infty,r\to\infty).$$

Hence the asymptotic value of the Kirchhoff index of iterated para-line graphs of a regular graph is dependent on the structure of G.

Proof. Let m_k and n_k denote the numbers of edges and vertices of $C^k(G)$, respectively. Then $m_k=\frac{1}{2}nr^{k+1}$ and $n_k=nr^k$, $k=0,1,\ldots$. By Lemma 3.5, we have

$$Kf(C^k(G))=\frac{n_{k-1}r(m_{k-1}-n_{k-1}+1)}{r+2}+n_{k-1}(m_{k-1}-n_{k-1})+r(r+2)Kf(C^{k-1}(G)).\tag{11}$$

Let $y_k=Kf(C^k(G))$, $f_k=r(r+2)$, and $g_k=\frac{n_{k-1}r(m_{k-1}-n_{k-1}+1)}{r+2}+n_{k-1}(m_{k-1}-n_{k-1})$, $k=1,2,\ldots$. Hence we have the following recurrence:

$$y_{k+1}=f_ky_k+g_k,\quad k\ge 0.$$

By Lemma 2.2,

$$\begin{aligned}Kf(C^{k+1}(G))&=y_{k+1}\\&=\left\{Kf(G)+\sum_{i=0}^{k}\left[\frac{n_{k-1}r(m_{k-1}-n_{k-1}+1)}{r+2}+n_{k-1}(m_{k-1}-n_{k-1})\right]r^{-i-1}(r+2)^{-i-1}\right\}r^{k+1}(r+2)^{k+1}\\&=\frac{r}{r+2}\sum_{i=0}^{k}r^{k-i}(r+2)^{k-i}n_i(m_i-n_i+1)+\sum_{i=0}^{k}r^{k-i}(r+2)^{k-i}n_i(m_i-n_i)+r^{k+1}(r+2)^{k+1}Kf(G).\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{i=0}^{k}r^{k-i}(r+2)^{k-i}n_i(m_i-n_i+1)\\&=r^k(r+2)^k\sum_{i=0}^{k}\frac{nr^i\left(\frac{1}{2}nr^{i+1}-nr^i+1\right)}{r^i(r+2)^i}=r^k(r+2)^k\sum_{i=0}^{k}\left[\frac{n^2r^{i+1}}{2(r+2)^i}-\frac{n^2r^i}{(r+2)^i}+\frac{n}{(r+2)^i}\right]\\&\longrightarrow r^k(r+2)^k\left[\frac{rn^2}{2}\frac{1}{1-\frac{r}{R+2}}-\frac{n^2}{1-\frac{r}{r+2}}+\frac{n}{1-\frac{1}{r+2}}\right]\quad(k\to\infty)\end{aligned}$$

$$= r^k(r+2)^k\left[\frac{n^2r(r+2)}{4} - \frac{n^2(r+2)}{2} + \frac{n(r+2)}{r+1}\right] = r^k(r+2)^k\left[\frac{n^2(r-2)(r+2)}{4} + \frac{n(r+2)}{r+1}\right].$$

Similarly,

$$\sum_{i=0}^{k} r^{k-i}(r+2)^{k-i}n_i(m_i - n_i) \longrightarrow \frac{n^2(r-2)(r+2)}{4}r^k(r+2)^k, \quad (k \to \infty).$$

Hence

$$\begin{aligned} Kf(C^{k+1}(G)) &\sim \frac{r}{r+2}r^k(r+2)^k\left[\frac{n^2(r-2)(r+2)}{4} + \frac{n(r+2)}{r+1}\right] \\ &\quad + r^k(r+2)^k\frac{n^2(r-2)(r+2)}{4} + r^{k+1}(r+2)^{k+1}Kf(G) \quad (k \to \infty) \\ &= \left[\frac{n^2(r-2)(r+1)}{2r(r+2)} + \frac{n}{(r+1)(r+2)} + Kf(G)\right]r^{k+1}(r+2)^{k+1}. \end{aligned}$$

So the theorem follows. □

Finally, we would like to point out that, in the expression of Theorem 3.3, the asymptotic behavior only depends on the number of vertices and regularity of the initial graph G, but, in Theorem 3.6, it depends on the structure of G. In the case of the energy of iterated graphs (the energy of a graph G is defined as the sum of the absolute values of eigenvalues of G), in [11,14], the authors showed that its asymptotic behavior has a strong form, as follows.

Theorem 3.7 (*Ramane, Walikar, et al., [11]*)**.** *Let G be an r-regular graph with n vertices. Then, for $k \geq 1$, the energy $\mathcal{E}(L^{k+1}(G))$ of $L^{k+1}(G)$ can be expressed by*

$$\mathcal{E}(L^{k+1}(G)) = 2n(r-2)\prod_{i=0}^{k-1}(2^ir - 2^{i+1} + 2).$$

Theorem 3.8 (*Walikar, Ramane et al., [14]*)**.** *Let G be an r-regular graph with n vertices. Then, for $k \geq 1$, the energy $\mathcal{E}(\overline{L^{k+1}(G)})$ of the complement of $L^{k+1}(G)$ can be expressed by*

$$\mathcal{E}(\overline{L^{k+1}(G)}) = \left[\frac{n}{2^{k-1}}\prod_{i=0}^{k-1}(2^ir - 2^{i+1} + 2) - 4\right](2^kr - 2^{k+1} + 1) - 2.$$

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